

3. Teoría de la torsión

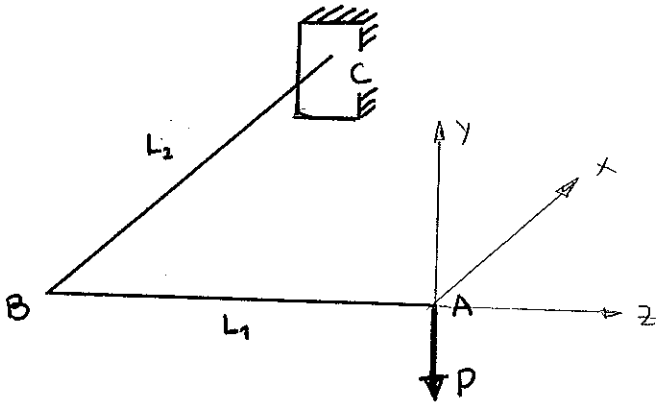
3.1

● Sección circular

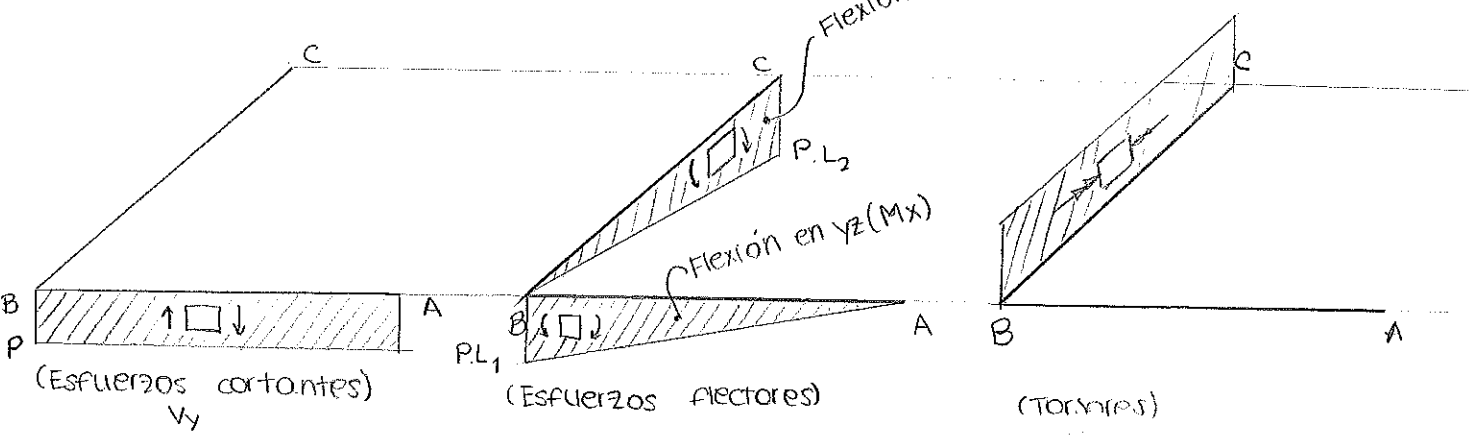
● E, G, I, R

$I_p = 2I$

(se desprecia el efecto de los esfuerzos cortantes en la deformación)

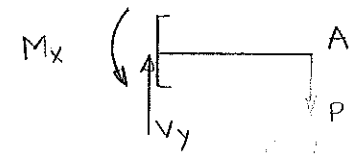


1) DIAGRAMAS DE ESFUERZOS:

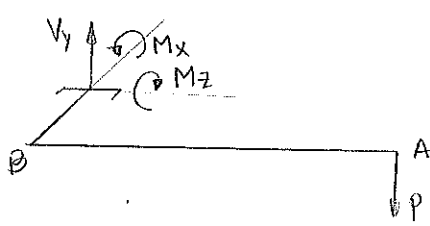


AB ($0 \leq z \leq L_1$)

BC ($0 \leq x \leq L_2$)



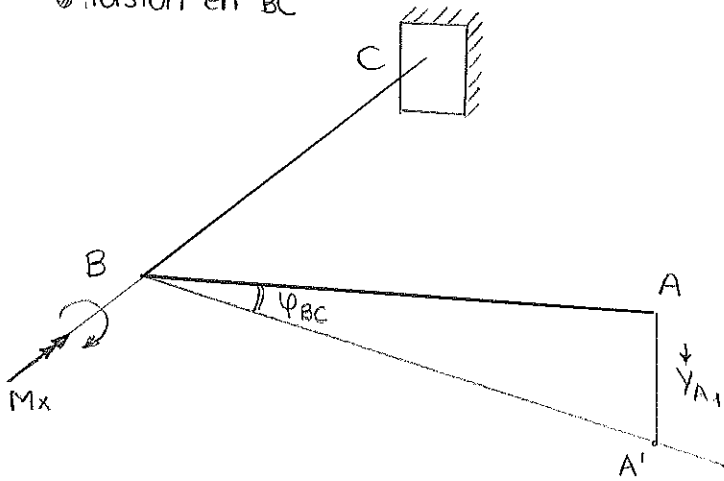
$M_x = P \cdot z$
 $V_y = P$



$M_x = P \cdot L_1$
 $M_z = P \cdot x$
 $V_y = P$

2) DEFORMADA

● Torsión en BC

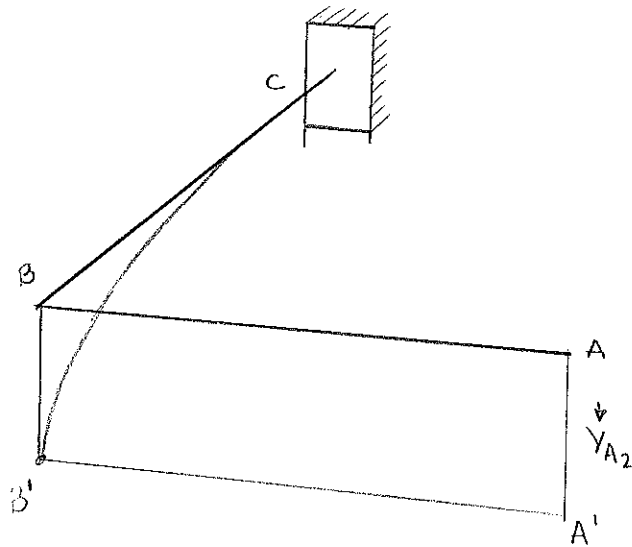


$$\downarrow Y_{A1} = \varphi_{BC} \cdot L_1$$

$$\varphi_{BC} = \frac{M_x (\Delta x)_{BC}}{G \cdot I_p} = \frac{P \cdot L_1 (L_2 - 0)}{2 \cdot G \cdot I} = \frac{P \cdot L_1 \cdot L_2}{2 \cdot G \cdot I}$$

$$\downarrow Y_{A1} = \frac{P L_1^2 L_2}{2 G I}$$

● Flexión de BC

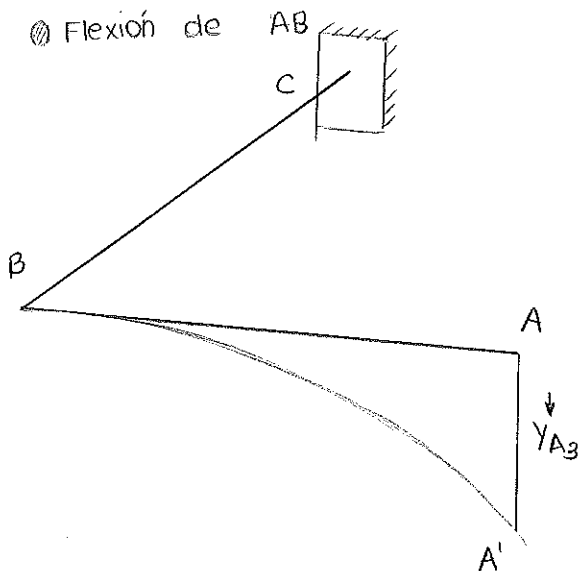


$$\downarrow Y_{A2} = AA' = BB'$$

$$BB' = \delta_{BC} = \frac{1}{EI} \left[\frac{1}{2} P \cdot L_2 \cdot L_2 \cdot \frac{2}{3} L_2 \right] = \frac{P L_2^3}{3EI}$$

$$\downarrow Y_{A2} = \frac{P L_2^3}{3EI}$$

● Flexión de AB



$$\downarrow Y_{A3} = AA'$$

$$AA' = \delta_{AB} = \frac{1}{EI} \left[\frac{1}{2} P L_1 \cdot L_1 \cdot \frac{2}{3} L_1 \right] = \frac{P L_1^3}{3EI}$$

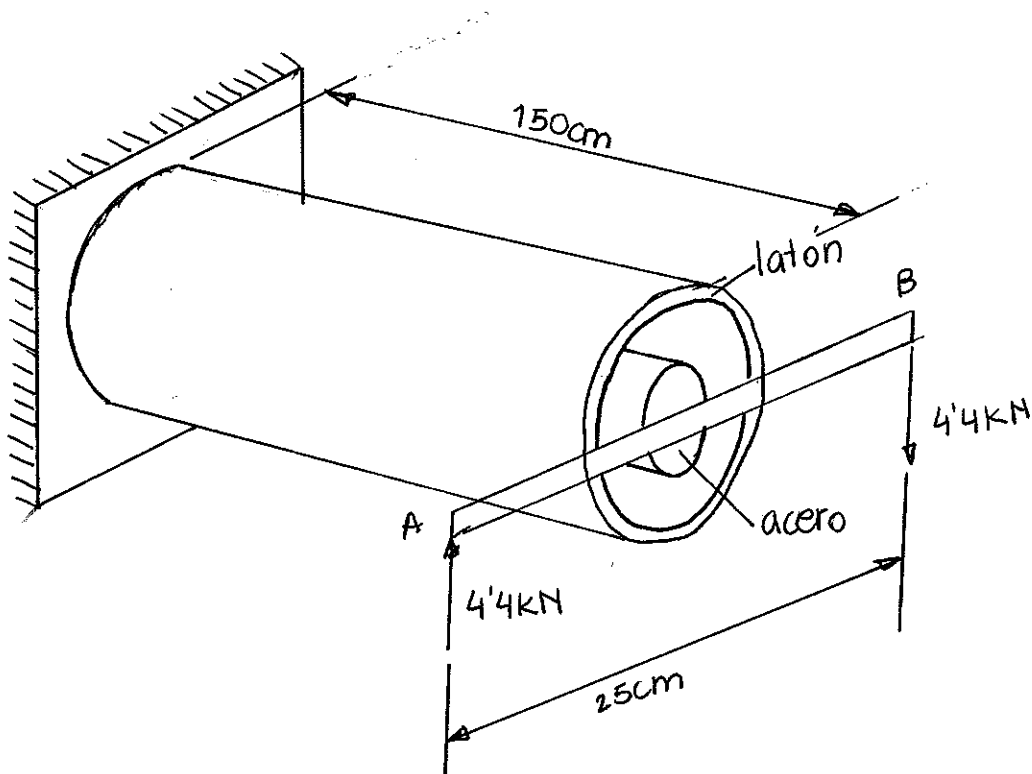
$$\downarrow Y_{A3} = \frac{P L_1^3}{3EI}$$

$$\downarrow Y_A = \sum Y_{Ai} = \frac{P L_1^2 L_2}{2 G I} + \frac{P \cdot L_2^3}{3 E I} + \frac{P L_1^3}{3 E I}$$

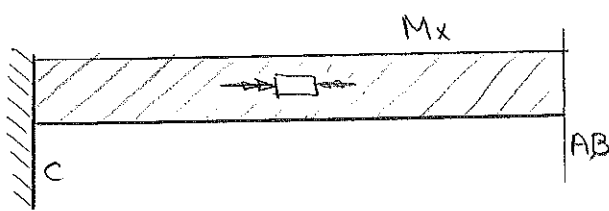
8.2

● Eje latón : $\phi_e = 15\text{cm}$, $\phi_i = 10\text{cm}$; $G_L = 35\text{GPa}$

● Eje acero : $\phi = 5\text{cm}$; $G_A = 80\text{GPa}$



1) DIAGRAMA DE ESFUERZOS.



$M_x = 1100\text{ N.m}$

● En cada sección $M_x = M_A + M_L$

Ec. compatibilidad: $\psi_A = \psi_L$

(1) $\psi_A = \frac{M_A \cdot L}{G_A \cdot I_{PA}}$

(2) $\psi_L = \frac{M_L \cdot L}{G_L \cdot I_{PL}}$

$I_{PA} = \frac{\pi R^4}{2} = 6'135923 \cdot 10^{-7} \text{ m}^4$

$I_{PL} = \frac{\pi(R_e^4 - R_i^4)}{2} = 9,988350 \cdot 10^{-5} \text{ m}^4$

$$\frac{M_A \cdot L}{G_A \cdot I_{PA}} = \frac{M_L \cdot L}{G_L \cdot I_{PL}} \Rightarrow \boxed{M_A = \frac{M_L \cdot G_A \cdot I_{PA}}{G_L \cdot I_{PL}}}$$

$$M_x = M_A + M_L$$

$$1100 \text{ N.m} = \frac{M_L + G_A \cdot I_{PA}}{G_L \cdot J_R} + M_L \Rightarrow M_L = 1062,633 \text{ Nm}$$

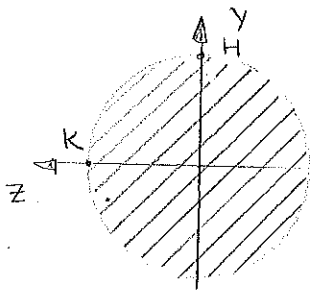
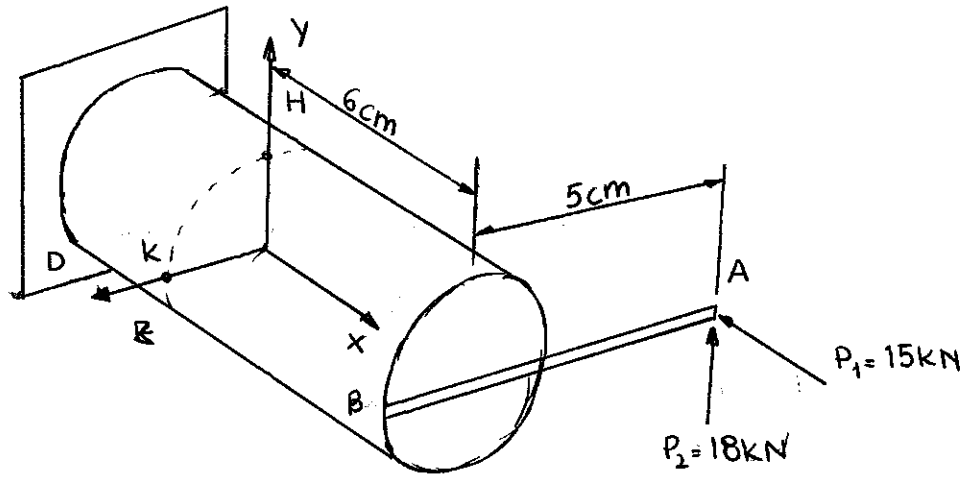
$$\varphi_L = \frac{M_L \cdot L}{G_L J_{PL}} = 1,142 \cdot 10^{-3} \text{ rad}$$

$$\varphi = \varphi_L = 1,142 \cdot 10^{-3} \text{ rad}$$

3.3

$\otimes R_{BD} = 2\text{cm}$

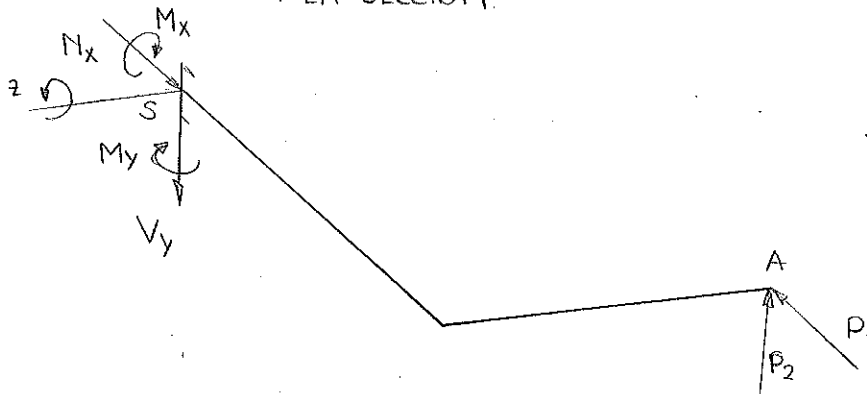
$\otimes \sigma_F = 240\text{MPa}$



$$I_y = I_z = \frac{\pi R^4}{4} = 1'25664 \cdot 10^{-7} \text{ m}^4$$

$$I_p = 2I = 2'51327 \cdot 10^{-7} \text{ m}^4$$

1) ESFUERZOS EN LA SECCIÓN.



$$\begin{cases} N_x = P_1 = 15 \text{ kN} \\ V_y = P_2 = 18 \text{ kN} \\ M_x = P_2 \cdot 5\text{cm} = 18 \text{ kN} \cdot 5\text{cm} = 900 \text{ N}\cdot\text{m} \\ M_y = P_1 \cdot 5\text{cm} = 15 \text{ kN} \cdot 5\text{cm} = 750 \text{ N}\cdot\text{m} \\ M_z = P_2 \cdot 6\text{cm} = 18 \text{ kN} \cdot 6\text{cm} = 1080 \text{ N}\cdot\text{m} \end{cases}$$

2) TENSIONES EN LOS PUNTOS K y H

(K)

$$\sigma_{xx} = \sigma_{xx}^{(N_x)} + \sigma_{xx}^{(M_y)} + \sigma_{xx}^{(M_z)}$$

$$\sigma_{xx}^{(N_x)} = \frac{N_x}{A} = \frac{15 \text{ kN}}{\pi \cdot (2\text{cm})^2} = 11'937 \text{ MPa} (-)$$

$$\sigma_{xx}^{(M_y)} = \frac{M_y \cdot z}{I_y} = \frac{750 \text{ N}\cdot\text{m} \cdot 1\text{cm}}{1'25664 \cdot 10^{-7} \text{ m}^4} = 119'366 \text{ MPa} (+)$$

$$\sigma_{xx}^{(M_z)} = \frac{M_z \cdot y}{I_z} = 0$$

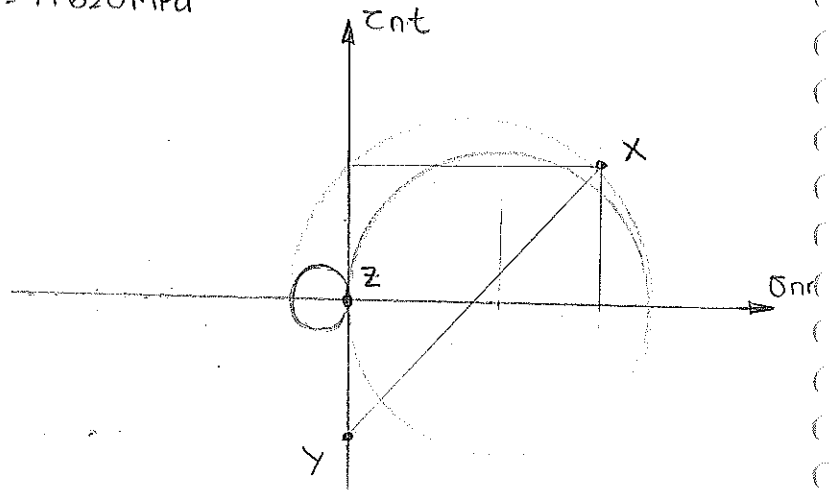
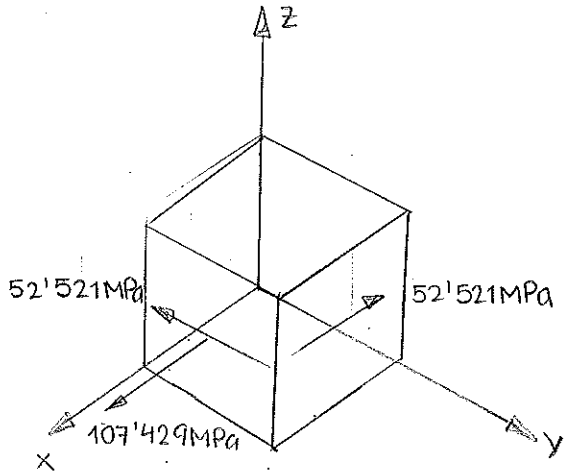
$\sigma_{xx}^{(K)} = 107,429 \text{ MPa} (+)$

$$\tau_{xy} = \tau_{xy}^{(V)} + \tau_{xy}^{(M)}$$

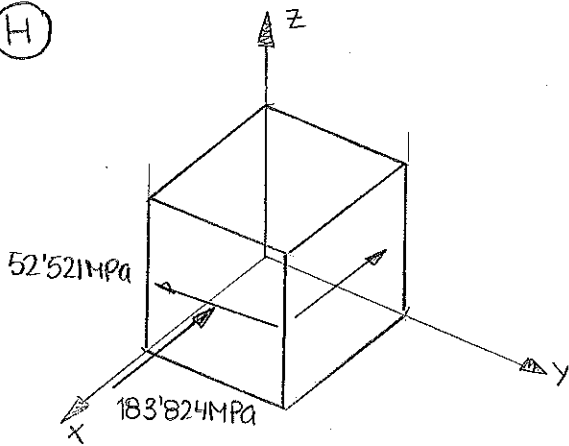
$$\tau_{xy}^{(V)} = \frac{4}{3} \frac{V_y}{A} = \frac{4}{3} \frac{18 \text{ kN}}{\pi (2 \text{ cm})^2} = 19'099 \text{ MPa}$$

$$\tau_{xy}^{(M)} = \frac{M_x R}{I_p} = \frac{900 \text{ Nm} \cdot 2 \text{ cm}}{2'51327 \cdot 10^{-7} \text{ m}^4} = 71'620 \text{ MPa}$$

$$\tau_{xy}^{(k)} = 52'521 \text{ MPa}$$



(H)



$$\sigma_{xx}^{(H)} = \sigma_{xx}^{(N_x)} + \sigma_{xx}^{(M_y)} + \sigma_{xx}^{(M_z)}$$

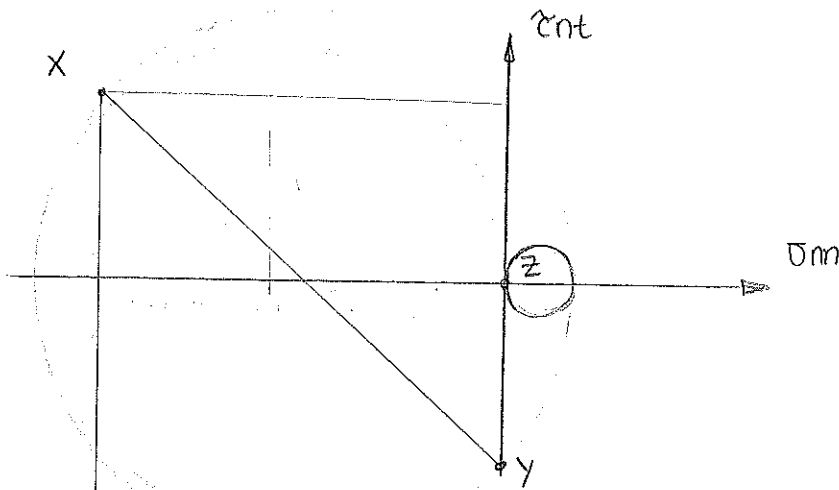
$$\sigma_{xx}^{(N_x)} = \frac{N_x}{A} = \frac{15 \text{ kN}}{\pi (2 \text{ cm})^2} = 11'937 \text{ MPa} (-)$$

$$\sigma_{xx}^{(M_y)} = \frac{M_y \cdot z}{I_y} = 0$$

$$\sigma_{xx}^{(M_z)} = \frac{M_z \cdot y}{I_z} = \frac{1080 \text{ Nm} \cdot 2 \text{ cm}}{1'25664 \cdot 10^{-7} \text{ m}^4} = 171'887 \text{ MPa} (-)$$

$$\sigma_{xx}^{(H)} = 183'824 \text{ MPa} (-)$$

$$\tau_{xy}^{(H)} = \tau_{xy}^{(V)} + \tau_{xy}^{(M)} = \tau_{xy}^{(k)} = 52'521 \text{ MPa}$$



3) COEFICIENTES DE SEGURIDAD.

$$\textcircled{K} \text{ Tresca: } \sqrt{\sigma_{xx}^2 + 4\tau_{xy}^2} = \sigma_{eq}$$

$$\sigma_{eq}^{(K)} = \sqrt{(107,429 \text{ MPa})^2 + 4(52,521 \text{ MPa})^2} = 150,249 \text{ MPa}$$

$$\boxed{n^{(K)} = \frac{\sigma_f}{\sigma_{eq}^{(K)}} = 1,6}$$

$$\sigma_{eq}^{(H)} = \sqrt{(183,824 \text{ MPa})^2 + 4 \left(\frac{52,521 \text{ MPa}}{71,620} \right)^2} = 211,718 \text{ MPa}$$

71,620 233,043

$$\boxed{n^{(H)} = \frac{\sigma_f}{\sigma_{eq}^{(H)}} = 1,3} \quad 1,03$$

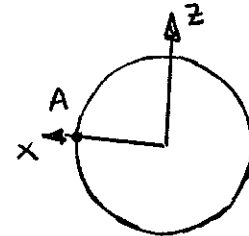
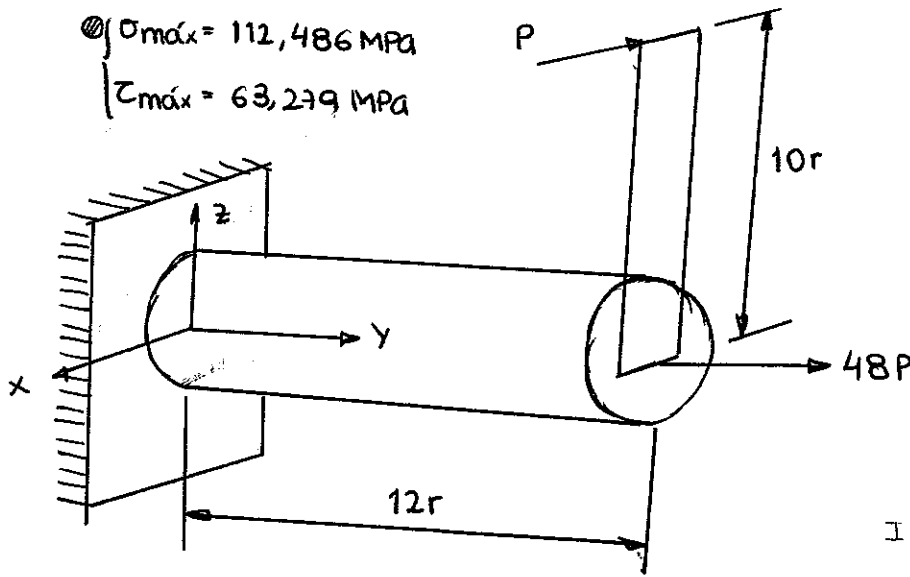


34

$r = 5,08 \text{ cm}$

Punto más desfavorable: A

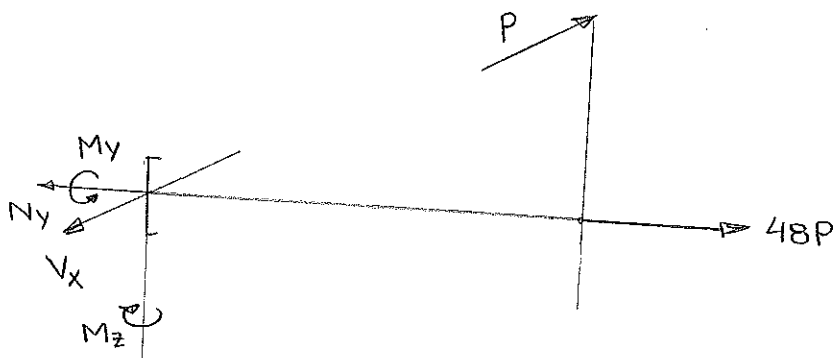
$\sigma_{\text{máx}} = 112,486 \text{ MPa}$
 $\tau_{\text{máx}} = 63,279 \text{ MPa}$



$I_z = I_x = \frac{\pi R^4}{4} = 5,23052 \cdot 10^{-6} \text{ m}^4$

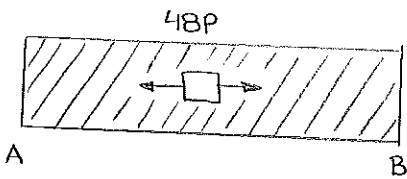
$I_p = 2I = 1,046104 \cdot 10^{-5} \text{ m}^4$

1) DIAGRAMA DE ESFUERZOS.

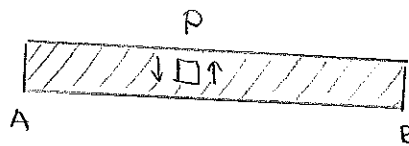


$N_y = 48P$
 $V_x = P$
 $M_y = P \cdot 10r$
 $M_z = P \cdot y$

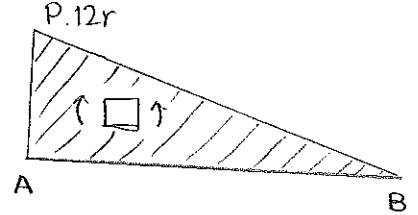
Esfuerzos normales



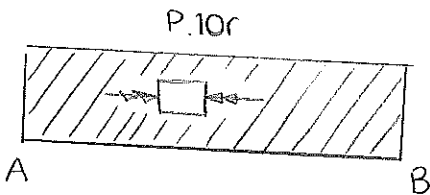
Esfuerzos cortantes



Esfuerzos flectores



Esfuerzos torsores



Deducimos que la sección más desfavorable es la del empotramiento.

2) TENSIONES EN EL EMPOTRAMIENTO

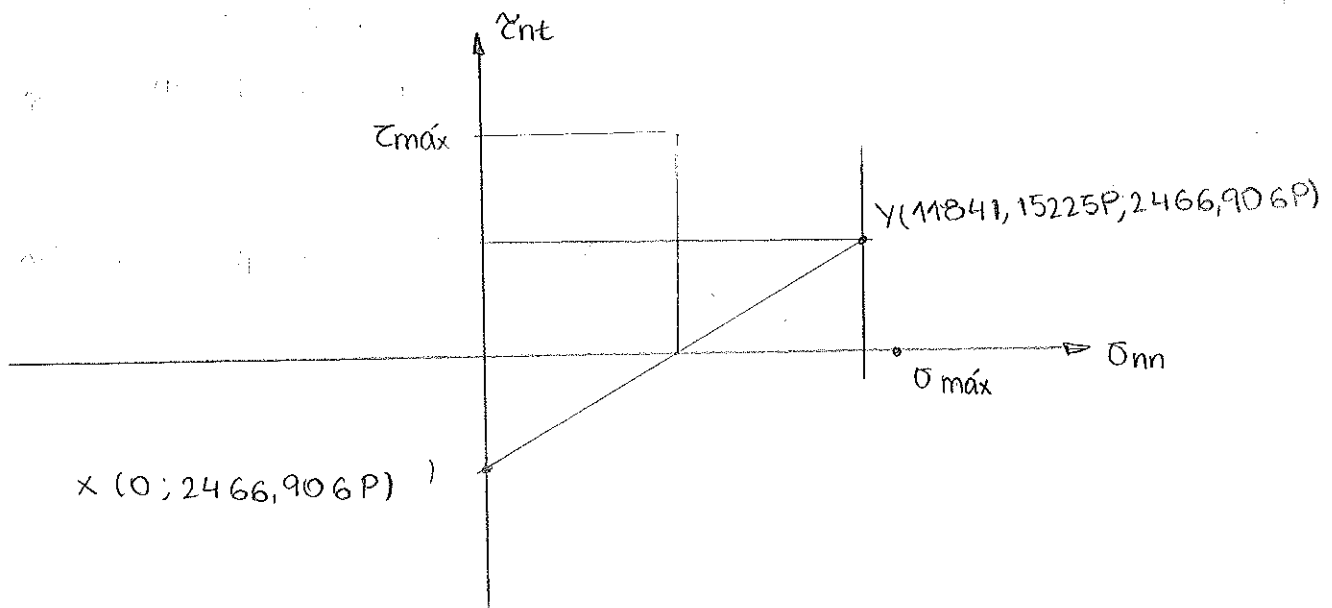
$$\textcircled{A} \quad \sigma_{yy}^{(A)} = \sigma_{yy}^{(My)} + \sigma_{yy}^{(Mz)}$$

$$\sigma_{yy}^{(My)} = \frac{My}{A} = \frac{48P}{\pi(5,08\text{cm})^2} = 5920,576 P \text{ m}^{-2} (+)$$

$$\sigma_{yy}^{(A)} = 11841,15225 P \text{ m}^{-2} (+)$$

$$\sigma_{yy}^{(Mz)} = \frac{Mz \cdot x}{I_z} = \frac{P \cdot 12r \cdot r}{5'230518 \cdot 10^{-6} \text{m}^4} = 5920,576 P \text{ m}^{-2} (+)$$

$$\tau_{yz}^{(A)} = \tau_{yz}^{(My)} = \frac{My \cdot r}{I_p} = \frac{P \cdot 10r \cdot r}{1'046104 \cdot 10^{-5} \text{m}^4} = 2466,906 P \text{ m}^{-2}$$



$$\sigma_{\text{máx}} = 11841,15225P + \frac{\sqrt{(11841,15225P)^2 + (2 \cdot 2466,906P)^2} - 11841,15225P}{2} = 12334,533P \text{ m}^{-2}$$

$$\sigma_{\text{máx}} = 12334,533 P \text{ m}^{-2} = 112,4186 \text{ MPa}$$

$$P_{\text{máx}}^{\sigma} = 9119,599 \text{ N}$$

$$\tau_{\text{máx}} = 2466,906P + \frac{\sqrt{(11841,15225P)^2 + (2 \cdot 2466,906P)^2} - 2 \cdot 2466,906P}{2} = 6413,957 P \text{ m}^{-2}$$

$$\tau_{\text{máx}} = 6413,957 P \text{ m}^{-2} = 63,279 \text{ MPa}$$

$$P_{\text{máx}}^{\tau} = 9865,828 \text{ N}$$

$$P_{\text{máx}}^{\sigma} < P_{\text{máx}}^{\tau} \Rightarrow \boxed{P_{\text{máx}} = P_{\text{máx}}^{\sigma} = 9119,599 \text{ N}}$$

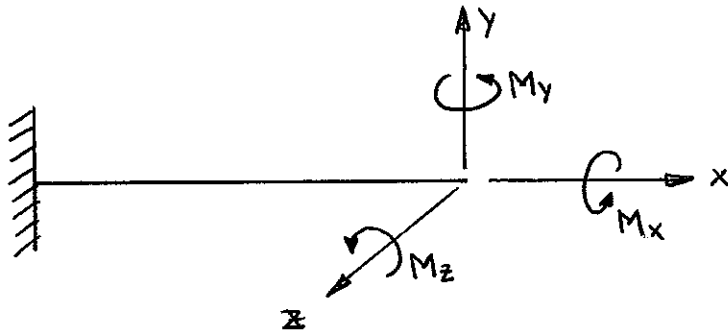
3.5 ● Sección circular: $\phi = 4\text{cm}$

● $M_x = 700\text{ N}\cdot\text{m}$ (TORSIÓN)
 ● $M_y = 300\text{ N}\cdot\text{m}$, $M_z = 400\text{ N}\cdot\text{m}$ (FLEXIÓN)

● $\sigma_f = 240\text{ MPa}$

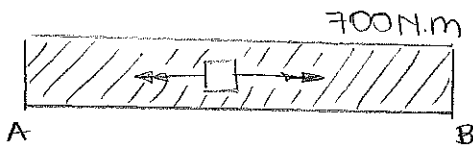
$$I_y = I_z = \frac{\pi R^4}{4} = 1'2566 \cdot 10^{-7} \text{ m}^4$$

$$I_p = 2I = 2'51327 \cdot 10^{-7} \text{ m}^4$$

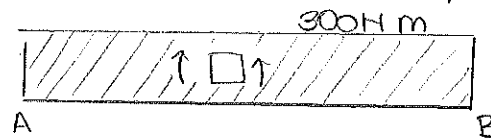


1) DIAGRAMAS DE ESFUERZOS.

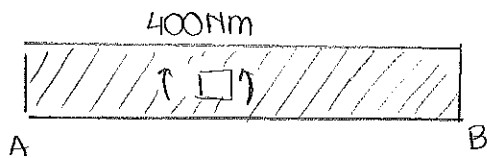
Esfuerzos torsores



Esfuerzos flectores (M_y)

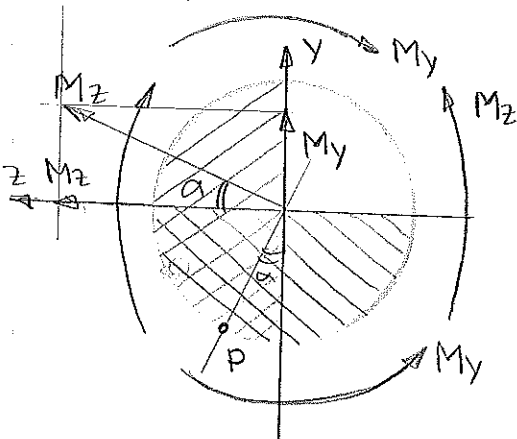


Esfuerzos flectores (M_z)



● Todos las secciones soportan los mismos esfuerzos luego analizamos una al azar.

2) TENSIONES EN LA SECCIÓN



● Por la torsión sabemos que el punto más desfavorable está en la periferia.

● Calculamos la dirección del momento flector resultante y así encontramos el punto más desfavorable

$$\text{tg } \alpha = \frac{M_y}{M_z} = 0.75 \Rightarrow \alpha = 36'8699$$

El punto P será el más desfavorable:

$$\begin{cases} y_p = -R \cdot \cos \alpha \\ z_p = R \cdot \sin \alpha \end{cases}$$

$$\sigma_{xx}^{(P)} = \sigma_{xx}^{(M_1)} + \sigma_{xx}^{(M_2)}$$

$$\sigma_{xx}^{(M_1)} = \frac{M_1 \cdot z}{I_y} = \frac{300 \text{ N} \cdot \text{m} \cdot 2 \text{ cm} \cdot \sin \alpha}{1'2566 \cdot 10^{-7} \text{ m}^4} = 28,649 \text{ MPa}$$

$$\sigma_{xx}^{(M_2)} = \frac{M_2 \cdot y}{I_z} = \frac{400 \text{ N} \cdot \text{m} \cdot 2 \text{ cm} \cdot \cos \alpha}{1'2566 \cdot 10^{-7} \text{ m}^4} = 50,931 \text{ MPa}$$

$$\sigma_{xx}^{(P)} = 79,580 \text{ MPa}$$

$$\tau_{xy}^{(P)} = \frac{M_x \cdot R}{I_p} = \frac{700 \text{ N} \cdot \text{m} \cdot 2 \text{ cm}}{2'51327 \cdot 10^{-7} \text{ m}^4} = 55'704 \text{ MPa}$$

$$\text{TRESCA} \Rightarrow \sigma_{eq} = \sqrt{\sigma_{xx}^2 + 4\tau_{xy}^2} = 136'911 \text{ MPa}$$

$$n = \frac{\sigma_f}{\sigma_{eq}} = \frac{240 \text{ MPa}}{136'911 \text{ MPa}} = 1'753$$

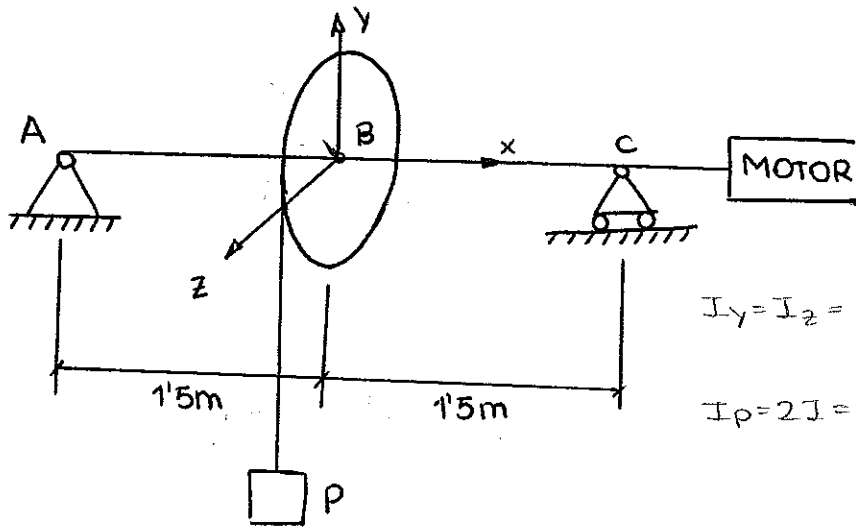
coeficiente de seguridad: $n = 1'753$

3.6 Sección circular: $\phi = 15\text{cm}$

$\phi_p = 1\text{m}$

$Q = 100\text{cv}$; $n = 100\text{rpm}$

(Se desprecia el peso de la polea y el efecto del esfuerzo cortante)



$$I_y = I_z = \frac{\pi R^4}{4} = 2'485 \cdot 10^{-5} \text{m}^4$$

$$I_p = 2I = 4,9701 \cdot 10^{-5} \text{m}^4$$

con la potencia del motor calculamos el valor de la carga P igualando el par motor y el resistente

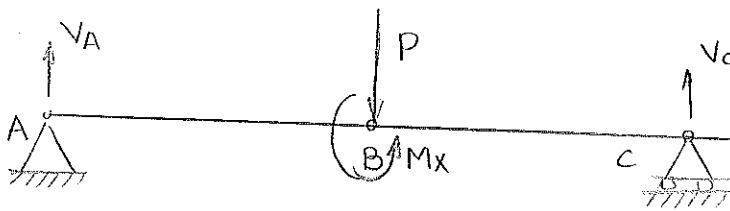
1) PAR MOTOR = PAR RESISTENTE

$$P_R = P \cdot 0'5\text{m}$$

$$P_M = \frac{Q}{\omega} = \frac{736 \frac{\text{Nm}}{\text{s}\cdot\text{cv}} \cdot 100\text{cv}}{100\text{rpm} \cdot \frac{2\pi \cdot \text{rad}}{1\text{rev}} \cdot \frac{1\text{min}}{60\text{s}}} = 7028,282 \text{ N}\cdot\text{m} = M_x$$

$$0'5P\text{m} = 7028,282 \text{ N}\cdot\text{m} \Rightarrow P = 14,056 \text{ kN}$$

sustituimos la polea por un momento torzor y una carga.



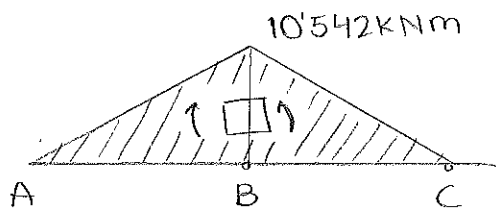
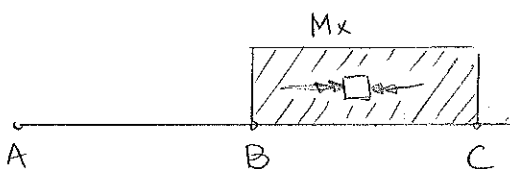
$$V_A + V_C = P \Rightarrow V_A = 7'028 \text{ kN}$$

$$P \cdot 1'5 - V_C \cdot 3 = 0 \Rightarrow V_C = 7'028 \text{ kN}$$

2) DIAGRAMAS DE ESFUERZOS

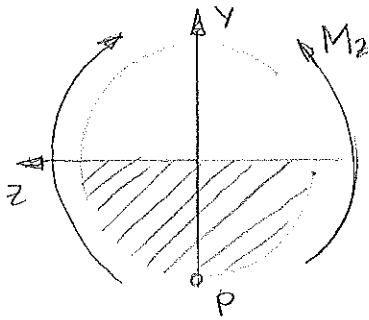
Esfuerzos torsores

Esfuerzos flectores



⊙ Veremos que la sección más desfavorecida del eje es la central (B).

3) TENSIONES EN LA SECCIÓN B.



⊙ El punto más desfavorecido de la sección es el A

$$\sigma_{xx}^{(P)} = \sigma_{xx}^{(M2)}$$

$$\sigma_{xx}^{(P)} = 31'817 \text{ MPa}$$

$$\sigma_{xx}^{(M2)} = \frac{M_z \cdot y}{I_z} = \frac{10'542 \text{ kNm} \cdot 7'5 \text{ cm}}{2'485 \cdot 10^{-5} \text{ m}^4} = 31'817 \text{ MPa}$$

$$\tau_{xz}^{(P)} = \tau_{xz}^{(Mx)}$$

$$\tau_{xz}^{(P)} = 10'606 \text{ MPa}$$

$$\tau_{xz}^{(Mx)} = \frac{M_x \cdot R}{I_p} = \frac{7028'282 \text{ N.m} \cdot 7'5 \text{ cm}}{4'9701 \cdot 10^{-5} \text{ m}^4}$$

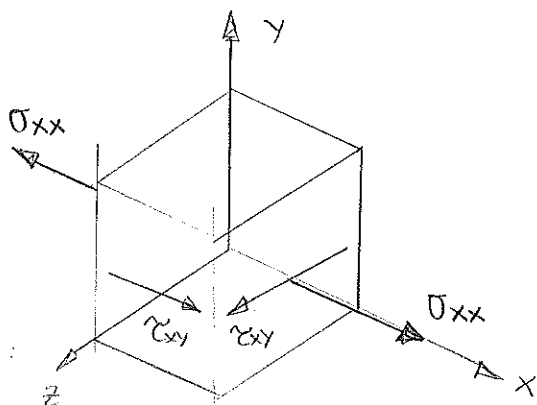
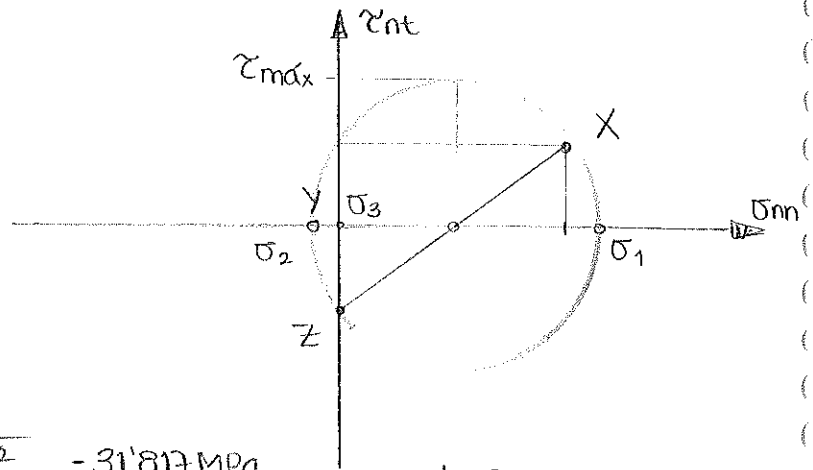


DIAGRAMA DE LOS CÍRCULOS DE MOHR



$$\sigma_1 = 31'817 \text{ MPa} + \frac{\sqrt{(31'817 \text{ MPa})^2 + 4(10'606 \text{ MPa})^2}}{2} - 31'817 \text{ MPa} = 35'028 \text{ MPa}$$

$$\sigma_2 = -\left(\sqrt{(31'817 \text{ MPa})^2 + 4(10'606 \text{ MPa})^2} - 35'028 \text{ MPa}\right) = -3'212 \text{ MPa}$$

$$\sigma_3 = 0$$

$$\tau_{\text{máx}} = \frac{\sqrt{(31'817 \text{ MPa})^2 + 4(10'606 \text{ MPa})^2}}{2} = 19'120 \text{ MPa}$$

$$\sigma_1 = 35'028 \text{ MPa} \quad \sigma_2 = -3'212 \text{ MPa} \quad \sigma_3 = 0 \quad ; \quad \tau_{\text{máx}} = 19'120 \text{ MPa}$$

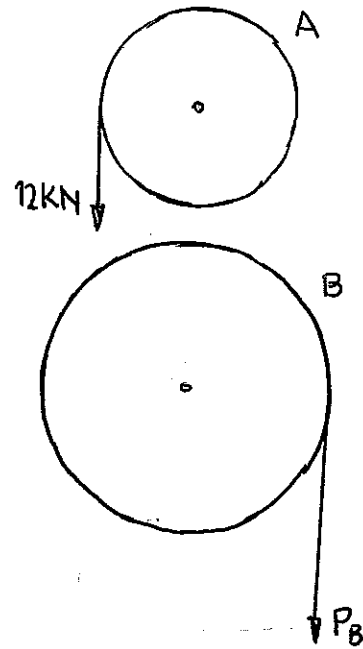
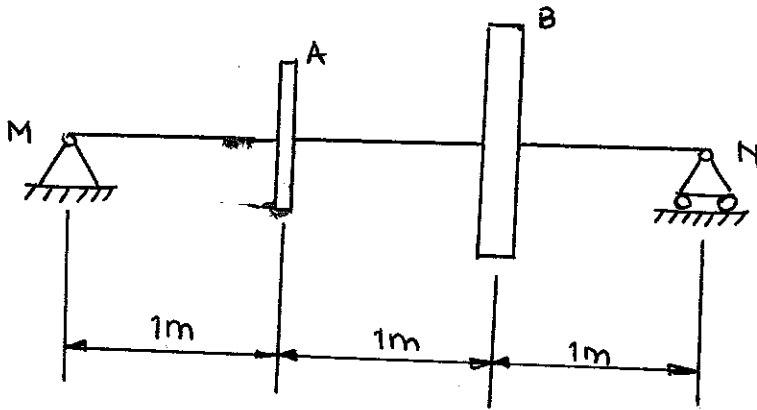
3.7 Sección circular

$R_A = 40\text{cm}; R_B = 60\text{cm}$

$P_A = 12\text{KN}$

$\sigma_{adm} = 160\text{MPa}$

(Se desprecian las cargas tangenciales debidas al esfuerzo cortante)



1) EQUILIBRIO DE LOS PARES.

$P_A \cdot R_A = P_B \cdot R_B$

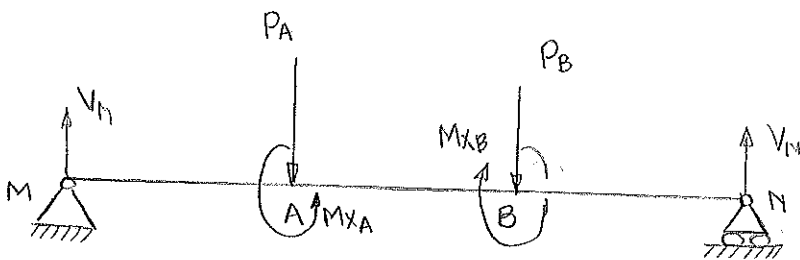
$12\text{KN} \cdot 40\text{cm} = P_B \cdot 60\text{cm}$

$P_B = 8\text{KN}$

$M_{xA} = 12\text{KN} \cdot 40\text{cm} = 4800\text{N}\cdot\text{m}$

$M_{xB} = 8\text{KN} \cdot 60\text{cm} = 4800\text{N}\cdot\text{m}$

Las poleas se pueden sustituir por una carga y un momento forsor



$P_A + P_B = V_M + V_N$

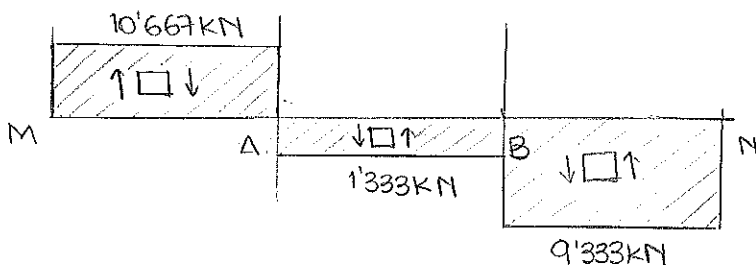
$V_M \cdot 3\text{m} - P_A \cdot 2\text{m} - P_B \cdot 1\text{m} = 0$

$V_M = 10,667\text{KN}$

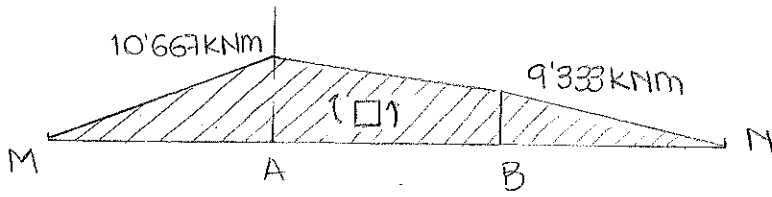
$V_N = 9,333\text{KN}$

2) DIAGRAMAS DE ESFUERZOS

Esfuerzos cortantes

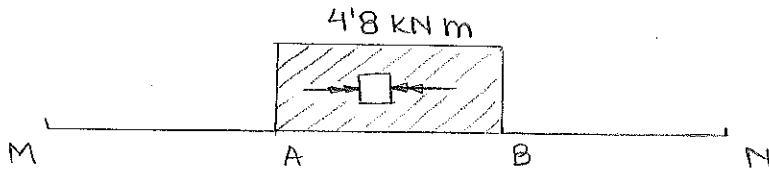


Esfuerzos flectores

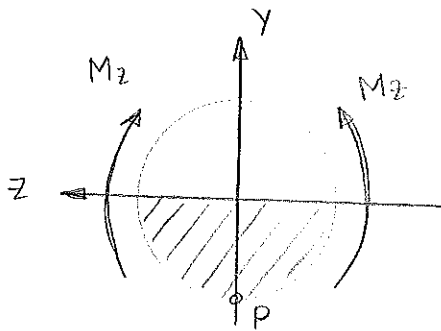


⊙ La sección más desfavorable del eje es la A

Esfuerzos torsores



3) TENSIONES EN LA SECCIÓN



⊙ Punto más desfavorecido: P

$$I_y = I_z = \frac{\pi R^4}{4}$$

$$I_p = 2I = \frac{\pi R^4}{2}$$

$$\sigma_{xx}^{(P)} = \sigma_{xx}^{(M_z)}$$

$$\sigma_{xx}^{(M_z)} = \frac{M_z \cdot y}{I_z} = \frac{10'667 \text{ kNm} \cdot R}{\frac{\pi R^4}{4}} = 13581'222 \frac{1}{R^3} \text{ N/m}$$

$$\sigma_{xx}^{(P)} = \frac{13581'222}{R^3} \text{ N/m}$$

$$\tau_{xy}^{(P)} = \tau_{xy}^{(M_x)}$$

$$\tau_{xy}^{(M_x)} = \frac{M_x \cdot R}{I_p} = \frac{4'8 \text{ kNm} \cdot R}{\frac{\pi R^4}{2}} = 3055'775 \frac{1}{R^3} \text{ N/m}$$

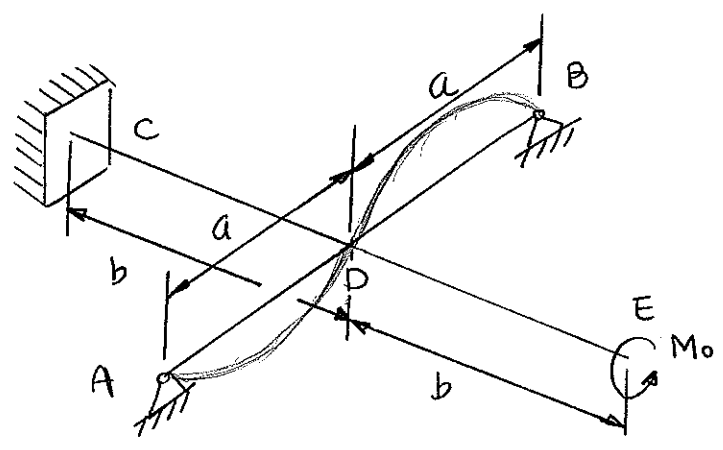
$$\tau_{xy}^{(P)} = \frac{3055'775}{R^3} \text{ N/m}$$

TRESCA: $\sigma_{eq} = \sqrt{\sigma_{xx}^2 + 4\tau_{xy}^2}$

$$\sigma_{eq}^{(P)} = \frac{1}{R^3} \sqrt{(13581'222 \text{ N/m})^2 + 4(3055'775 \text{ N/m})^2} = \frac{14892,973}{R^3} \text{ N/m} = 160 \text{ MPa}$$

$$R = 0'0453 \text{ m} = 4'532 \text{ cm}$$

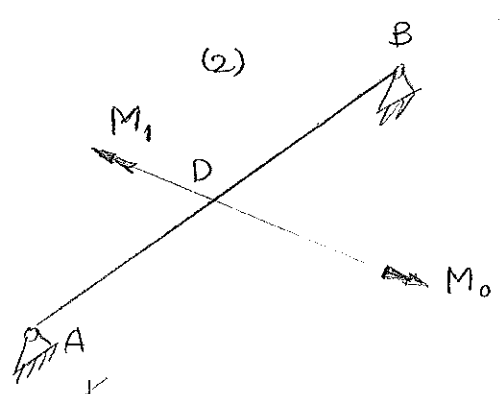
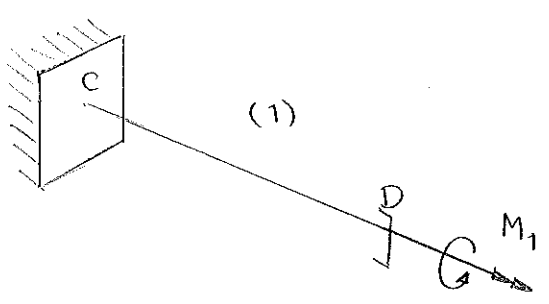
$$\phi = 9'064 \text{ cm}$$



● características del material
 $E, G, \nu = \frac{1}{3}$

ESTRUCTURA HIPERESTÁTICA.

$$\left. \begin{aligned} E &= 2G(1+\nu) = 2G \frac{4}{3} = \frac{8}{3}G \\ I &= \frac{I_p}{2} \\ EI &= \frac{4}{3}GI_p \end{aligned} \right\}$$



● EC compatibilidad

$$\varphi_D^{CD} = \theta_D^{ADB}$$

$$(1) \varphi_D^{CD} = \frac{M_1 \cdot b}{G \cdot I_p}$$

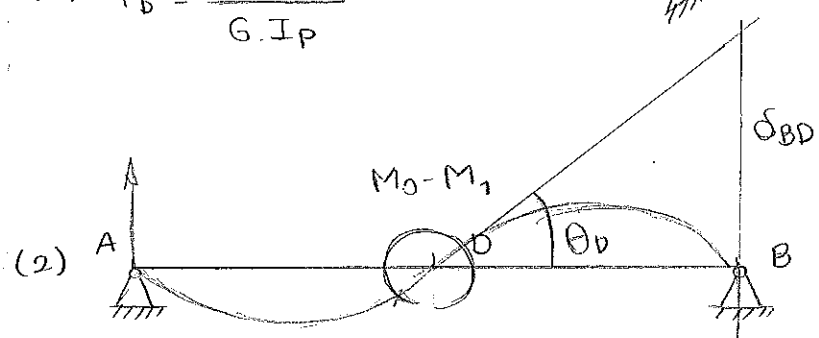
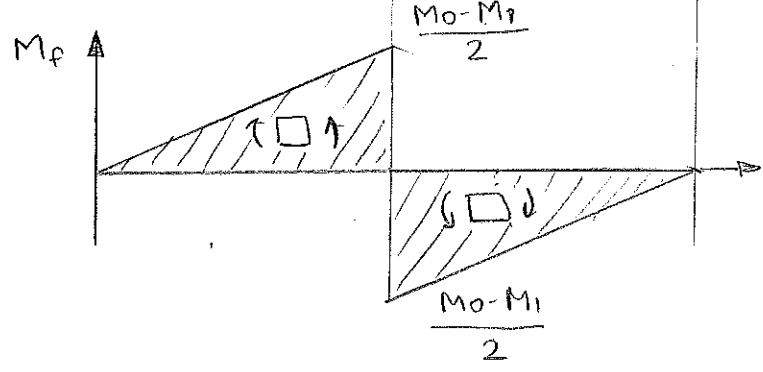


Diagrama de esfuerzos flectores.



$$\theta_D^{ADB} = \theta_D = \frac{\delta_{BD}}{a}$$

$$\delta_{BD} = \frac{1}{EI} \left[\frac{1}{2} \frac{M_0 - M_1}{2} \cdot a \cdot \frac{2}{3} a \right] = \frac{M_0 - M_1}{6EI} a^2$$

$$\theta_D^{ADB} = \frac{M_0 - M_1}{6EI} a = \frac{M_0 - M_1}{6 \cdot \frac{4}{3} GI_p} a =$$

$$= \frac{(M_0 - M_1)a}{8 GI_p}$$

⊗ EC compatibilidad

$$\frac{M_1 \cdot b}{G I_p} = \frac{(M_0 - M_1) a}{8 G I_p}$$

$$M_1 (8b + a) = M_0 a$$

$$\boxed{M_1 = \frac{M_0 a}{8b + a}}$$

$$\varphi_E = \varphi_{BC} + \varphi_{ED} = \frac{M_1 b}{G I_p} + \frac{M_0 \cdot b}{G I_p} = \left(\frac{M_0 b}{a + 8b} + M_0 b \right) \cdot \frac{1}{G I_p} = \frac{M_0 b + M_0 a b + 8 M_0 b^2}{G I_p (a + 8b)}$$

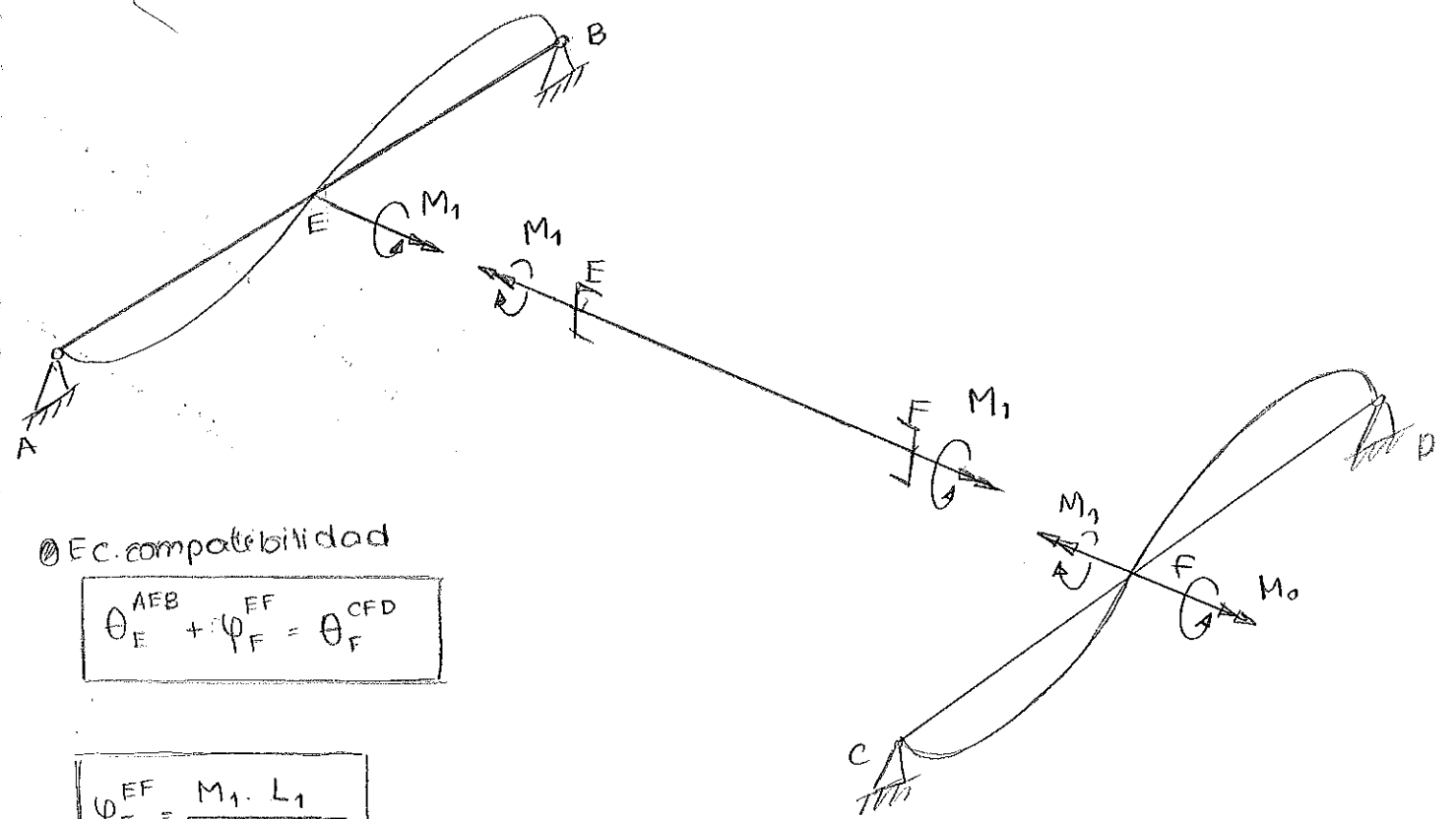
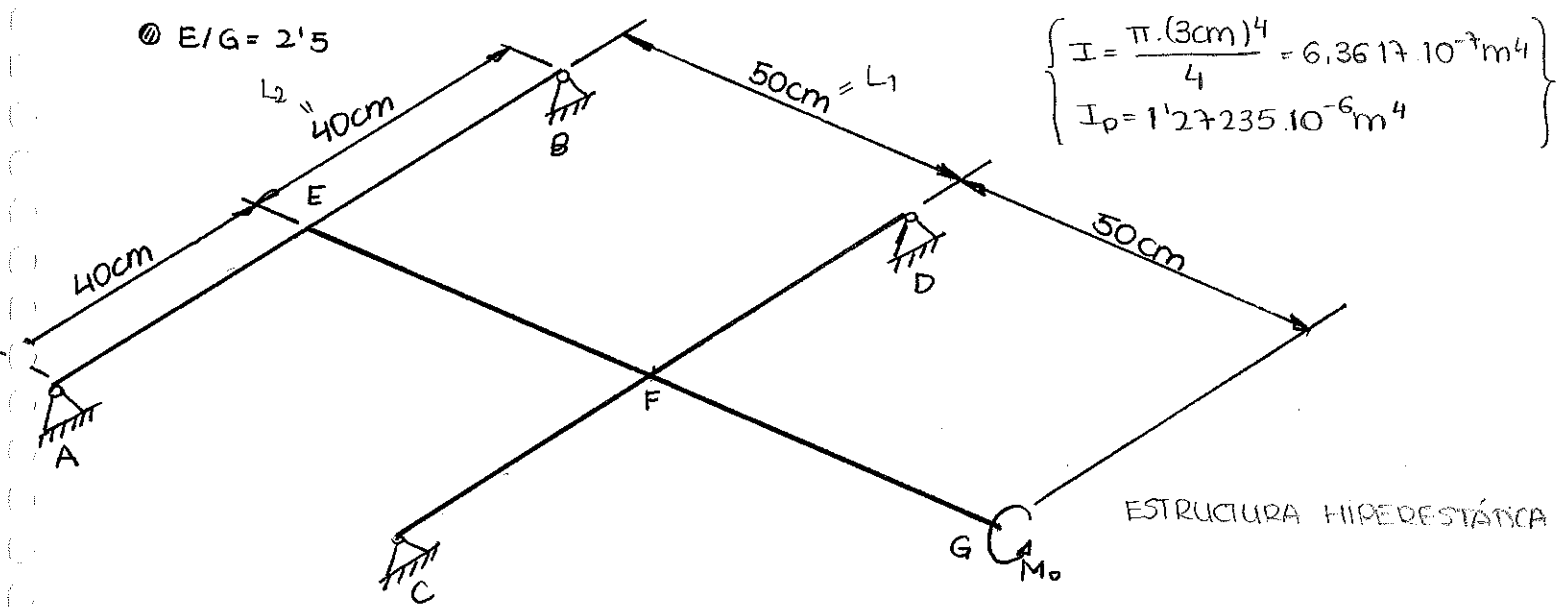
$$= \frac{2 M_0 b (a + 4b)}{G I_p (a + 8b)}$$

$$\boxed{\varphi_E = \frac{2 M_0 \cdot b (a + 4b)}{G I_p (a + 8b)}}$$

3.9

- Secciones circulares de $\phi = 6\text{cm}$ (Se desprecia el efecto del esfuerzo cortante)
- Tresca: $\sigma_F = 150\text{MPa}$
- $E/G = 2.5$

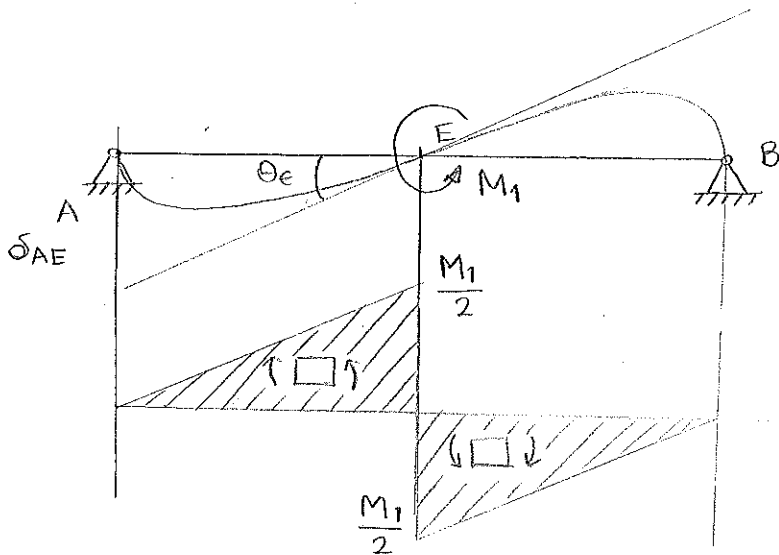
$$\left\{ \begin{aligned} I &= \frac{\pi \cdot (3\text{cm})^4}{4} = 6.3617 \cdot 10^{-7} \text{m}^4 \\ I_p &= 1.27235 \cdot 10^{-6} \text{m}^4 \end{aligned} \right\}$$



● Ec. compatibilidad

$$\theta_E^{AEB} + \psi_F^{EF} = \theta_F^{CFD}$$

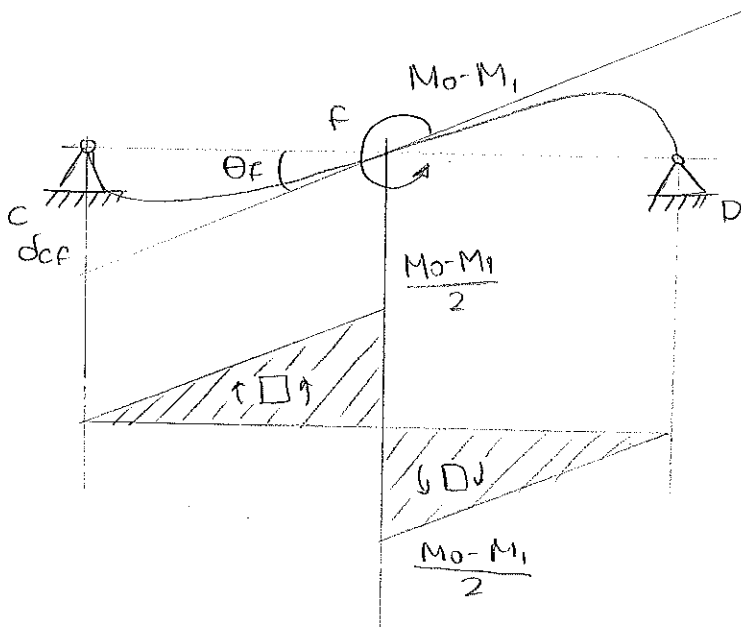
$$\psi_F^{EF} = \frac{M_1 \cdot L_1}{GI_p}$$



$$\theta_E^{AEB} = \frac{\delta_{AE}}{L_2}$$

$$\delta_{AE} = \frac{1}{EI} \left(\frac{1}{2} \cdot \frac{M_1}{2} \cdot L_2 \cdot \frac{2}{3} L_2 \right) = \frac{M_1 L_2^2}{6EI}$$

$$\theta_E^{AEB} = \frac{M_1 L_2}{6EI}$$



$$\theta_F^{CFD} = \frac{\delta_{CF}}{L_2}$$

$$\delta_{CF} = \frac{1}{EI} \left(\frac{1}{2} \cdot \frac{M_0 - M_1}{2} \cdot L_2 \cdot \frac{2}{3} L_2 \right) = \frac{(M_0 - M_1) L_2^2}{6EI}$$

$$\theta_F^{CFD} = \frac{(M_0 - M_1) L_2}{6EI}$$

⊗ Ec compatibilidad

$$\frac{M_1 L_2}{6EI} + \frac{M_1 L_1}{6I_p} = \frac{(M_0 - M_1) L_2}{6EI}$$

$$\frac{M_1 L_2}{6EI} + \frac{M_1 L_1}{\frac{E}{2.5} \cdot 2I} = \frac{(M_0 - M_1) L_2}{6EI}$$

$$M_1 \left(\frac{L_2}{6} + \frac{L_1 \cdot 5}{4} + \frac{L_2}{6} \right) = \frac{M_0 \cdot L_2}{6}$$

$$M_1 \left(\frac{4L_2 + 15L_1}{12} \right) = \frac{M_0 L_2}{6}$$

$$M_1 = \frac{2M_0 L_2}{4L_2 + 15L_1}$$

CÁLCULO DE LAS REACCIONES EN LOS APOYOS.

$$V_A = V_B = \frac{M_1}{2L_2} = \frac{M_0}{4L_2 + 15L_1} = \frac{10M_0}{91} \text{ (m}^{-1}\text{)}$$

$$V_D = V_D = \frac{M_0 - M_1}{2 \cdot L_2} = \frac{M_0}{2L_2} - \frac{M_0}{4L_2 + 15L_1} = \frac{415}{364} M_0 \text{ (m}^{-1}\text{)}$$

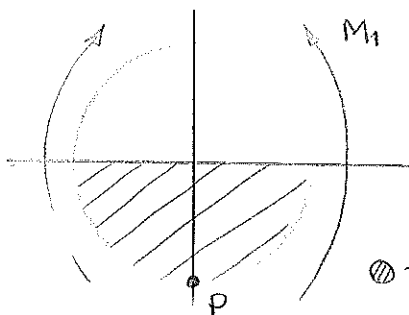
$$V_A = V_B = \frac{10 M_0}{91} \text{ (m}^{-1}\text{)}$$

$$V_C = V_D = \frac{415 M_0}{364} \text{ (m}^{-1}\text{)}$$

⊗ La sección más desfavorecida para la flexión es la E

⊗ La sección más desfavorecida para la torsión es cualquiera de la barra FG.

ANÁLISIS TENSIONAL DE LA SECCIÓN E



$$\sigma_{nn}^{(P)} = \sigma_{nn}^{(M_1)}$$

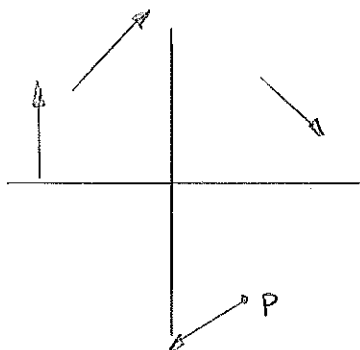
$$\sigma_{nn}^{(M_1)} = \frac{M_1 \cdot R}{I} = \frac{2M_0 \cdot L_2 \cdot R}{(4L_2 + 15L_1)I} = 4145,688 M_0 \left(\frac{1}{\text{m}^3} \right)$$

$$\otimes \text{ Tresca: } \sigma_{eq} = \sqrt{\sigma_{xx}^2 + 4\tau_{xy}^2} = \sigma_f$$

$$4145,688 M_0 \frac{1}{\text{m}^3} = 150 \text{ MPa}$$

$$M_0^{(f)} = 36182,173 \text{ Nm}$$

ANÁLISIS TENSIONAL DE LA SECCIÓN G



$$\tau_{nt}^{(P)} = \tau_{nt}^{(M_0)}$$

$$\tau_{nt}^{(M_0)} = \frac{M_0 \cdot R}{I_p} = 23578,418 M_0 \left(\frac{1}{\text{m}^3} \right)$$

$$\otimes \text{ Tresca: } \sigma_{eq} = \sqrt{\sigma_{xx}^2 + 4\tau_{xy}^2} = \sigma_f$$

$$47156,836 M_0 \frac{1}{\text{m}^3} = 150 \text{ MPa}$$

$$M_0^{(f)} = 3180,875 \text{ Nm}$$

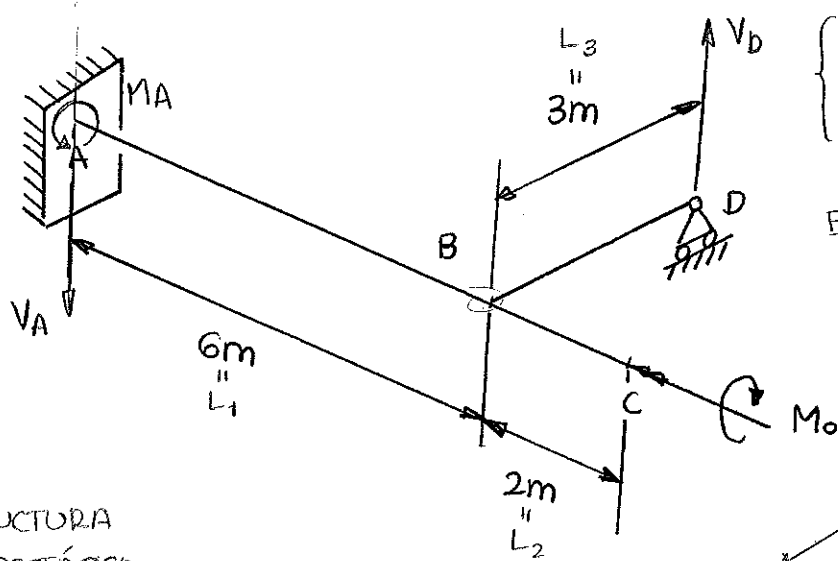
$$M_o^{(z)} < M_o^{(b)}$$

$$M_{o, \text{máx}} = M_o^{(z)} = 3180,875 \text{ N.m}$$

3.10

- Sección circular: $\phi = 20\text{cm}$
- $\sigma_f = 240\text{MPa}$, $n = 1'6$ (tresca)
- $E/G = 3$

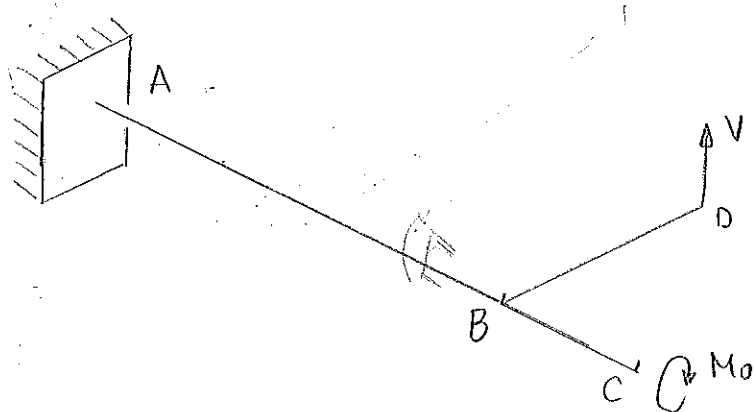
(se desprecia el efecto del esfuerzo cortante)



$$\left\{ \begin{aligned} I &= \frac{\pi R^4}{4} = 7'85398 \cdot 10^{-5} \text{m}^4 \\ I_p &= 2I = 1'570796 \cdot 10^{-4} \text{m}^4 \end{aligned} \right\}$$

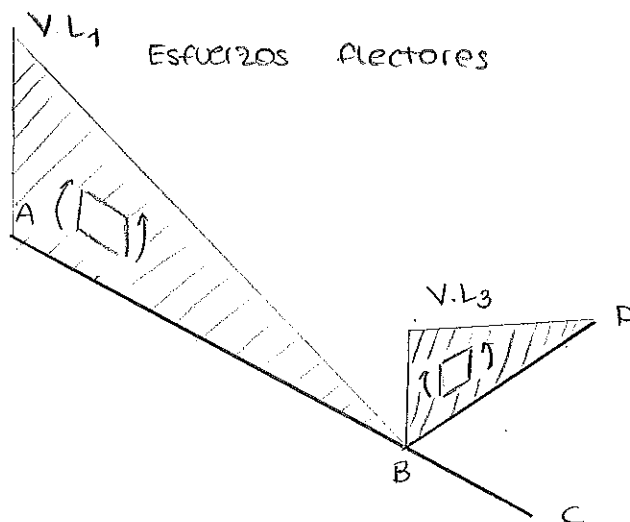
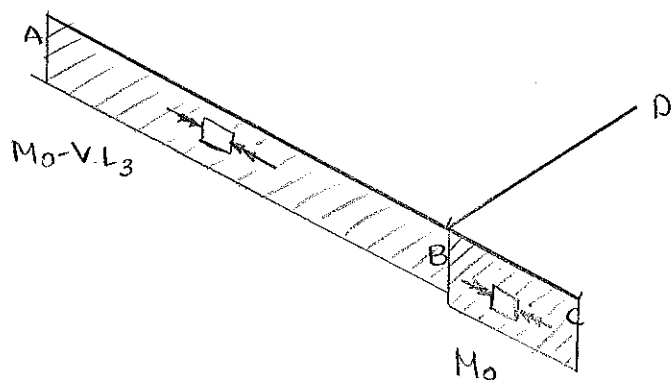
$$E \cdot I = \frac{3}{2} G I_p$$

ESTRUCTURA HIPERESTÁTICA



- Ec. compatibilidad: $Y_D = 0$

1) DIAGRAMAS DE ESFUERZOS.
Esfuerzos torsores



$$\left. \begin{aligned} \downarrow \\ Y_{D1} &= \varphi_{BA} \cdot L_3 \\ \varphi_{BA} &= \frac{(M_0 - V L_3) L_1}{G I_P} \end{aligned} \right\} \downarrow Y_{D1} = \frac{(M_0 - V L_3) L_1}{G I_P} \cdot L_3$$

$$\left. \begin{aligned} \downarrow \\ Y_{D2} &= -\delta_{BA} \\ \delta_{BA} &= \frac{1}{EI} \left[\frac{1}{2} V L_1^2 \cdot \frac{2}{3} L_1 \right] = \frac{V L_1^3}{3EI} \end{aligned} \right\} \downarrow Y_{D2} = -\frac{V L_1^3}{3EI}$$

$$\left. \begin{aligned} \downarrow \\ Y_{D3} &= -\delta_{DB} \\ \delta_{DB} &= \frac{1}{EI} \left[\frac{1}{2} V L_3 \cdot L_3 \cdot \frac{2}{3} L_3 \right] = \frac{V L_3^3}{3EI} \end{aligned} \right\} \downarrow Y_{D3} = -\frac{V L_3^3}{3EI}$$

⊗ E.c. compatibilidad: $\Sigma y_i = 0$

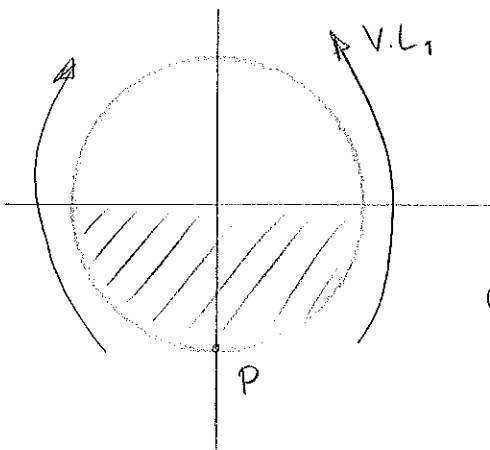
$$\frac{(M_0 - V L_3) L_1}{G I_P} \cdot L_3 - \frac{2 V L_1^3}{9 G I_P} - \frac{2 V L_3^3}{9 G I_P} = 0$$

$$V \cdot \left(L_1 L_3^2 + \frac{2}{9} L_1^3 + \frac{2}{9} L_3^3 \right) = L_1 \cdot M_0 \cdot L_3$$

$$\boxed{V = \frac{L_1 \cdot L_3}{L_1 L_3^2 + \frac{2}{9} L_1^3 + \frac{2}{9} L_3^3} \cdot M_0 = \frac{M_0}{16} \cdot (m^{-1})}$$

⊗ Las secciones más desfavorecidas son la A y cualquiera del tramo BC.

ANÁLISIS TENSIONAL DE LA SECCIÓN A



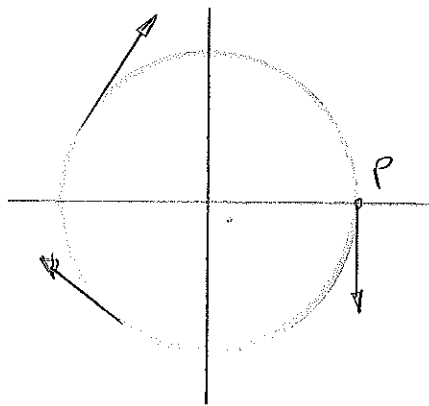
$$\sigma_{xx}^{(P)} = \frac{V L_1 \cdot R}{I} = 1273,2410 M_0 \left(\frac{1}{m^3} \right)$$

$$\tau_{xy}^{(P)} = \frac{(M_0 - V L_3) R}{I_P} = 318,3410 M_0 \left(\frac{1}{m^3} \right)$$

$$\sigma_{eq} = \sqrt{\sigma_{xx}^2 + 4\tau_{xy}^2} = 1423,526 M_0 \cdot \frac{1}{m^3} = \frac{240 \text{MPa}}{1'6} = \frac{\sigma_f}{n}$$

$$M_{0, \text{máx}}^{(A)} = 105372,183 \text{ Nm}$$

ANÁLISIS TENSIONAL DE LA SECCIÓN C



$$\chi^{(P)} = \frac{M_o \cdot R}{I_p} = 636,620 M_o \left(\frac{1}{m^3} \right)$$

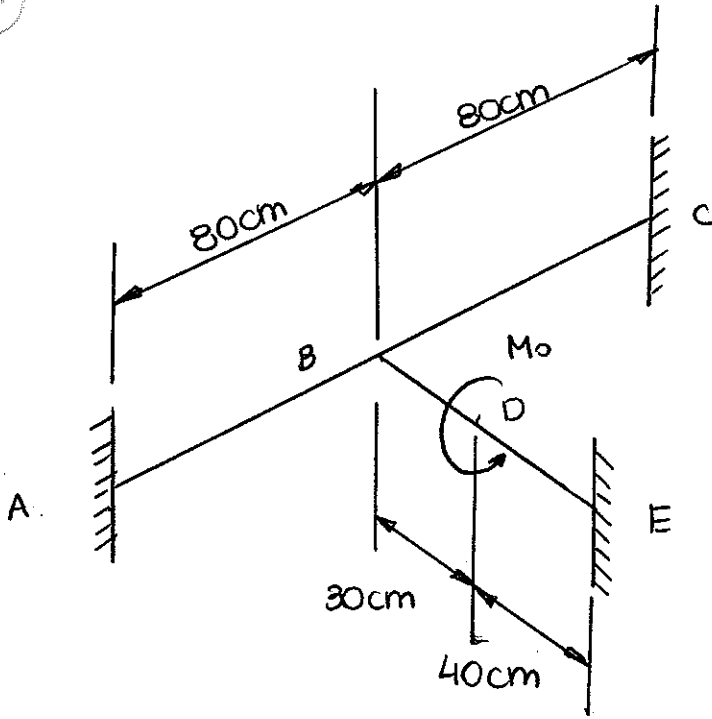
$$\textcircled{1} \text{ Tresca: } \sigma_{eq} = 1273,240 M_o \frac{1}{m^3} = \frac{240 \text{MPa}}{1,6}$$

$$M_o, \text{máx}^{(c)} = 117809,7 \text{Nm}$$

$$M_o, \text{máx}^{(c)} > M_o, \text{máx}^{(A)}$$

$M_o, \text{máx}^{(A)} = M_o, \text{máx}^{(A)} = 105372,183 \text{Nm}$
--





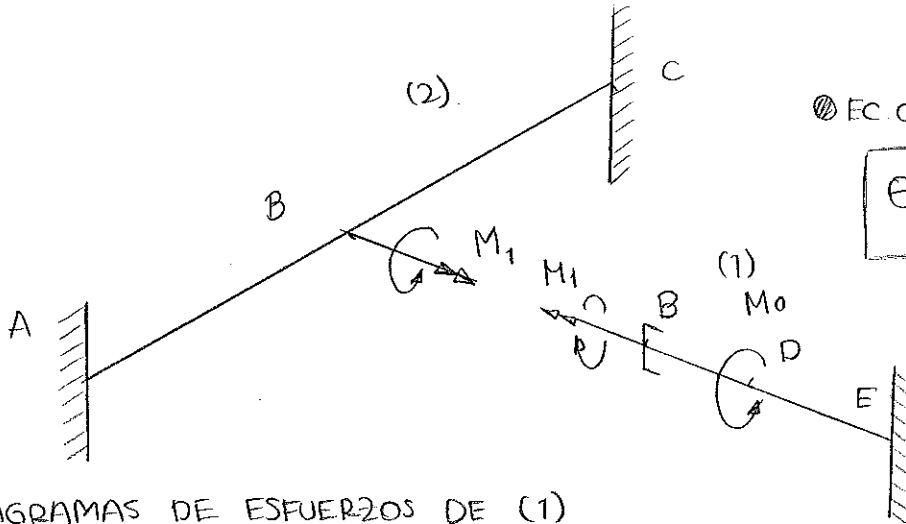
- Sección circular: $\phi = 8\text{cm}$
- $\sigma_f = 150\text{ MPa}$; $n = 1,5$ (Tresca)
- $E/G = 2,5$

ESTRUCTURA HIPERESTÁTICA

$$I = \frac{\pi R^4}{4} = 2,01062 \cdot 10^{-6} \text{m}^4$$

$$I_p = 2I = 4,02124 \cdot 10^{-6} \text{m}^4$$

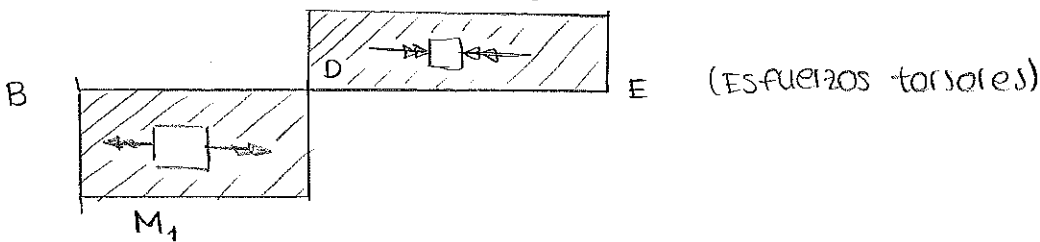
$$EI = 2,5G \cdot \frac{I_p}{2} = \frac{5}{4} GJ_p$$



● Ec. compatibilidad

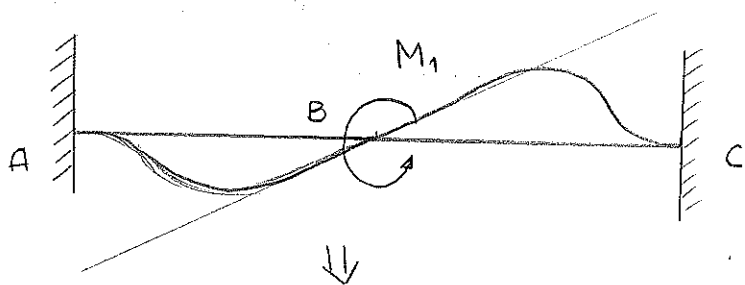
$$\theta_B^{ABC} = \varphi_B^{BE}$$

1) DIAGRAMAS DE ESFUERZOS DE (1)

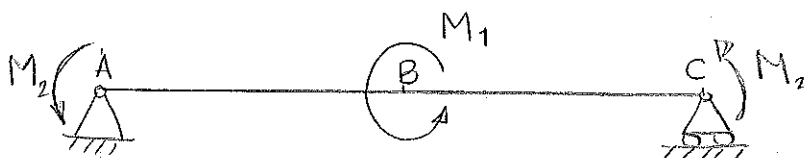


$$\varphi_B^{BE} = -\varphi_{BD} + \varphi_{DE} = -\frac{M_1 \cdot 30\text{cm}}{GJ_p} + \frac{(M_0 - M_1) \cdot 40\text{cm}}{GJ_p} = \frac{40\text{cm} \cdot M_0 - 70\text{cm} \cdot M_1}{GJ_p}$$

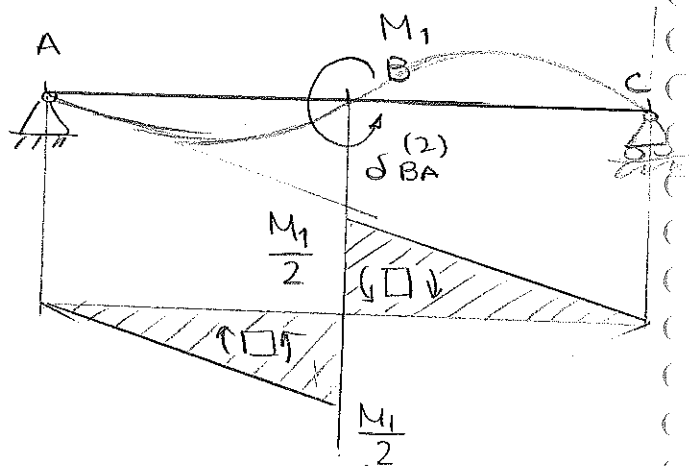
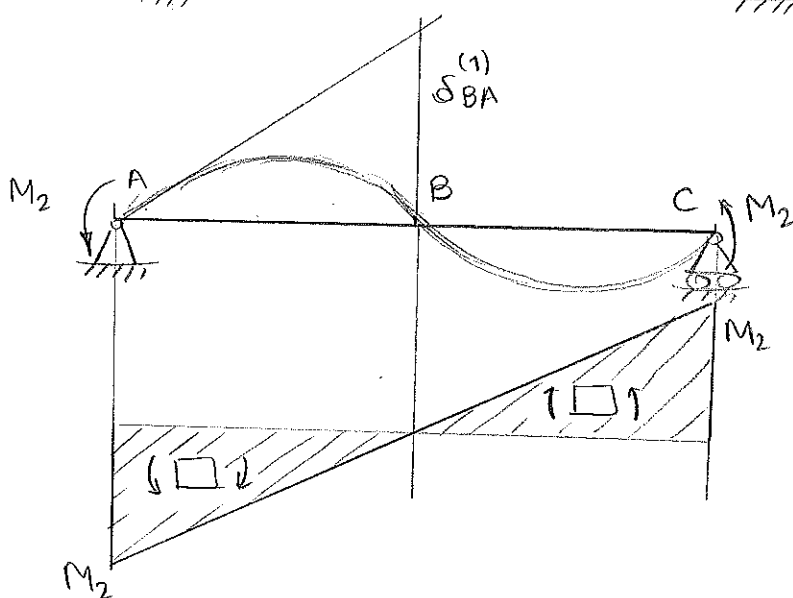
2) DIAGRAMAS DE ESFUERZOS DE (2)



ESTRUCTURA HIPERESTÁTICA



⊙ Ec compatibilidad
 $\delta_{BA} = 0$



$$\delta_{BA} = \delta_{BA}^{(1)} - \delta_{BA}^{(2)} = 0$$

$$\delta_{BA}^{(1)} = \frac{1}{EI} \left[\frac{1}{2} M_2 \cdot 80\text{cm} \cdot \frac{2}{3} \cdot 80\text{cm} \right] = \frac{1}{EI} \cdot \frac{(80\text{cm})^2 M_2}{3}$$

$$\delta_{BA}^{(2)} = \frac{1}{EI} \left[\frac{1}{2} \frac{M_1}{2} \cdot 80\text{cm} \cdot \frac{1}{3} \cdot 80\text{cm} \right] = \frac{1}{EI} \cdot \frac{(80\text{cm})^2 M_1}{12}$$

$$\frac{(80\text{cm})^2 M_2}{3EI} = \frac{(80\text{cm})^2 M_1}{12EI}$$

$$M_2 = \frac{1}{4} M_1$$

$$\overset{\curvearrowleft}{\Theta}_B^{ABC} = -\Theta_B^{(1)} + \Theta_B^{(2)}$$

$$\Theta_B^{(1)} = \frac{\delta_{AB}^{(1)}}{80\text{cm}}$$

$$\delta_{AB}^{(1)} = \frac{1}{EI} \left[\frac{1}{2} \cdot \frac{M_1}{4} \cdot 80\text{cm} \cdot \frac{1}{3} \cdot 80\text{cm} \right] = \frac{(80\text{cm})^2 M_1}{24 EI} = \frac{(80\text{cm})^2 M_1}{30 G I_p}$$

$$\Theta_B^{(1)} = \frac{80\text{cm} M_1}{30 G I_p}$$

$$\Theta_B^{(2)} = \frac{\delta_{AB}^{(2)}}{80\text{cm}}$$

$$\delta_{AB}^{(2)} = \frac{1}{EI} \left[\frac{1}{2} \cdot \frac{M_1}{2} \cdot 80\text{cm} \cdot \frac{2}{3} \cdot 80\text{cm} \right] = \frac{(80\text{cm})^2 M_1}{6 EI} = \frac{4 \cdot (80\text{cm})^2 M_1}{30 G I_p}$$

$$\Theta_B^{(2)} = \frac{4 \cdot 80\text{cm} M_1}{30 G I_p}$$

$$\overset{\curvearrowleft}{\Theta}_B^{ABC} = \frac{80\text{cm} \cdot M_1}{10 G I_p}$$

● Ec compatibilidad:

$$\frac{40\text{cm} M_0 - 70\text{cm} M_1}{G I_p} = \frac{80\text{cm} M_1}{10 G I_p}$$

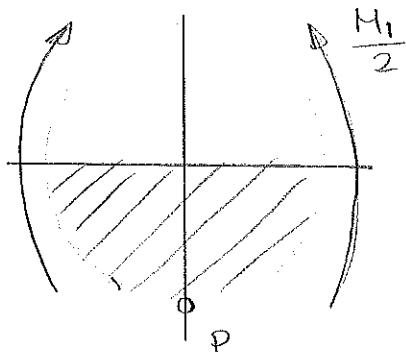
$$40 \cdot M_0 - 70 M_1 = 8 M_1$$

$$M_1 = \frac{40}{78} M_0 = \frac{20}{39}$$

● La sección que mayor esfuerzo flector soporta es la B

● La sección que mayor esfuerzo torsor es cualquiera del tramo BD.

ANÁLISIS TENSIONAL DE LA SECCIÓN B.



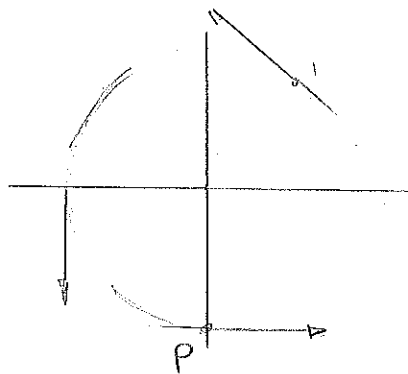
$$\sigma_{xx}^{(P)} = \frac{M_1 \cdot R}{2 \cdot I} = \frac{20 M_0 R}{78 I} = 5101,118 M_0 \left(\frac{1}{m^3} \right)$$

$$\text{Tresca: } \sigma_{eq} = \sqrt{\sigma_{xx}^{(P)} + 4 \tau_{xy}^{(P)}} = \frac{\sigma_f}{n}$$

$$5101,118 M_0 \left(\frac{1}{m^3} \right) = \frac{150 \text{ MPa}}{1,5}$$

$$\boxed{(B) M_0, \text{máx} = 19603,545 \text{ Nm}}$$

ANÁLISIS TENSIONAL DE LA SECCIÓN B



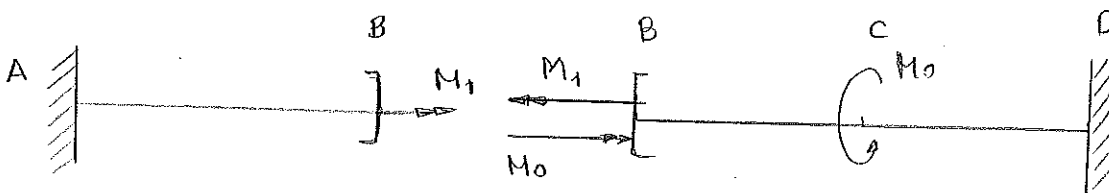
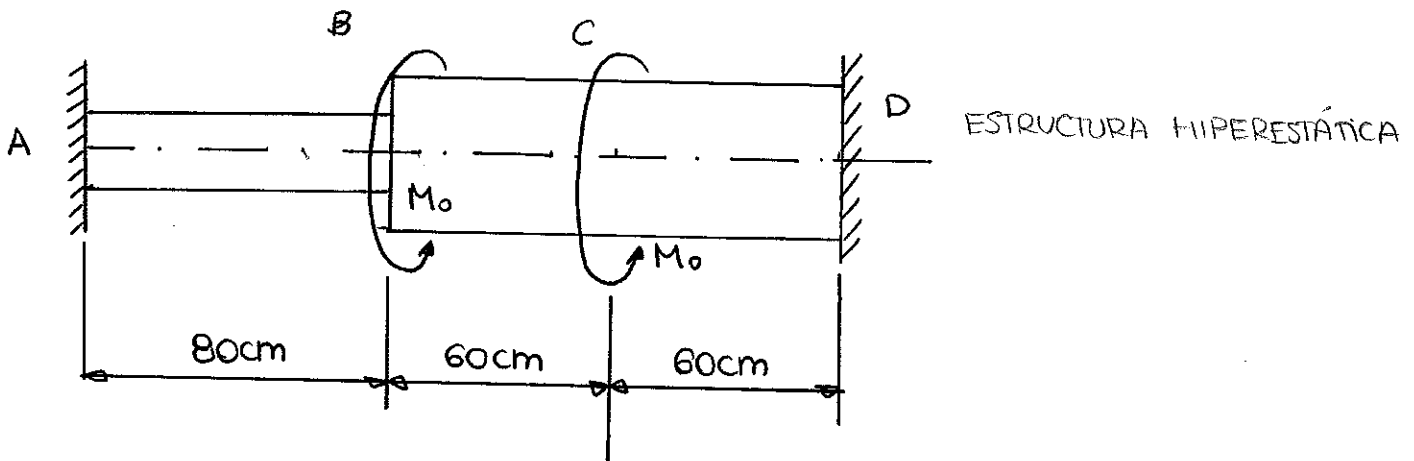
$$\tau_{nt}^{(P)} = \frac{11 M_1 \cdot R}{I_p} = \frac{20 R}{39 I_p} M_0 = 5101,118 M_0 \left(\frac{1}{m^3} \right)$$

$$\text{Tresca: } \sigma_{eq} = 10202,236 M_0 \frac{1}{m^3} = \frac{150 \text{ MPa}}{1,5}$$

$$\boxed{(B) M_0, \text{máx} = 9801,77 \text{ Nm}}$$

3.12

● Sección circular

● $G = 80 \text{ GPa}$ ● $I_p^{AB} = 1000 \text{ cm}^4$; $I_p^{BC} = 2000 \text{ cm}^4$ ● Empotramiento en D: $k = 10^2 \text{ MN.cm/rad}$.● $M_0 = 1 \text{ MN.cm}$ 

● EC. compatibilidad:

$$\varphi_{BA} = \varphi_{BD}$$

$$\varphi_{BA} = \frac{M_1 \cdot 80 \text{ cm}}{G I_p^{AB}}$$

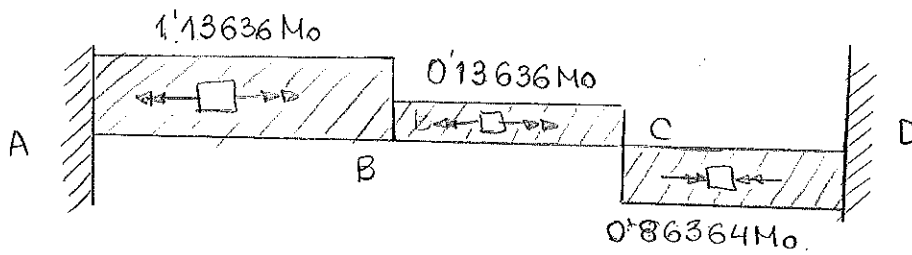
$$\varphi_{BD} = \varphi_{BC} + \varphi_{CD} + \varphi_{emp} = \frac{(M_0 - M_1) \cdot 60 \text{ cm}}{G I_p^{BC}} + \frac{(2M_0 - M_1) \cdot 60 \text{ cm}}{G I_p^{(ec)}} + \frac{12M_0 - M_1}{10^2 \frac{\text{MN.cm}}{\text{rad}}}$$

$$\frac{M_1 \cdot 80 \text{ cm}}{G I_p^{AB}} = \frac{(3M_0 - 2M_1) \cdot 60 \text{ cm}}{G I_p^{BC}} + \frac{2M_0 - M_1}{10^2 \text{ MN.cm}}$$

$$M_1 \left(\frac{80 \text{ cm}}{G I_p^{AB}} + \frac{2 \cdot 60 \text{ cm}}{G I_p^{BC}} + \frac{1}{10^2 \text{ MN.cm}} \right) = \frac{3M_0 \cdot 60 \text{ cm}}{G I_p^{BC}} + \frac{2M_0}{10^2 \text{ MN.cm}}$$

$$M_1 = 1.13636 M_0$$

1) DIAGRAMA DE MOMENTOS TORSORES.



La sección de máximo giro es la C.

2) GIRO DE C

$$\varphi_c = \varphi_{cd} + \varphi_{empotramiento} = \frac{0,86364 M_0 \cdot 60 \text{ cm}}{G J_p} + \frac{0,86364 M_0}{10^2 \text{ MN} \cdot \text{cm/rad}} =$$

$$= 0,011875 \text{ rad}$$

$$\varphi_c = 1,1875 \cdot 10^{-2} \text{ rad}$$

3.13 Redondo macizo : $L_{AC} = 1.4\text{m}$; $\phi_{AC} = 8\text{cm}$

Tubo: $L_{BC} = 0.8\text{m}$; $\phi_e = 12\text{cm}$; $\phi_i = 10\text{cm}$

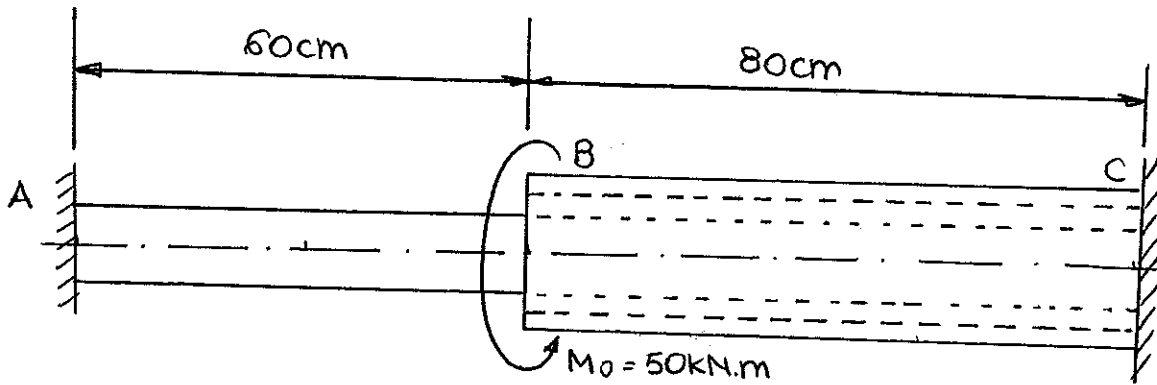
$$I_{PAB} = \frac{\pi R^4}{2} = 4.02124 \cdot 10^{-6} \text{m}^4$$

$$I_{Ptubo} = \frac{\pi (R_e^4 - R_i^4)}{2} = 1.054 \cdot 10^{-5} \text{m}^4$$

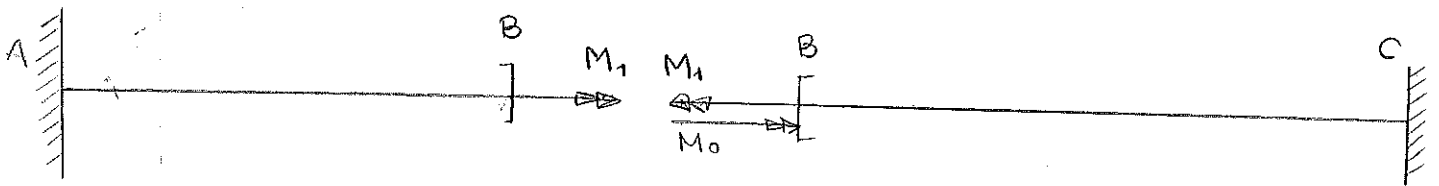
Empotramiento A: $K = 400 \text{KN.m/rad}$

$M_0 = 50 \text{KN.m}$

$G = 80 \text{GPa}$.



ESTRUCTURA HIPERESTÁTICA



Ec. compatibilidad: $\varphi_B^{AB} = \varphi_B^{BC}$

$$\boxed{\varphi_B^{AB} = \varphi_{BA} + \varphi_{\text{empotramiento}} = \frac{M_1 \cdot L_{AB}}{G I_{PAB}} + \frac{M_1}{K}}$$

(BC) $M_{\text{redondo}} + M_{\text{tubo}} = M_0 - M_1$ (1)

Ec. compatibilidad: $\varphi_{\text{redondo}} = \varphi_{\text{tubo}}$

$$\left. \begin{aligned} \varphi_{\text{redondo}} &= \frac{M_{\text{redondo}} \cdot L_{BC}}{G I_{\text{redondo}}} \\ \varphi_{\text{tubo}} &= \frac{M_{\text{tubo}} \cdot L_{BC}}{G I_{\text{tubo}}} \end{aligned} \right\} M_{\text{redondo}} = \frac{M_{\text{tubo}} \cdot I_{\text{redondo}}}{I_{\text{tubo}}}$$

$$(1) \frac{M_{\text{tubo}} I_{\text{redondo}}}{I_{\text{tubo}}} + M_{\text{tubo}} = M_0 - M_1 \Rightarrow M_{\text{tubo}} = \frac{I_{\text{tubo}}}{I_{\text{redondo}} + I_{\text{tubo}}} (M_0 - M_1)$$

$$\varphi_B^{BC} = \frac{I (M_0 - M_1) L_{BC}}{(I_{\text{redondo}} + I_{\text{tubo}}) \cdot G}$$

Ec. compatibilidad

$$\frac{M_1 \cdot L_{AB}}{G I_{PAB}} + \frac{M_1}{K} = \frac{(M_0 - M_1) L_{BC}}{(I_{PAB} + I_{P-tubo}) G}$$

$$M_1 = 6797,06 \text{ Nm}$$

1) ANÁLISIS DE UNA SECCIÓN DEL TUBO

$$M_{tubo} = 31271,992 \text{ Nm}$$

$$\tau_{tubo} = \frac{M_{tubo} R_e}{I_{P-tubo}} = 178,019 \text{ MPa}$$

2) ANÁLISIS DE UNA SECCIÓN DEL TRAMO MÁS DESFAVORECIDO DEL REDONDO.

$$(AB) M_x = M_1 = 6797,06 \text{ Nm}$$

$$(BC) M_x = 11930,948 \text{ Nm}$$

$$\tau_{redondo} = \frac{M_{redondo}^{BC} R}{I_{P-redondo}} = 118,679 \text{ MPa}$$

$$\tau_{\text{máx, tubo}} = 178,019 \text{ MPa}$$

$$\tau_{\text{máx, redondo}} = 118,679 \text{ MPa}$$

3.14

● Sección circular $\begin{cases} \phi_{BE} = d\sqrt{2} \\ \phi = d \end{cases} \quad (d=4\text{cm})$

● $E = 200\text{GPa}; G = 80\text{GPa}$

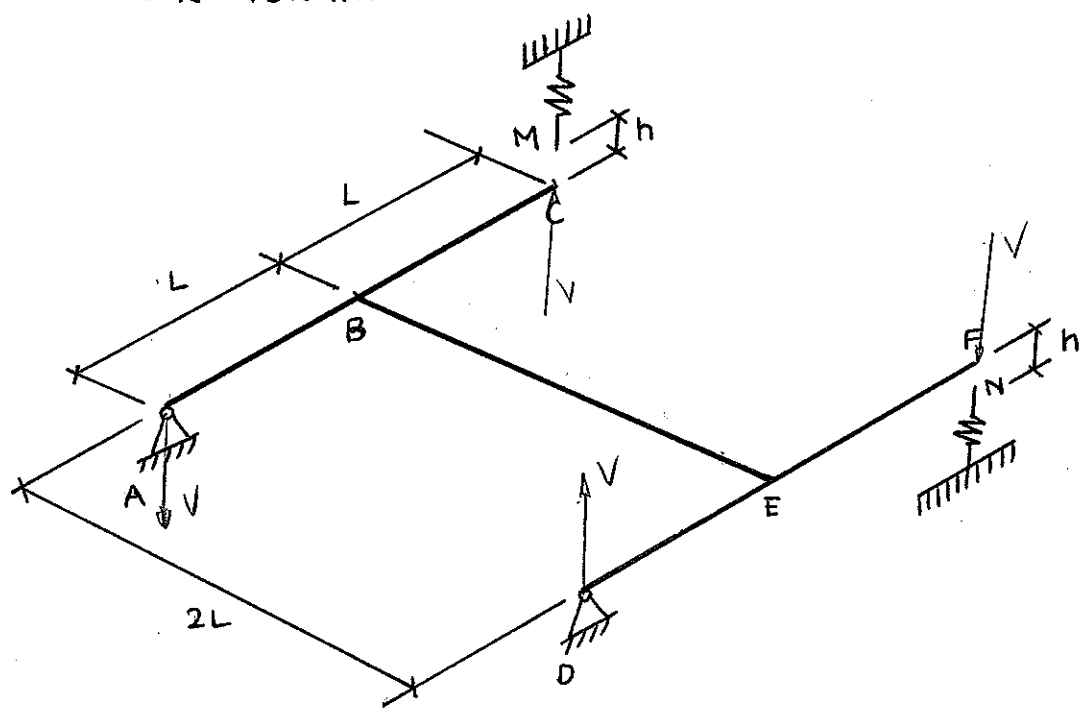
$$I_{BE} = \frac{\pi R_{BE}^4}{4} = 5,02655 \cdot 10^{-7} \text{ m}^4; I_{PBE} = 1,00531 \cdot 10^{-6} \text{ m}^4$$

● $L = 1\text{m}$

$$I = 1'25664 \cdot 10^{-7} \text{ m}^4; I_p = 2'51327 \cdot 10^{-7} \text{ m}^4$$

● $h = 2\text{cm}$

● $K = 40\text{kN/m}$

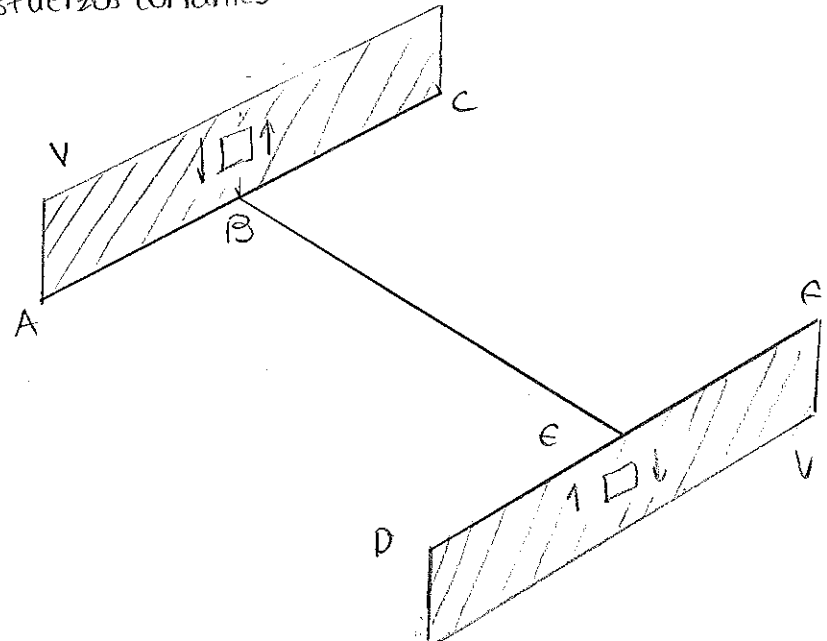


● Las cargas que provocan los muelles son: $V_C = K(h - \uparrow y_C)$ y $V_F = K(h - \downarrow y_F)$
 Por simetría $V_C = V_F = V$

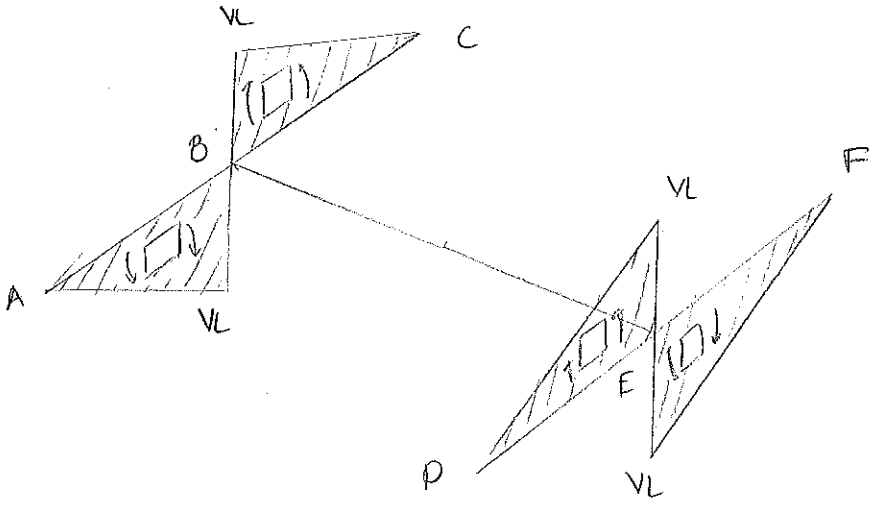
● Ec. compatibilidad: $y_C = y_D = h - \frac{V}{K}$

1) DIAGRAMA DE ESFUERZOS:

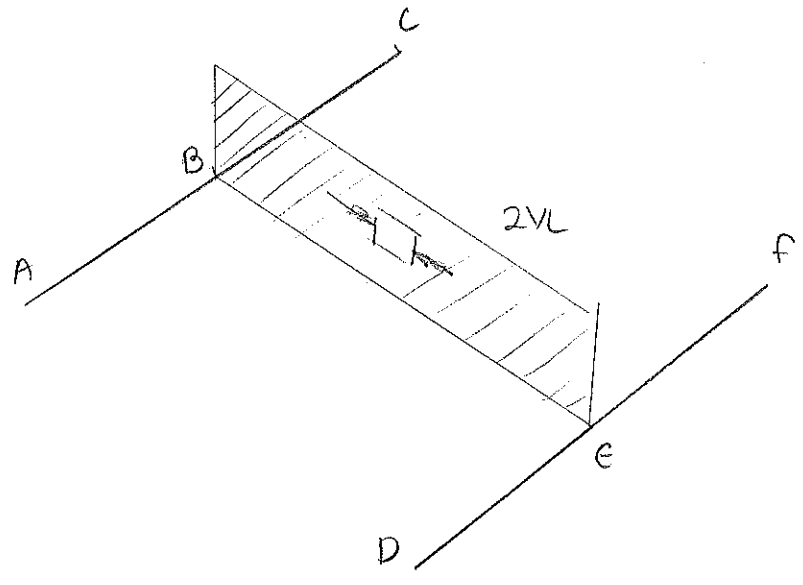
Esfuerzos cortantes



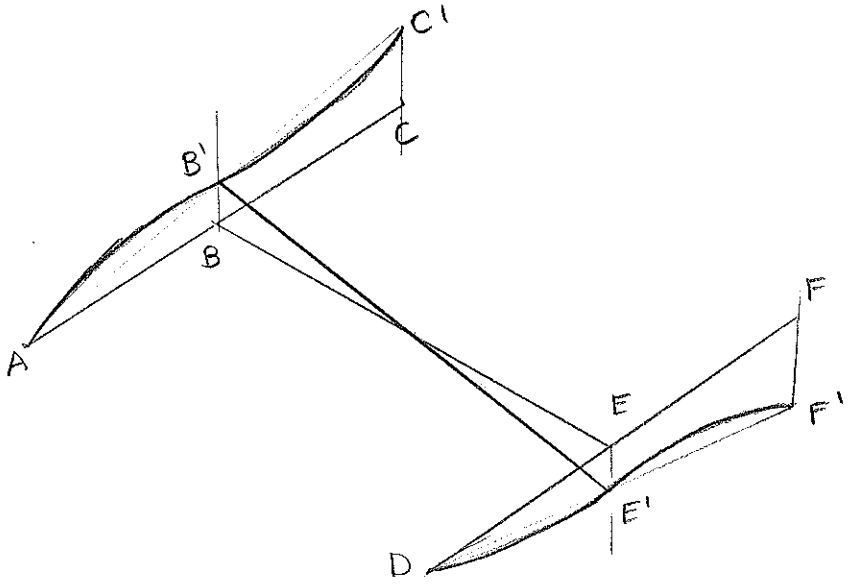
Esfuerzos flectores

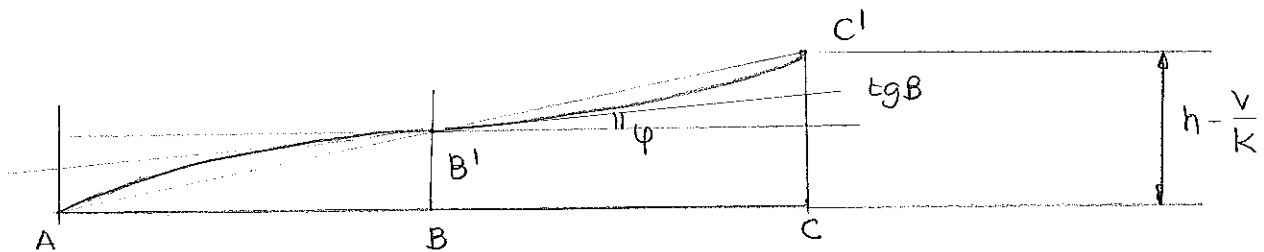


Esfuerzos torsionales



2) DIBUJO APROXIMADO DE LA DEFORMADA





$$CC' = \delta_{CB} + \varphi \cdot L + (\varphi L + \delta_{AB}) = 2\delta + 2\varphi L$$

⊙ Ec. compatibilidad

$$h - \frac{V}{K} = 2\delta + 2\varphi L$$

$$\begin{cases} \delta = \delta_{CB} = \frac{1}{EI} \left(\frac{1}{2} VL \cdot L \cdot \frac{2}{3} L \right) = \frac{VL^3}{3EI} \\ \varphi = \frac{2VL \cdot L}{G \cdot I_p} \end{cases}$$

$$h - \frac{V}{K} = \frac{2VL^3}{3EI} + 2 \frac{2VL^3}{GI_p} \Rightarrow \boxed{V = 197,508 \text{ N}}$$

$$\delta_M = \delta_N = \frac{V}{K} = 4,9377 \cdot 10^{-3} \text{ m} = 0,494 \text{ cm}$$

$$\boxed{\delta_M = \delta_N = 0,494 \text{ cm}}$$

⊙ Las secciones más desfavorables de la estructura son:

* Flexión: B y E

* Torsión: Tramo BE

$$\sigma_{\text{máx}}^{\text{pl}} = \frac{VL \cdot R}{I} = 31'434 \text{ MPa}$$

$$\tau_{\text{máx}}^{\text{t}} = \frac{2VL \cdot R_{BE}}{I_{PBE}} = 11'114 \text{ MPa}$$

$$\tau_{\text{máx, torsión}} = 11'114 \text{ MPa (tramo BE)}$$

$$\sigma_{\text{máx, flexión}} = 31'434 \text{ MPa (secciones B y E)}$$



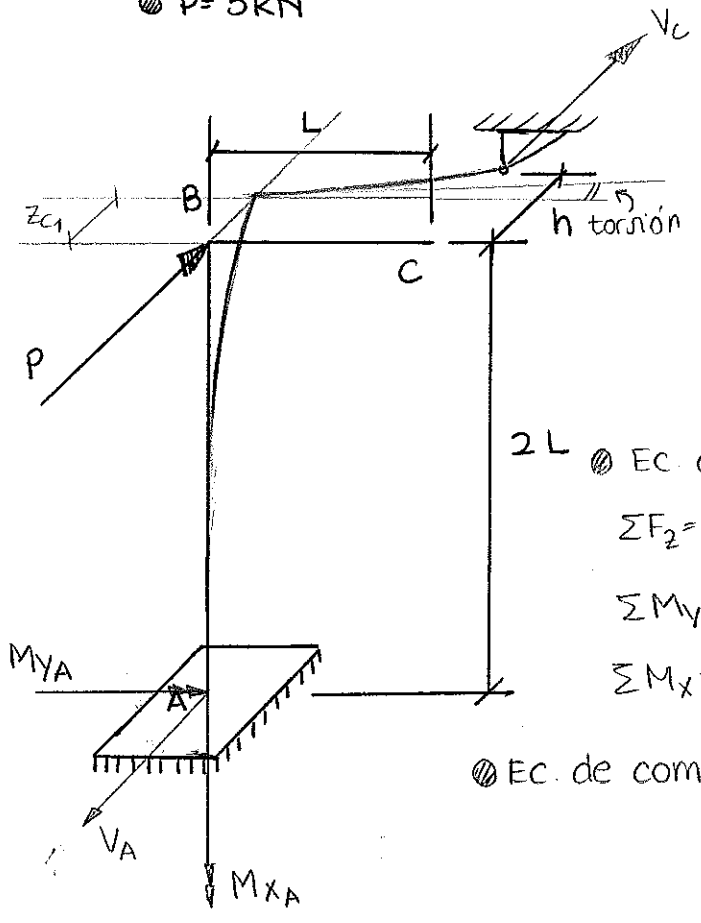
3.15

- Sección circular de $\phi = d = 8\text{cm}$
- $h = 2\text{cm}$, $L = 1\text{m}$
- $E = 200\text{GPa}$, $\nu = 0.25$
- $P = 5\text{KN}$

$$G = \frac{E}{2(1+\nu)} = 80\text{GPa}$$

$$I = \frac{\pi R^4}{4} = 2,010619 \cdot 10^{-6}\text{m}^4$$

$$J_p = 2 \cdot I = 4,0212386 \cdot 10^{-6}\text{m}^4$$



2L ● Ec. de equilibrio

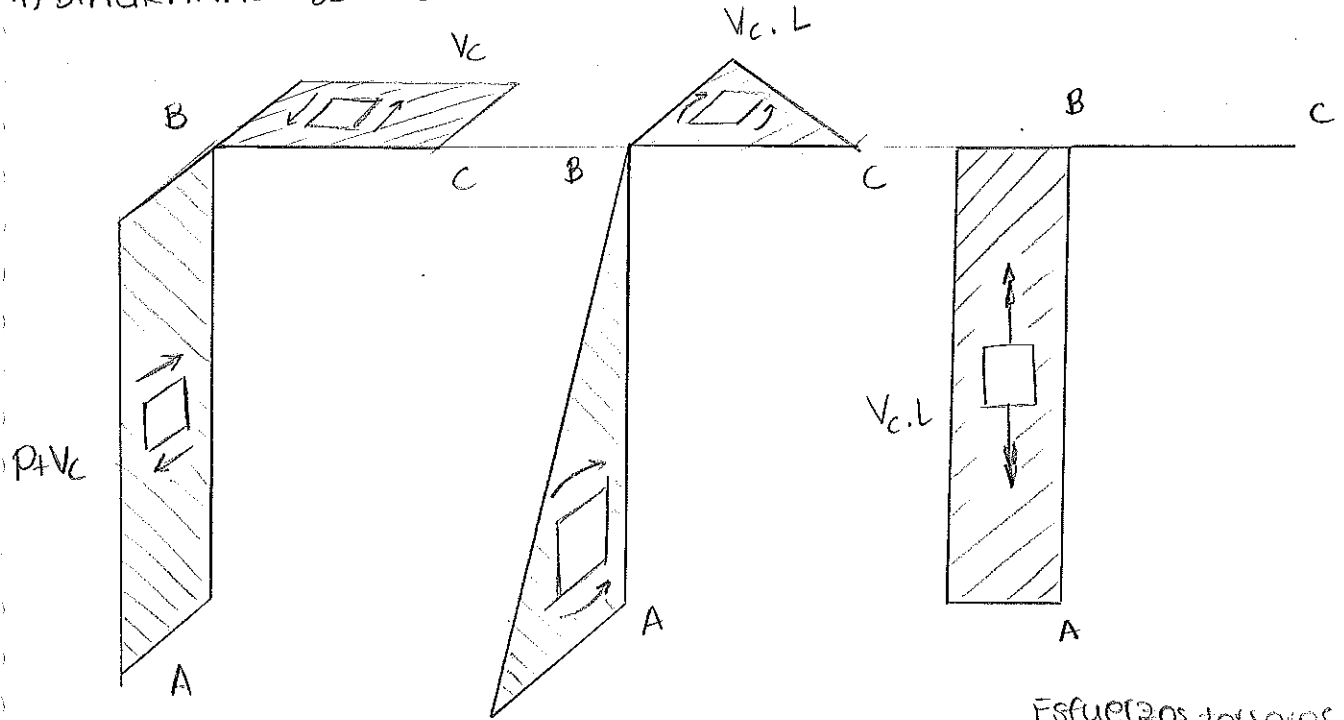
$$\sum F_z = 0: P + V_c = V_A$$

$$\sum M_y = 0: M_{yA} - 2L \cdot P - 2L \cdot V_c = 0 \Rightarrow M_{yA} = 2L(P + V_c)$$

$$\sum M_x = 0: M_{xA} - V_c \cdot L = 0 \Rightarrow M_{xA} = V_c \cdot L$$

● Ec. de compatibilidad: $z_c = h$

1) DIAGRAMAS DE ESFUERZOS.



Esfuerzos cortantes

$2L(P + V_c)$ Esfuerzos flectores

Esfuerzos torsores

$$z_{c_1}(\text{flexión AB}) = \delta_{BA} = \frac{1}{EI} \left[\frac{1}{2} 2L(P+V_c) \cdot 2L \cdot \frac{2}{3} 2L \right] = \frac{8(P+V_c)L^3}{3EI}$$

$$z_{c_2}(\text{flexión BC}) = \delta_{CB} = \frac{1}{EI} \left[\frac{1}{2} V_c L \cdot L \cdot \frac{2}{3} L \right] = \frac{V_c L^3}{3EI}$$

$$z_{c_3}(\text{torsión AB}) = \varphi_{BA} \cdot L = \frac{V_c L \cdot 2L}{GI_p} \cdot L = \frac{2V_c L^3}{GI_p}$$

⑦ EC compatibilidad.

$$h = \frac{8(P+V_c)L^3}{3EI} + \frac{V_c L^3}{3EI} + \frac{2V_c \cdot L^3}{GI_p}$$

$$V_c = -961,974 \text{ N}$$

2) ANÁLISIS DEL NUDO B.

$$z_B = \delta_{BA} = \frac{8(P+V_c)L^3}{3EI} = 0,02678 \text{ m} = 2,678 \text{ cm}$$

$$\varphi_B = \varphi_{BA} = \frac{V_c L \cdot 2L}{GI_p} = 5,981 \cdot 10^{-3} \text{ rad}$$

Traslación de B: $z_B = 2,678 \text{ cm}$

Giro de B: $\varphi_B = 5,981 \cdot 10^{-3} \text{ rad}$

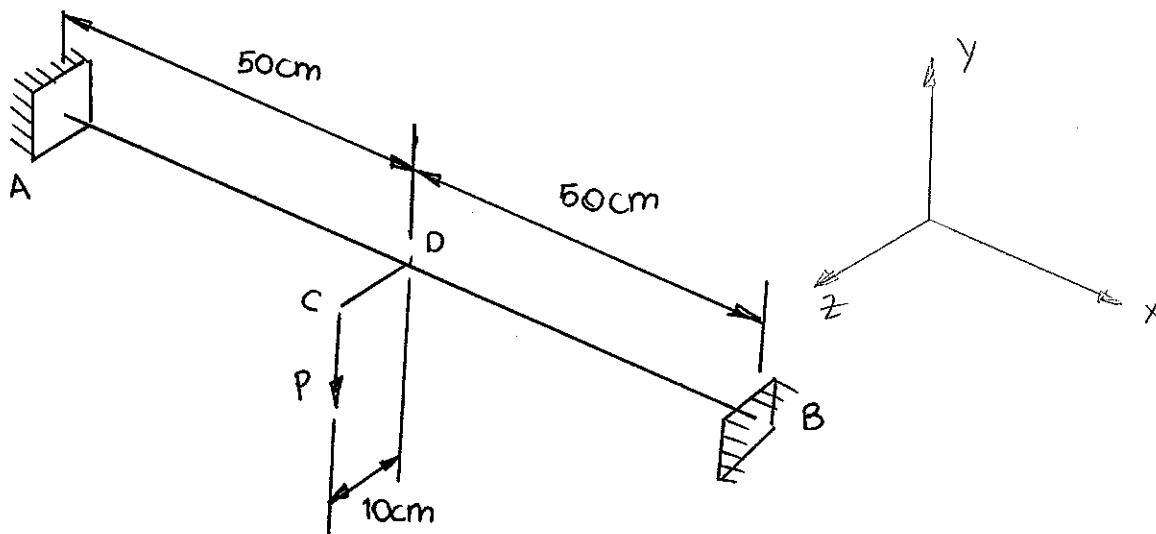
3.16

● $P = 10 \text{ kN}$

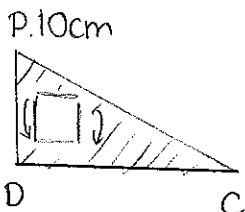
● Tresca: $\sigma_f = 240 \text{ MPa}$, $n = 1.2$

● Sección circular

(se desprecia el efecto del esfuerzo cortante)

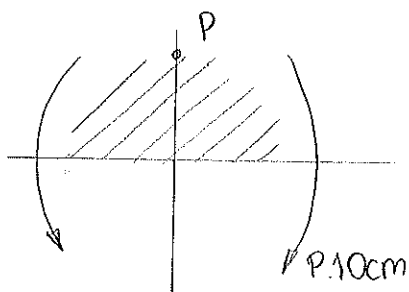


CD



● Sección más comprometida del tramo CD: D

● Análisis de la sección D



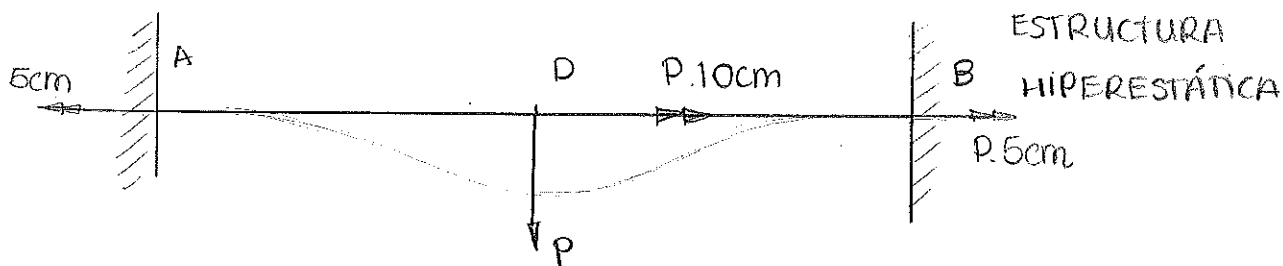
$$\sigma_{zz} = \frac{P \cdot 10 \text{ cm} \cdot R}{I} = \frac{4P \cdot 10 \text{ cm}}{\pi R^3}$$

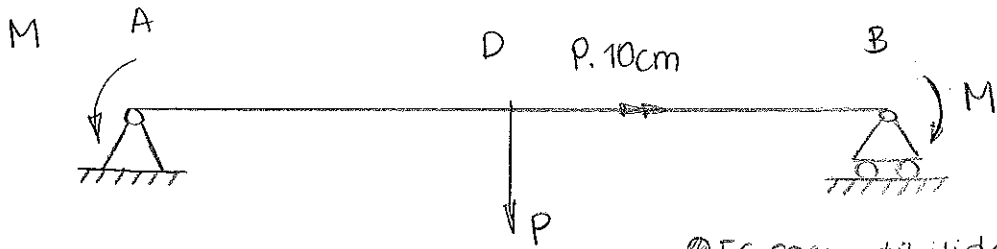
● Tresca: $\sigma_{eq} = \sqrt{\sigma_{zz}^2 + 4\sigma_{zx}^2} = \frac{4P \cdot 10 \text{ cm}}{\pi R^3} = \frac{\sigma_f}{n}$

$$\frac{4 \cdot P \cdot 10 \text{ cm}}{\pi R^3} = \frac{240 \text{ MPa}}{1.2}$$

$$R = 0.0185 \text{ m}$$

ADB

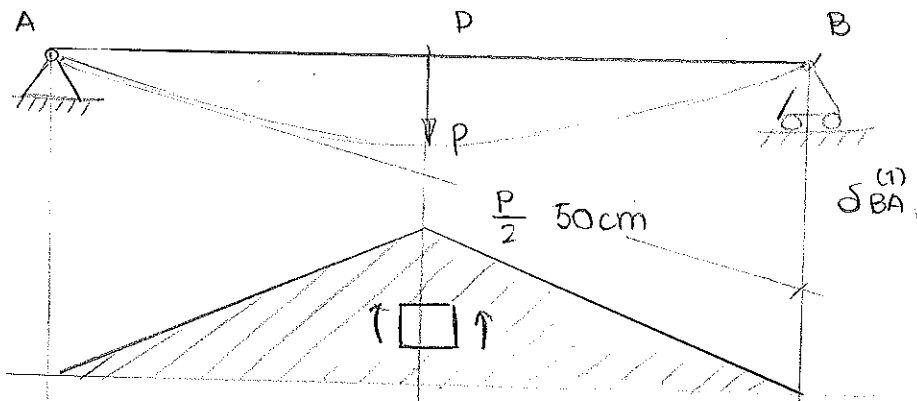




⊙ Ec compatibilidad: $\delta_{BA} = 0$

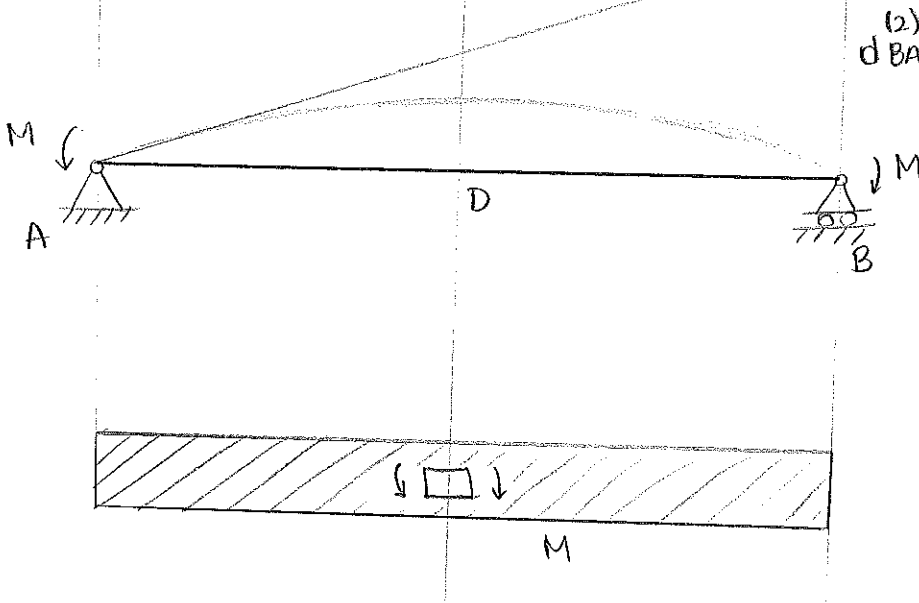
1) ANÁLISIS DE LA FLEXIÓN DE LA ESTRUCTURA

(1)



$$\delta_{BA}^{(1)} = \frac{1}{EI} \left[\frac{1}{2} \cdot \frac{P}{2} \cdot 50 \text{ cm} \cdot 100 \text{ cm} \cdot 50 \text{ cm} \right]$$

(2)



$$\delta_{BA}^{(2)} = \frac{1}{EI} [M \cdot 100 \text{ cm} \cdot 50 \text{ cm}]$$

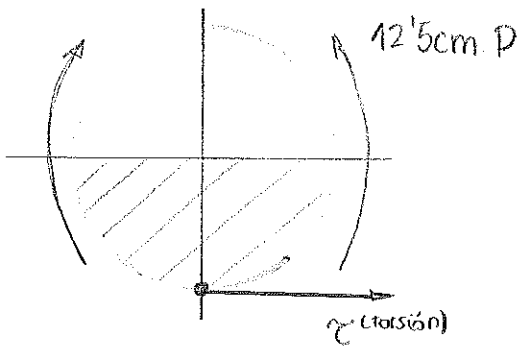
$$\delta_{BA} = \delta_{BA}^{(1)} - \delta_{BA}^{(2)} = 0$$

$$\frac{1}{EI} \left[\frac{P}{4} 50\text{cm} \cdot 100\text{cm} \cdot 50\text{cm} \right] = \frac{1}{EI} \left[M \cdot 100\text{cm} \cdot 50\text{cm} \right]$$

$$M = 50\text{cm} \cdot P \cdot \frac{1}{4} = 12.5\text{cm} P$$

⊙ secciones más comprometidas: A, D, B: Analizamos una cualquiera

2) ANÁLISIS DE LA SECCIÓN D.



$$\sigma_{xx} = \frac{12.5\text{cm} P R}{I} = \frac{12.5\text{cm} P R}{\frac{\pi R^4}{4}} = \frac{50\text{cm} P}{\pi R^3}$$

$$\tau_{xz} = \frac{5\text{cm} \cdot P \cdot R}{I_p} = \frac{10\text{cm} P}{\pi R^3}$$

⊙ Tresca: $\sigma_{eq} = \sqrt{\sigma_{xx}^2 + 4\tau_{xz}^2} = \frac{\sigma_f}{n}$

$$\frac{10\sqrt{29}\text{cm} P}{\pi R^3} = \frac{240\text{MPa}}{1.2}$$

$$R = 0.02046\text{m}$$

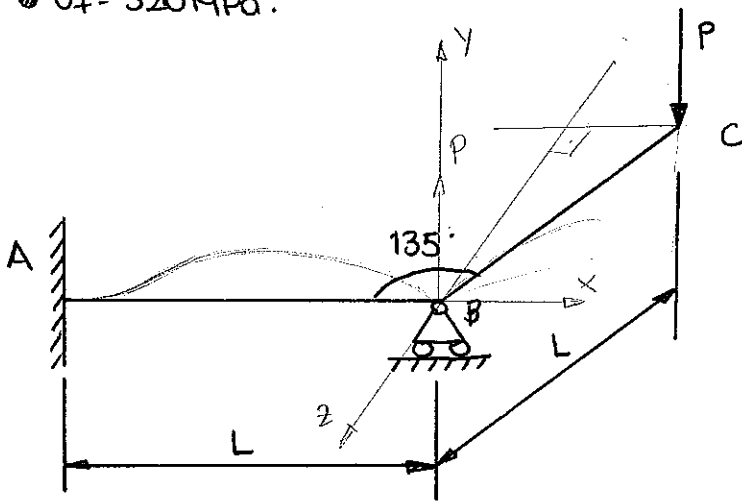
$$R^{(ADB)} > R^{(CD)} \Rightarrow R = R^{(ADB)}$$

$$d = 2 \cdot R^{(ADB)} = 0.0409\text{m} = 4.09\text{cm}$$



- 3.17
- Sección circular $d=4\text{cm}$
 - $P=3\text{kN}$
 - $L=50\text{cm}$
 - $E=200\text{GPa}$, $G=80\text{GPa}$
 - $\sigma_f=320\text{MPa}$.
- (Se desprecia el efecto del esfuerzo cortante)

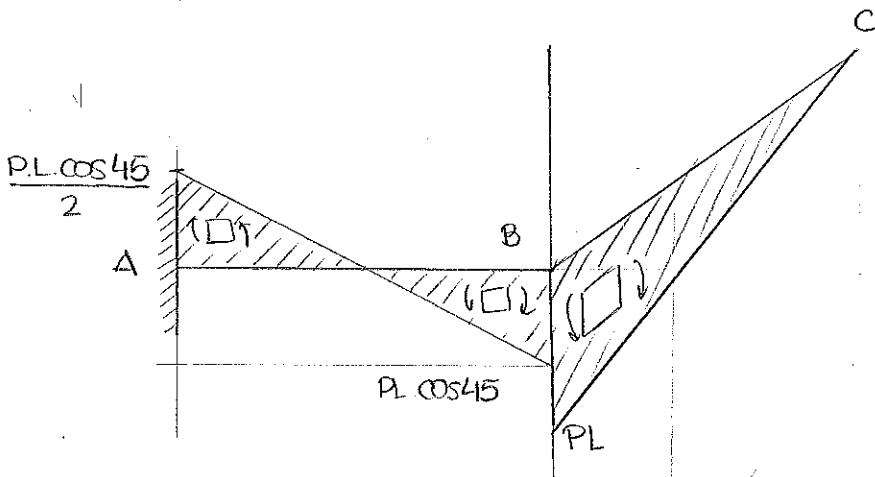
$$\left\{ \begin{aligned} I &= \frac{\pi R^4}{4} = 1256637 \cdot 10^{-7} \text{ m}^4 \\ I_p &= 2I = 2,51327 \cdot 10^{-7} \text{ m}^4 \end{aligned} \right\}$$



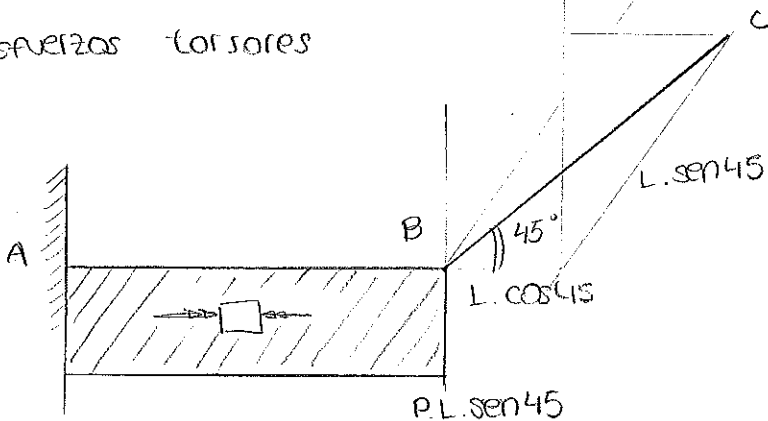
$$y_c = y_c (\text{flexión BC}) + y_c (\text{torsión AB}) + y_c (\text{flexión AB})$$

1) DIAGRAMAS DE ESFUERZOS

ESFUERZOS RECTORES



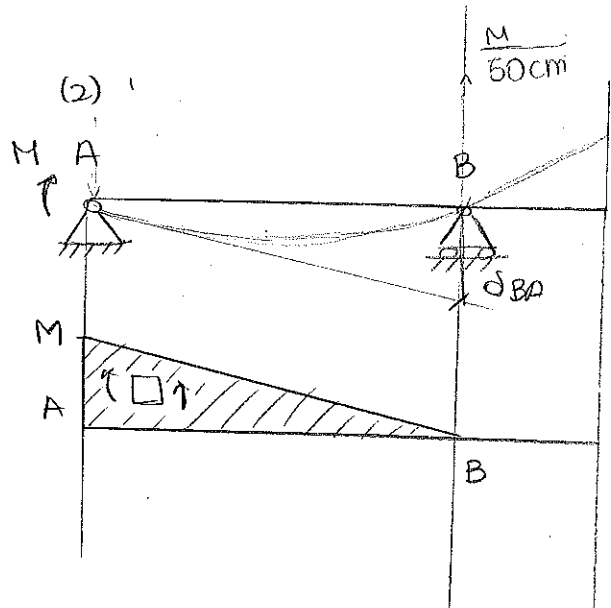
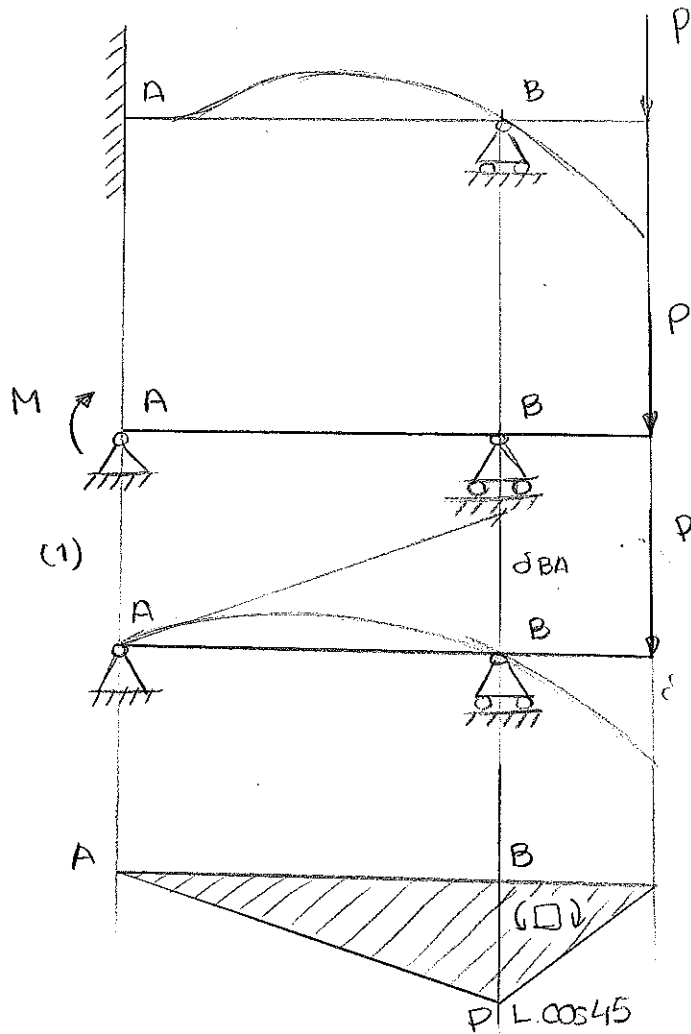
ESFUERZOS TORSORES



ESTRUCTURA HIPERESTÁTICA

⊙ EC compatibilidad

$$\delta_{BA} = 0.$$



$$\delta_{BA} = \delta_{BA}^{(1)} - \delta_{BA}^{(2)}$$

$$\delta_{BA}^{(1)} = \frac{1}{EI} \left(\frac{1}{2} P \cdot L \cos 45 \cdot L \cdot \frac{L}{3} \right) = \frac{1}{EI} \frac{\sqrt{2} PL^3}{12}$$

$$\delta_{BA}^{(2)} = \frac{1}{EI} \left(\frac{1}{2} M \cdot L \cdot \frac{2}{3} L \right) = \frac{1}{EI} \frac{ML^2}{3}$$

⊙ EC compatibilidad $\delta_{BA}^{(1)} = \delta_{BA}^{(2)}$

$$\boxed{\frac{\sqrt{2}}{4} PL = M}$$

$$\downarrow Y_{C1} \text{ (flexión BC)} = \delta_{CB} = \frac{1}{EI} \left(\frac{1}{2} PL \cdot L \cdot \frac{2}{3} L \right) = \frac{PL^3}{3EI}$$

$$\downarrow Y_{C2} \text{ (torsión AB)} = \varphi_{BA} \cdot L \cdot \text{sen } 45$$

$$\varphi_{BA} = \frac{P \cdot L \cdot \text{sen } 45 \cdot L}{GJ_p} = \frac{\sqrt{2} \cdot PL^2}{2GJ_p}$$

$$\downarrow Y_{C2} = \frac{PL^3}{2GJ_p}$$

$$\downarrow$$

$$Y_{C3} (\text{flexión AB}) = \theta_B \cdot L \cdot \cos 45$$

$$\theta_B = \theta_B^{(1)} - \theta_B^{(2)}$$

$$\theta_B^{(1)} = \frac{\delta_{AB}}{L} = \frac{1}{EI} \left(\frac{1}{2} PL \cdot \cos 45 \cdot L \cdot \frac{2}{3} \right) \frac{1}{L} = \frac{PL^2 \sqrt{2}}{6EI}$$

$$\theta_B^{(2)} = \frac{\delta_{AB}}{L} = \frac{1}{EI} \left(\frac{1}{2} \frac{\sqrt{2}}{4} PL \cdot L \cdot \frac{1}{3} \right) \frac{1}{L} = \frac{PL^2 \cdot \sqrt{2}}{24EI}$$

$$\theta_B = \frac{PL^2 \sqrt{2}}{6EI} - \frac{PL^2 \sqrt{2}}{24EI} = \frac{PL^2 \sqrt{2}}{8EI}$$

$$\downarrow$$

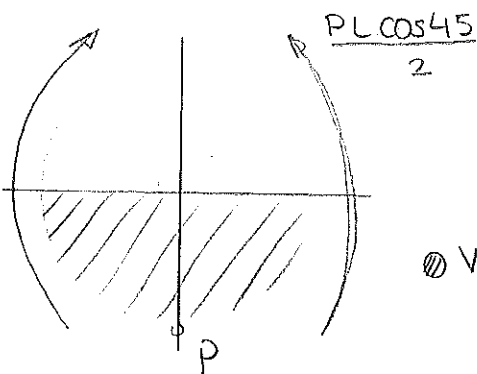
$$Y_{C3} = \frac{PL^3}{8EI}$$

$$Y_C = \frac{PL^3}{3EI} + \frac{PL^3}{26J_p} + \frac{PL^3}{8EI} = 0,016164 \text{ m}$$

$$Y_C = 1,6164 \text{ cm}$$

Las secciones más desfavorecidas son el empotramiento A (flexión + torsión) y la sección B (flexión BC)

2) ANÁLISIS DE LA SECCIÓN A



$$\sigma_{xx}^p = \frac{PL \cos 45 \cdot R}{2I} = 84,405 \text{ MPa}$$

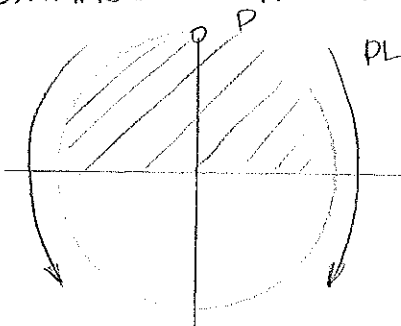
$$\tau_{xz}^p = \frac{R \cdot \sin 45 \cdot R}{J_p} = 84,405 \text{ MPa}$$

$$\text{Von-Mises: } \sigma_{eq} = \sqrt{\sigma_{xx}^2 + 3\tau_{xz}^2} = \frac{\sigma_f}{n}$$

$$168,810 \text{ MPa} = \frac{320 \text{ MPa}}{n}$$

$$n^{(A)} = 1,896$$

3) ANÁLISIS DE LA SECCIÓN B



$$\sigma_{zz}^p = \frac{P \cdot L \cdot R}{J} = 238,732 \text{ MPa}$$

$$\text{Von-Mises: } \sigma_{eq} = 238,732 \text{ MPa} = \frac{\sigma_f}{n} = \frac{320 \text{ MPa}}{n}$$

$$n^{(B)} = 1,340$$

$$n^{(B)} < n^{(A)}$$

$$n = n^{(B)} = 1'34$$

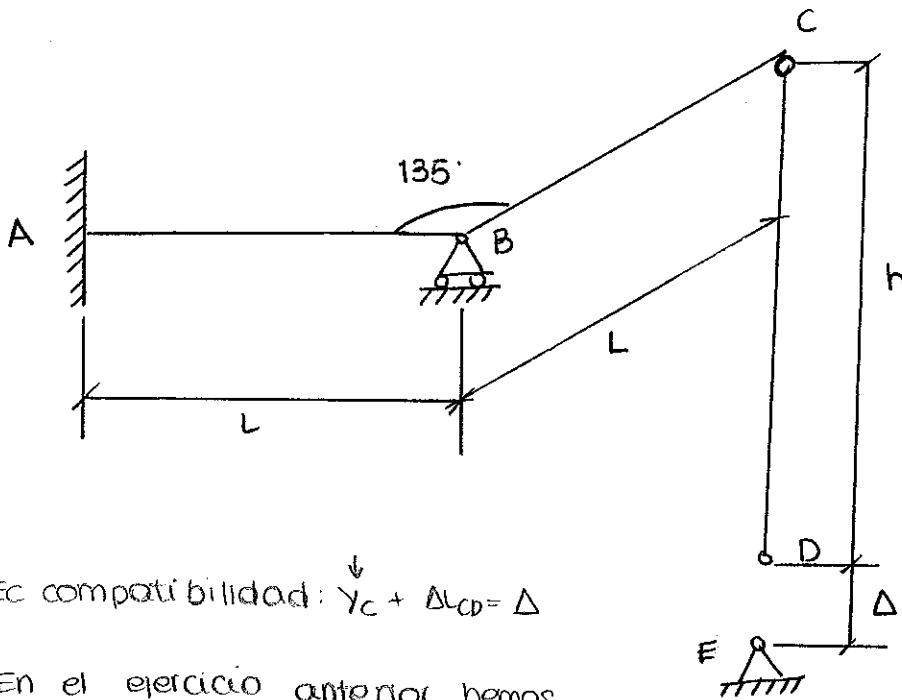
3.18 ● AB, BC: sección circular: $d = 4\text{ cm}$

● Tirante CD: sección $A_t = 5\text{ mm}^2$

● $L = 50\text{ cm}$, $h = 2\text{ m}$, $\Delta = 5\text{ mm}$

● $E = 200\text{ GPa}$, $G = 80\text{ GPa}$

$$\left\{ \begin{aligned} I &= \frac{\pi R^4}{4} = 1'256637 \cdot 10^{-7} \text{ m}^4 \\ I_p &= 2I = 2'513274 \cdot 10^{-7} \text{ m}^4 \end{aligned} \right.$$



● Ec compatibilidad: $y_c + \Delta_{CD} = \Delta$

● En el ejercicio anterior hemos

calculado el descenso de C cuando se aplica una carga P en dicho

punto, nos encontramos exactamente en esa situación siendo $N = P$,

la fuerza que aparece en el tirante al forzar que D y E coincidan

$$\downarrow y_c = \frac{NL^3}{3EI} + \frac{NL^3}{2GI_p} + \frac{NL^3}{8EI} \quad (3.17)$$

$$\left. \begin{aligned} \sigma_t &= \frac{N}{A_t} \\ \varepsilon_t &= \frac{\sigma_t}{E} \end{aligned} \right\} \varepsilon_{CD} = \frac{N}{A_t E} \Rightarrow \Delta_{LCD} = \varepsilon_{DD} \cdot h$$

● Ec compatibilidad

$$\frac{NL^3}{3EI} + \frac{NL^3}{2GI_p} + \frac{NL^3}{8EI} + \frac{N h}{A_t E} = \Delta$$

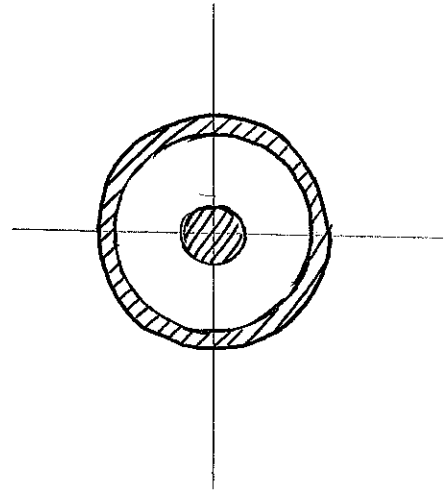
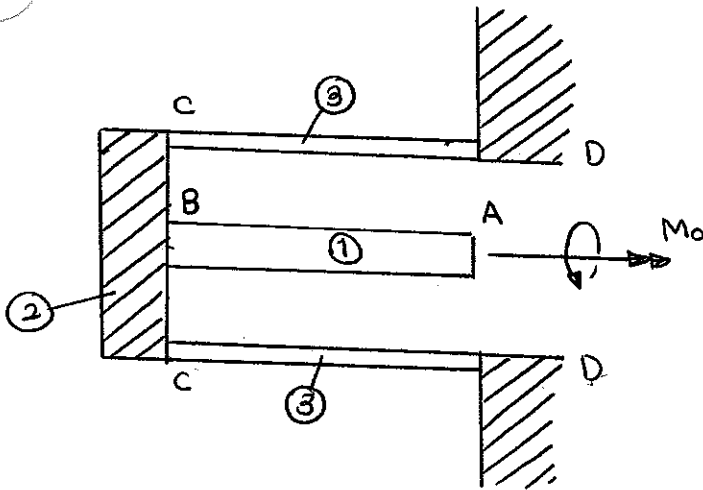
$$N = 676,7678\text{ N}$$

$$\sigma_t = \frac{N}{A_t} = 135,354\text{ MPa}$$

$$\downarrow y_c = \frac{NL^3}{3EI} + \frac{NL^3}{2GJ_p} + \frac{NL^3}{8EI} = 3'646 \cdot 10^{-3} \text{ m}$$

$$\downarrow y_c = 0,365 \text{ cm}$$

C-31



● $r_{eje} = r$

● $r_{tubo} = R; \text{espesor} = t.$

$I_{peje} = \frac{\pi r^4}{2}$

● $I_{p\text{tubo}} = 2\pi R^3 t$

$\varphi = \varphi_{AD} = \varphi_{AB} + \varphi_{CD}$

$\varphi_{AB} = \frac{M_x \cdot L}{G I_{peje}} = \frac{M_o L}{G \frac{\pi r^4}{2}} = \frac{2M_o L}{G \pi r^4}$

$\varphi_{CD} = \frac{M_x \cdot L}{G I_{p\text{tubo}}} = \frac{M_o \cdot L}{G 2\pi R^3 t}$

$\varphi_{AD} = \frac{M_o L}{G \pi} \left(\frac{2}{r^4} + \frac{1}{2R^3 t} \right)$

$$K = \frac{M_o}{\varphi_{AD}} = \frac{\pi \cdot G}{M_o L \left(\frac{2}{r^4} + \frac{1}{2R^3 t} \right)}$$



C-3.2

$E = 200 \text{ GPa}$

$E/G = 2.5$

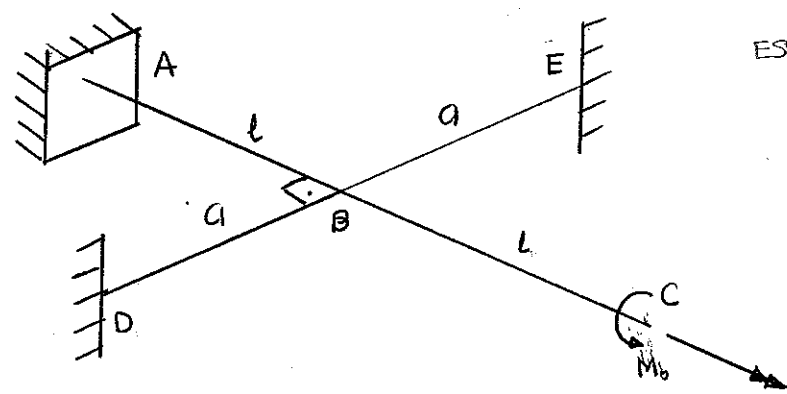
$\sigma_f = 260 \text{ MPa}$

sección circular de diámetro d .

$l = 1 \text{ m}$

$a = 60 \text{ cm}$

$M_0 = 4 \text{ kN.m}$

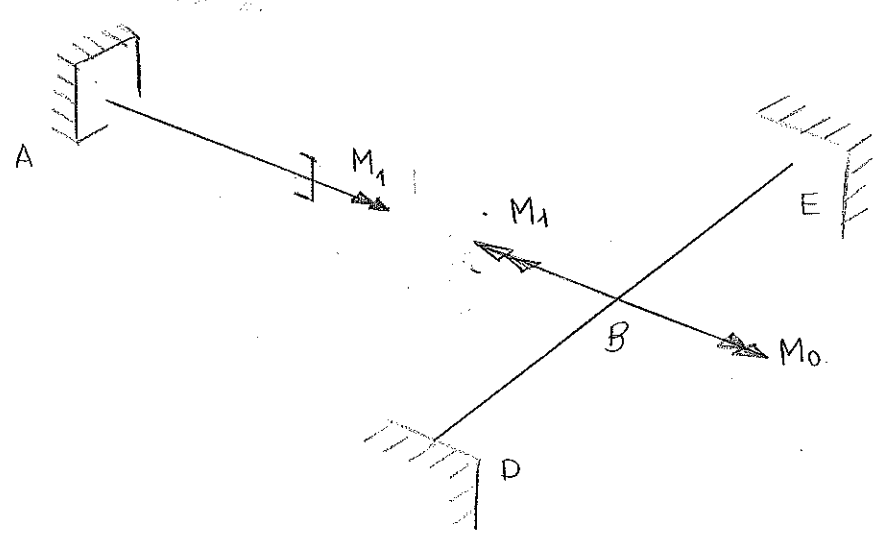


ESTRUCTURA HIPERESTÁTICA

$$I_p = \frac{\pi \left(\frac{d}{2}\right)^4}{2}$$

$$I = \frac{\pi \left(\frac{d}{2}\right)^4}{4}$$

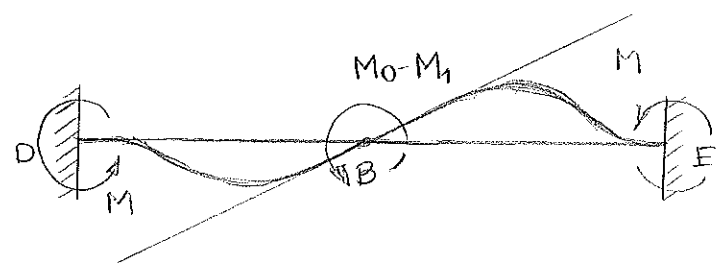
Incógnita: esfuerzo de sección M_1



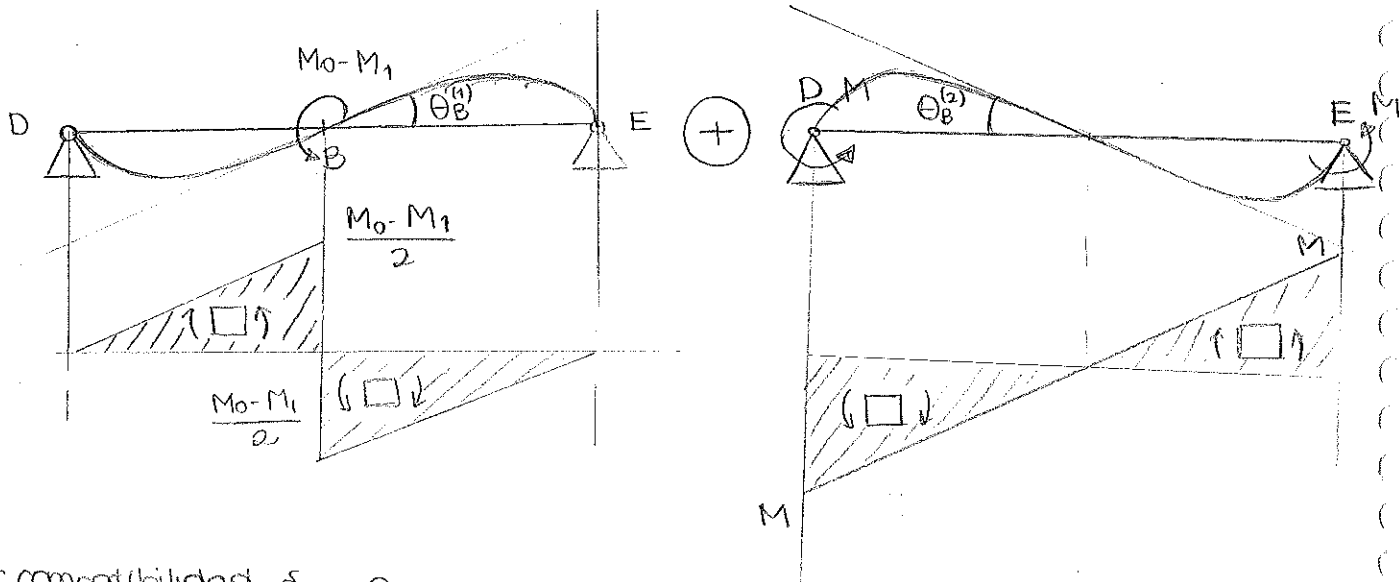
Ecuación de compatibilidad $\varphi_B^{(AB)} = \theta_B^{(DBE)}$

$$\varphi_B^{(AB)} = \frac{M_1 \cdot L}{G \cdot I_p}$$

$\theta_B^{(DBE)}$



ESTRUCTURA HIPERESTÁTICA



⊙ Ec compatibilidad $\delta_{BD} = 0$

$$\delta_{BD} = \frac{1}{EI} \left(\frac{1}{2} \frac{M_0 - M_1}{2} a \cdot \frac{1}{3} a - \frac{1}{2} M \cdot a \cdot \frac{2}{3} a \right) = 0$$

$$\frac{M_0 - M_1}{12} a^2 = \frac{M}{3} a^2$$

$$M = \frac{M_0 - M_1}{4}$$

$$\begin{aligned} \theta_B^{(DBE)} &= \theta_B^{(1)} - \theta_B^{(2)} = \frac{\delta_{DB}}{a} = \frac{1}{EIa} \left(\frac{1}{2} \frac{M_0 - M_1}{2} \cdot a \cdot \frac{2}{3} a - \frac{1}{2} \cdot \frac{M_0 - M_1}{4} \cdot a \cdot \frac{1}{3} a \right) = \\ &= \frac{1}{EI} (M_0 - M_1) \left(\frac{a}{6} - \frac{a}{24} \right) = \frac{M_0 - M_1}{8EI} a \end{aligned}$$

⊙ Ecuación de compatibilidad anterior

$$\left. \begin{aligned} \frac{M_1 \cdot L}{G \cdot I_p} = \frac{M_0 - M_1}{8EI} a \\ I_p = 2J \end{aligned} \right\} \begin{aligned} \frac{M_1 L}{2G} = \frac{M_0 - M_1}{8E} a \\ \boxed{M_1 = 226,415 \text{ Nm}} \end{aligned}$$

Von Mises $\sigma_{eq} = \sqrt{\sigma_{xx}^2 + 3\tau_{xy}^2}$

AB $\tau_{xy} = \frac{M_1 \cdot L}{I_p} = \frac{M_1 \cdot L}{2J} \Rightarrow \sigma_{eq} = \sqrt{3} \cdot \frac{M_1 L}{2J} = \frac{260 \text{ MPa}}{1,5} \Rightarrow R = 0,011 \text{ m} \Rightarrow \phi = 0,022 \text{ m} = 2,2 \text{ cm}$

DBE $\sigma_{xx} = \frac{M_2 \cdot y}{I} = \frac{(M_0 - M_1) R}{I} = \sigma_{eq} = \frac{260 \text{ MPa}}{1,5} \Rightarrow R = 0,03 \text{ m} \Rightarrow \phi = 0,06 \text{ m} = 6 \text{ cm}$

El diámetro ha de ser de 6cm

$$\varphi_{CA} = \varphi_{BA} + \varphi_{CB} = \frac{M_1 \cdot L}{GI_p} + \frac{M_0 \cdot L}{GI_p} = 4,15 \cdot 10^{-2} \text{ rad}$$

$$\varphi = 4,15 \cdot 10^{-2} \text{ rad}$$



C-3.3

● $G = 80 \text{ GPa}$

● D: empotramiento flexible $k = 200 \text{ m.kN/rad}$

● $\varphi_{\text{rot}}^h = 0,012 \text{ rad}$

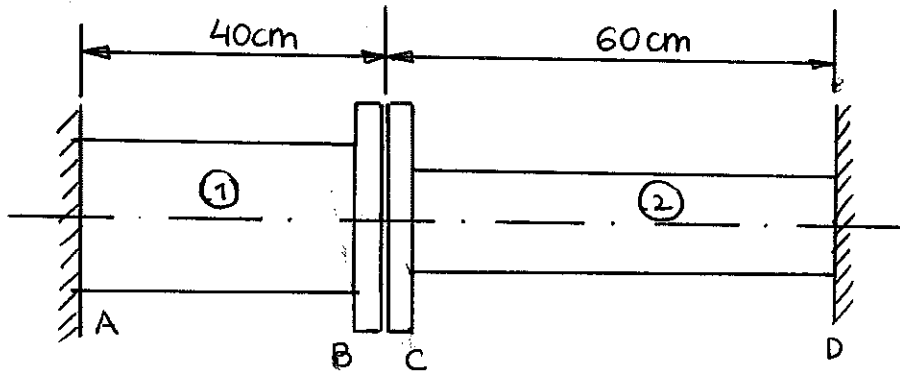
(1) $M_B = 8 \text{ m.kN}$

(2) $M_C = 8 \text{ m.kN}$

$$I_{PBA} = \frac{\pi R^4}{2} = 4,021239 \cdot 10^{-6} \text{ m}^4$$

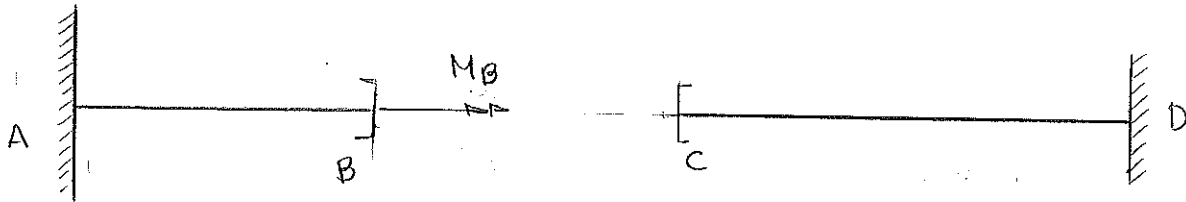
$$I_{PCD} = \frac{\pi R^4}{2} = 1,272345 \cdot 10^{-6} \text{ m}^4$$

● sección circular $\begin{cases} d_1 = 8 \text{ cm} \\ d_2 = 6 \text{ cm} \end{cases}$



(1) $M_B = 8 \text{ m.kN}$

ESTRUCTURA HIPERESTÁTICA:



$$\varphi_{BA} = \frac{M_B \cdot L}{G I_{PAB}} = 9,947 \cdot 10^{-3} \text{ rad} < \varphi_h \rightarrow \text{La barra CD no trabaja a torsión.}$$

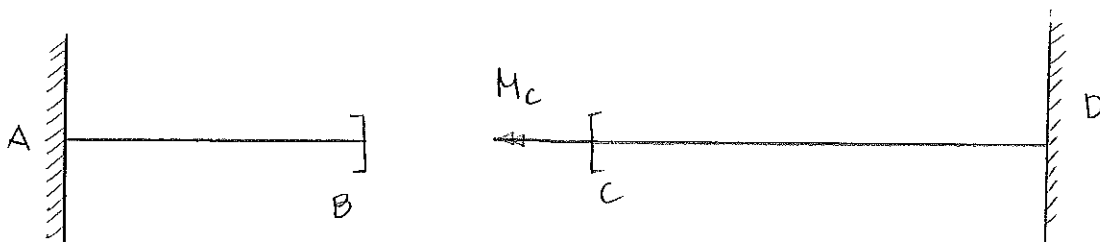
$$\tau_{\text{máx}}^{(1)} = \frac{M_B R_1}{I_{PAB}} = 79,577 \text{ MPa}; \quad \tau_{\text{máx}}^{(2)} = 0 \text{ MPa}$$

$$\tau_{\text{máx}}^{(1)} = 79,577 \text{ MPa}$$

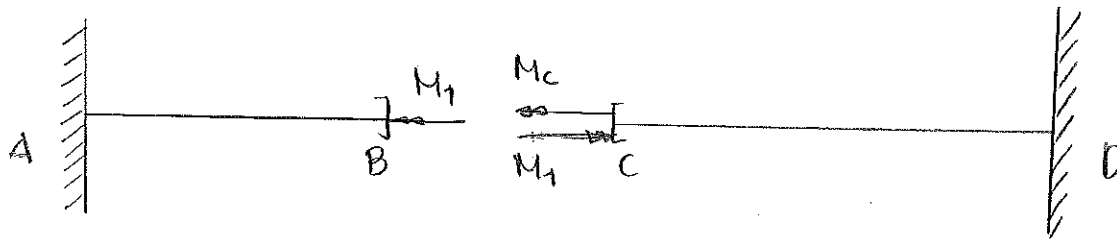
$$\tau_{\text{máx}}^{(2)} = 0 \text{ MPa}$$

(2) $M_C = 8 \text{ m.kN}$

ESTRUCTURA HIPERESTÁTICA



$$\varphi_{CD} = \frac{M_c \cdot 60\text{cm}}{G I_{PCD}} = 0'0472 \text{ rad} > \varphi_n: \text{ambas barras trabajan a torsión.}$$



⊙ Ec compatibilidad: $\varphi_{BA} + \varphi_n = \varphi_{CD} + \varphi_p$

$$\varphi_{CD} = \frac{(M_c - M_1) \cdot 60\text{cm}}{G I_{PCD}}$$

$$\varphi_{BA} = \frac{M_1 \cdot 40\text{cm}}{G I_{PBA}}$$

$$\frac{M_1 \cdot 40\text{cm}}{G \cdot I_{PBA}} + 0'012 = \frac{(M_c - M_1) \cdot 60\text{cm}}{G \cdot I_{PCD}} + \frac{M_c - M_1}{K} \Rightarrow M_1 = 6191,865 \text{ Nm}$$

$$\tau_{\text{máx}}^{(1)} = \frac{M_1 \cdot R_1}{I_{PAB}} = 61'592 \text{ MPa}$$

$$\tau_{\text{máx}}^{(2)} = \frac{(M_c - M_1) R_2}{I_{PCD}} = 42'633 \text{ MPa}$$

$\tau_{\text{máx}}^{(1)} = 61'592 \text{ MPa}$ $\tau_{\text{máx}}^{(2)} = 42'633 \text{ MPa}$

C-3.4

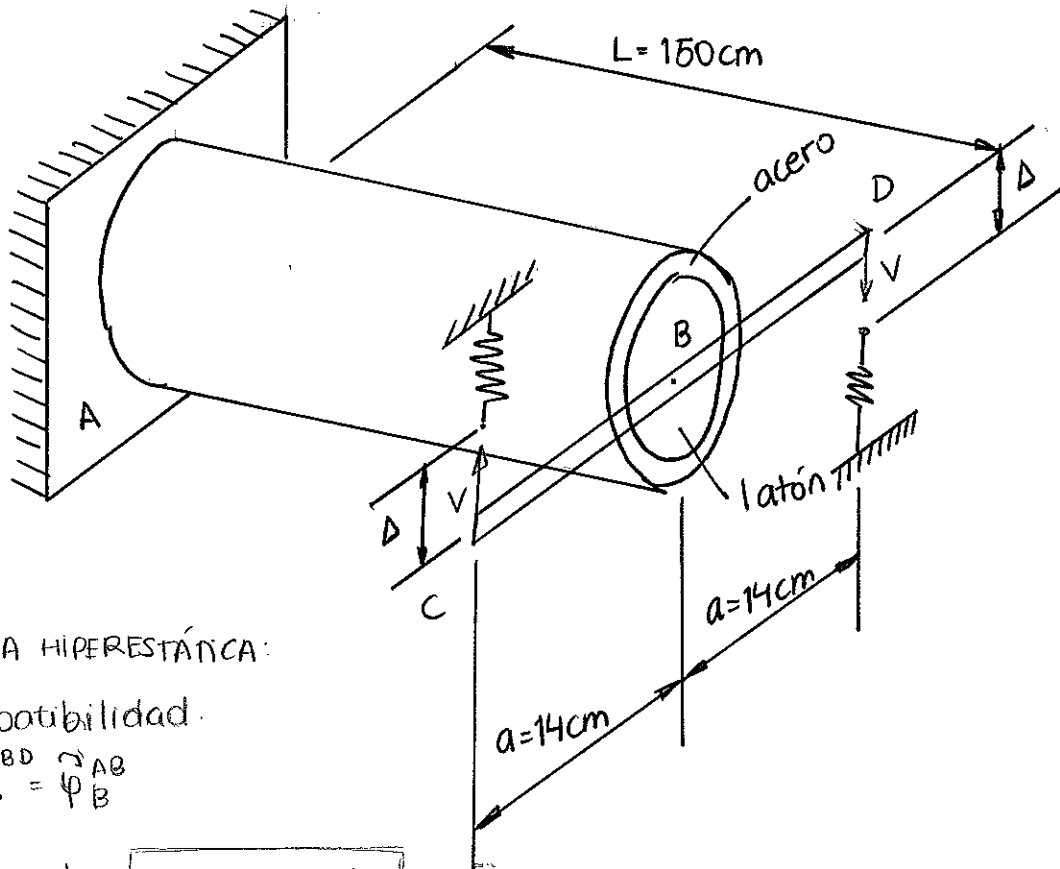
● Acero: $d_a = 6\text{cm}$ } sección circular.
 ● Latón: $d_l = 4\text{cm}$

● $k = 400 \frac{\text{kN}}{\text{m}}$, $\Delta = 3\text{cm}$

● $G_A = 80\text{GPa}$; $G_L = 35\text{GPa}$.

$$I_{PA} = \frac{\pi(R_a^4 - R_l^4)}{2} = 1'021'018 \cdot 10^{-6} \text{m}^4$$

$$I_{PL} = \frac{\pi R_l^4}{2} = 2'513'274 \cdot 10^{-7} \text{m}^4$$



ESTRUCTURA HIPERESTÁTICA:

● Ec. compatibilidad.

$$\vec{\theta}_B^{CBD} = \vec{\varphi}_B^{AB}$$

$$\vec{\theta}_B^{CBD} = \frac{CC'}{a} \quad \left\{ \begin{array}{l} \vec{\theta}_B^{CBD} = \frac{\Delta - \frac{V}{k}}{a} \\ CC' = \Delta - \frac{V}{k} \end{array} \right.$$

$$\left\{ \begin{array}{l} M_x = 2 \cdot V \cdot a = M_A + M_L \\ \varphi_A = \varphi_L \end{array} \right.$$

$$\varphi_A = \frac{M_A \cdot L}{G_A \cdot I_{PA}} \quad \left\{ \begin{array}{l} \varphi_A = \varphi_L : M_A = \frac{M_L \cdot G_A \cdot I_{PA}}{G_L \cdot I_{PL}} \\ M_L = 2Va \left(\frac{G_L \cdot I_{PL}}{G_A \cdot I_{PA} + G_L \cdot I_{PL}} \right) \end{array} \right.$$

$$\varphi_L = \frac{M_L \cdot L}{G_L \cdot I_{PL}}$$

$$M_L = 2Va \left(\frac{G_L \cdot I_{PL}}{G_A \cdot I_{PA} + G_L \cdot I_{PL}} \right)$$

$$\varphi_L = \frac{2Va \cdot L}{G_A \cdot I_{PA} + G_L \cdot I_{PL}} = \vec{\varphi}_B^{AB}$$

Ec. Compatibilidad

$$\frac{\Delta - \frac{V}{K}}{a} = \frac{2VaL}{G_A I_{PA} + G_L I_{PL}} \quad \boxed{V = 9524,165 \text{ N}}$$

$$\varphi_{BA} = \frac{2VaL}{G_A I_{PA} + G_L I_{PL}} = 0,0442 \text{ rad}$$

$$\boxed{\varphi_B^{(AB)} = \varphi_{BA} = 0,0442 \text{ rad}}$$

$$\tau_A = \frac{M_A \cdot R_A}{I_{PA}} = 70,738 \text{ MPa}$$

$$\boxed{\tau_{A,\text{máx}} = 70,738 \text{ MPa}}$$

$$\tau_L = \frac{M_L \cdot R_L}{I_{PL}} = 20,632 \text{ MPa}$$

$$\boxed{\tau_{L,\text{máx}} = 20,632 \text{ MPa}}$$

$$M_L = 259,269 \text{ Nm}$$

$$M_A = 2407,497 \text{ Nm}$$

2-3.5

● Sección circular : $d=20\text{cm}$

● TRESCA. $\sigma_f = 240\text{MPa}$

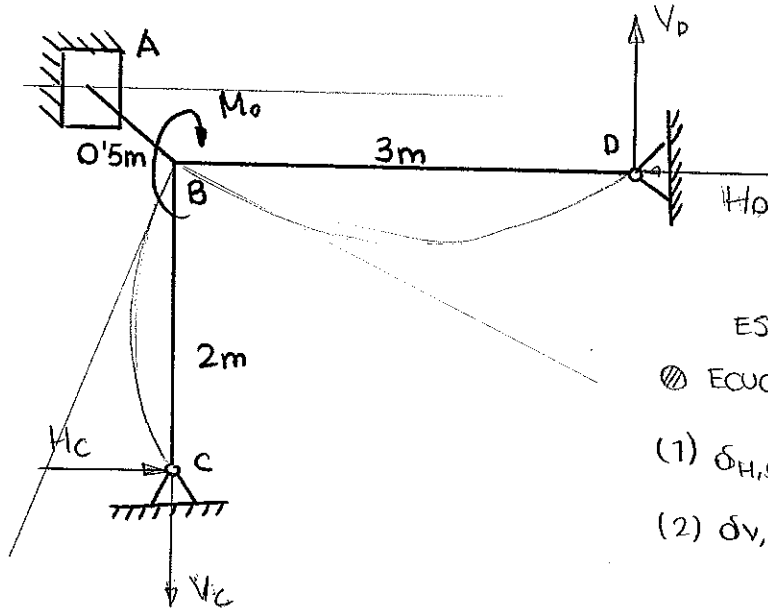
● $E = 200\text{GPa}$, $\nu = 0.25$

● $M_o = 205\text{KNm}$

$$G = \frac{E}{2(1+\nu)} = 80\text{GPa}$$

$$I = \frac{\pi R^4}{4} = 7.853982 \cdot 10^{-5} \text{m}^4$$

$$I_p = 2 \cdot I = 1.570796 \cdot 10^{-4} \text{m}^4$$



ESTRUCTURA HIPERESTÁTICA

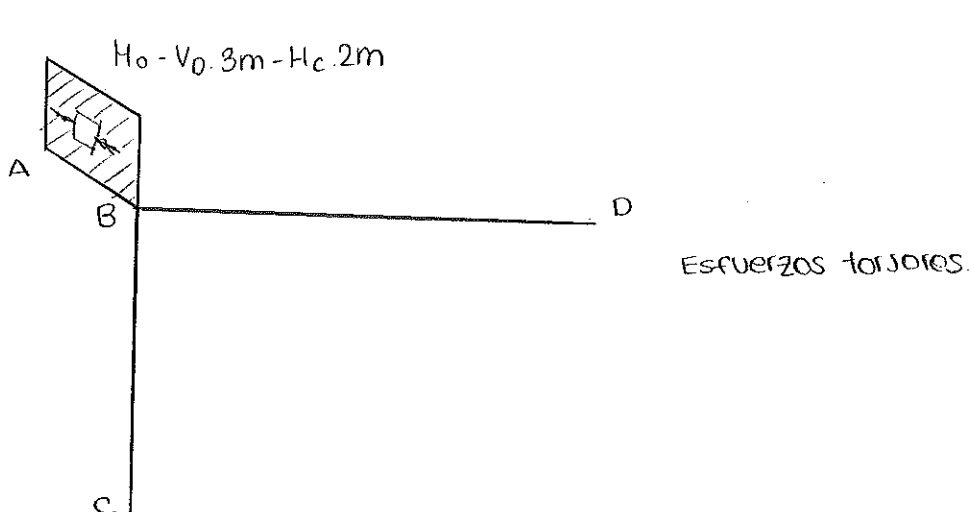
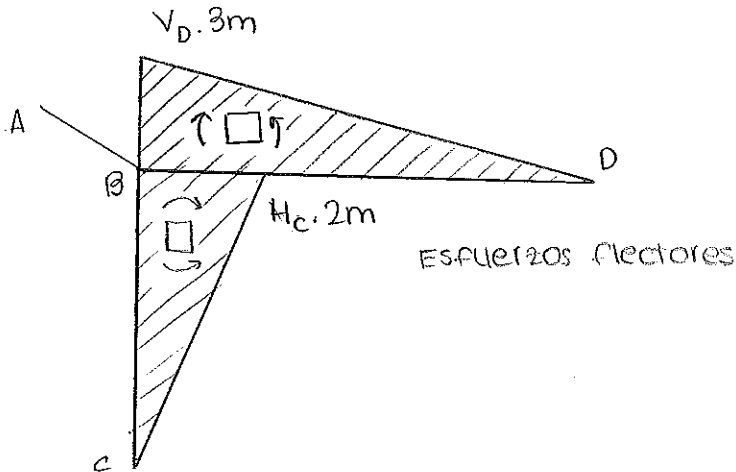
● Ecuaciones de compatibilidad:

(1) $\delta_{H,C} = 0$

(2) $\delta_{V,D} = 0$

1) DIAGRAMA DE ESFUERZOS

● Se desprecian los esfuerzos axiales, luego el punto B no se puede mover \Rightarrow no hay flexión en AB. $\Rightarrow M_A^r = 0$; $V_b = V_c$; $H_c = H_D$.



$$(1) \delta_{H,c} = \delta_{H,c}^{(fl.BC)} - \delta_{H,c}^{(tr.AB)} = 0$$

$$\delta_{H,c}^{(fl.BC)} = \frac{1}{EI} \left(\frac{1}{2} H_c \cdot 2m \cdot 2m \cdot \frac{2}{3} \cdot 2m \right) = \frac{H_c \cdot 8}{3EI}$$

$$\delta_{H,c}^{(tr.AB)} = \varphi_{BA} \cdot 2m$$

$$\varphi_{BA} = \frac{(M_0 - V_D \cdot 3m - H_c \cdot 2m) \cdot 0.5m}{G \cdot I_p}$$

$$\delta_{H,c}^{(tr.AB)} = \frac{(M_0 - V_D \cdot 3m - H_c \cdot 2m) \cdot 0.5m}{G \cdot I_p}$$

$$(1) \frac{H_c \cdot 8}{3EI} = \frac{(M_0 - V_D \cdot 3m - H_c \cdot 2m) \cdot 0.5m}{G \cdot I_p} : V_D = \left(\frac{8H_c \cdot G \cdot I_p}{0.5 \cdot 3EI} - M_0 + 2m \cdot H_c \right) \left(-\frac{1}{3} \right)$$

$$(2) \delta_{V,D} = \delta_{V,D}^{(fl.B,D)} - \delta_{V,c}^{(tr.AB)} = 0$$

$$\delta_{V,D}^{(fl.B,D)} = \frac{1}{EI} \left(\frac{1}{2} V_D \cdot 3m \cdot 3m \cdot \frac{2}{3} \cdot 3m \right) = \frac{9V_D}{EI}$$

$$\delta_{V,D}^{(tr.AB)} = \varphi_{BA} \cdot 3m$$

$$\varphi_{BA} = \frac{(M_0 - V_D \cdot 3m - H_c \cdot 2m) \cdot 0.5m}{G \cdot I_p}$$

$$\delta_{V,D}^{(tr.AB)} = \frac{(M_0 - V_D \cdot 3m - H_c \cdot 2m) \cdot 1.5m}{G \cdot I_p}$$

$$(2) \frac{9V_D}{EI} = \frac{(M_0 - V_D \cdot 3m - H_c \cdot 2m) \cdot 1.5m}{G \cdot I_p} : V_D = \frac{(M_0 - 2H_c)}{\frac{9G \cdot I_p}{15EI} + 3}$$

$$\left(\frac{8H_c \cdot G \cdot I_p}{3EI} - M_0 + 2H_c \right) \left(-\frac{1}{3} \right) = \frac{M_0 - 2H_c}{\frac{9G \cdot I_p}{15EI} + 3}$$

$$H_c = 37500,004 \text{ N} = 37,5 \text{ kN}$$

$$V_D = 16666,666 \text{ N} = 16,67 \text{ kN}$$

$$H_c = H_D = 37,5 \text{ kN}$$

$$V_c = V_D = 16,67 \text{ kN}$$

$$\varphi_{BA} = \frac{(M_0 - V_0 \cdot 3m - H_c \cdot 2m) \cdot 0.5}{G I_p} = 3.1831 \cdot 10^{-3} \text{ rad}$$

$$\varphi_B = \varphi_{BA} = 3.1831 \cdot 10^{-3} \text{ rad}$$

- Secciones más desfavorecidas: torsión (tramo AB)
flexión (B del tramo BC)

2) ANÁLISIS DE LA SECCIÓN A

$$\tau_{\max}^t = \tau_A = \frac{(M_0 - V_0 \cdot 3m - H_c \cdot 2m) \cdot R}{I_p} = 50.923 \text{ MPa}$$

$$\sigma_{eq} = \sqrt{\sigma_{xx}^2 + 4\tau_{xy}^2} = 101.846 \text{ MPa} = \frac{\sigma_f}{n} \rightarrow n^t = 2.356$$

3) ANÁLISIS DE LA SECCIÓN B

$$\sigma_{xx} = \frac{H_c \cdot 2m \cdot R}{I} = 95.493 \text{ MPa}$$

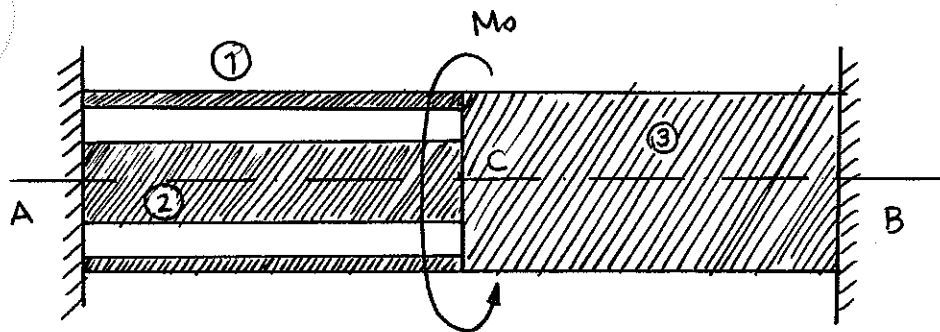
$$\sigma_{eq} = 95.493 \text{ MPa} = \frac{\sigma_f}{n} \rightarrow n^f = 2.513$$

$$n^t < n^f \rightarrow n = n^t$$

$$n = 2.356$$



C-3.6



(1) tubo de radio $R=8\text{cm}$ y $t=1\text{cm}$

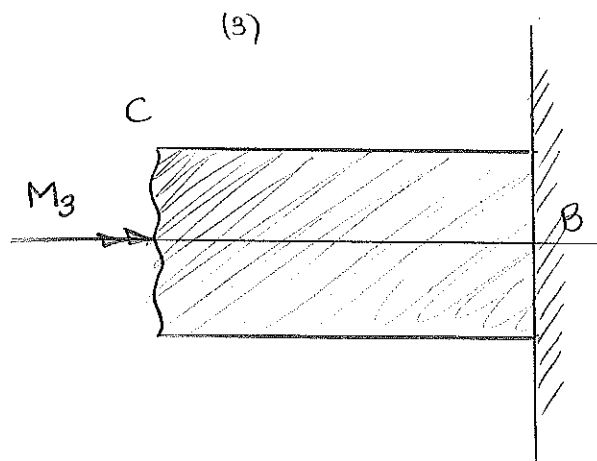
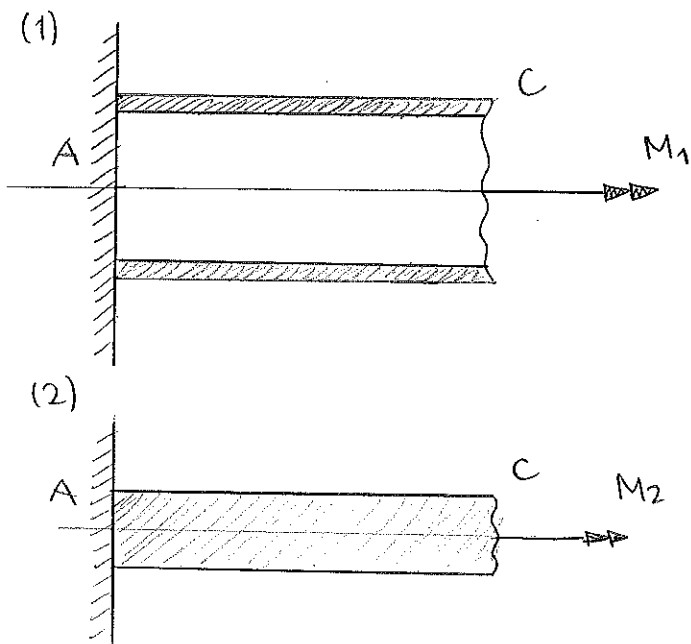
(2) cilindro de radio $\frac{R}{2} = 4\text{cm} = r$

(3) cilindro de radio $R = 8\text{cm}$

● $\sigma_p = 250\text{MPa}$

● $M_0 = 100\text{KN}\cdot\text{m}$

$$M_0 = M_1 + M_2 + M_3$$



Ec. compatibilidad:

$$\varphi_{CB}^{(3)} = \varphi_{CA}^{(1)}$$

$$\varphi_{CB}^{(3)} = \varphi_{CA}^{(2)}$$

$$\varphi_{CA}^{(1)} = \frac{M_1 \cdot L}{G I_{P1}}$$

$$I_{P1} = 2\pi t \cdot R^3 = 3'21699 \cdot 10^{-5} \text{m}^4 \text{ (A)}$$

$$\varphi_{CA}^{(2)} = \frac{M_2 \cdot L}{G I_{P2}}$$

$$I_{P2} = \frac{\pi r^4}{2} = 4'0212386 \cdot 10^{-6} \text{m}^4 \text{ (B)}$$

$$\varphi_{CB}^{(3)} = \frac{M_3 \cdot L}{G I_{P3}}$$

$$I_{P3} = \frac{\pi R^4}{2} = 6'43398155 \cdot 10^{-5} \text{m}^4 \text{ (C)}$$

$$\varphi_{CB}^{(3)} = \varphi_{CA}^{(1)} : \frac{M_1 \cdot L}{8 I_{P_1}} = \frac{M_3 \cdot L}{8 I_{P_3}} \Rightarrow M_1 = \frac{1}{2} M_3$$

$$\varphi_{CB}^{(3)} = \varphi_{CA}^{(2)} : \frac{M_2 \cdot L}{8 I_{P_2}} = \frac{M_3 \cdot L}{8 I_{P_3}} \Rightarrow M_2 = \frac{1}{16} M_3$$

$$M_1 + M_2 + M_3 = M_0$$

$$\begin{cases} M_3 = \frac{16}{25} M_0 = 0'64 M_0 \\ M_1 = 0'32 M_0 \\ M_2 = 0'04 M_0 \end{cases}$$

$$\begin{cases} M_1 = 0'32 M_0 \\ M_2 = 0'04 M_0 \\ M_3 = 0'64 M_0 \end{cases}$$

$$\tau_{xy, \max}^{(1)} = \frac{M_1 \cdot R}{I_{P_1}} = 79'577 \text{ MPa}$$

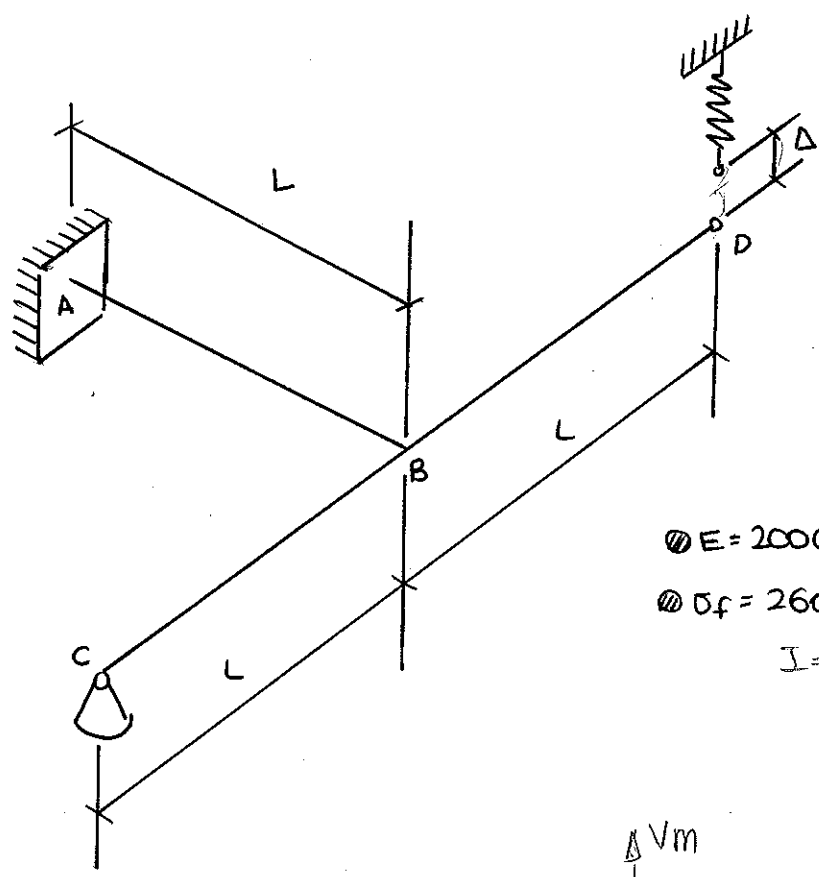
$$\tau_{xy, \max}^{(2)} = \frac{M_2 \cdot r}{I_{P_2}} = 39'7887 \text{ MPa}$$

$$\tau_{xy, \max}^{(3)} = \frac{M_3 \cdot R}{I_{P_3}} = 79'577 \text{ MPa}$$

$$\sigma_{eq} = \sqrt{4 \tau_{xy}} = 159,155 \text{ MPa}$$

$$n = \frac{\sigma_f}{\sigma_{eq}} = 1'571$$

$$n = 1'571$$



- sección circular
d = 8cm
- k = 2500 kN/m
- Δ = 2'5cm
- L = 100cm

- E = 200GPa, G = 80GPa
- σ_f = 260MPa.

$$I = \frac{\pi (4\text{cm})^4}{4} = 2,010619298 \cdot 10^{-6} \text{m}^4$$

● E_c compatibilidad.

$$\delta_D = \Delta - \delta_m$$

$$\delta_C = 0$$

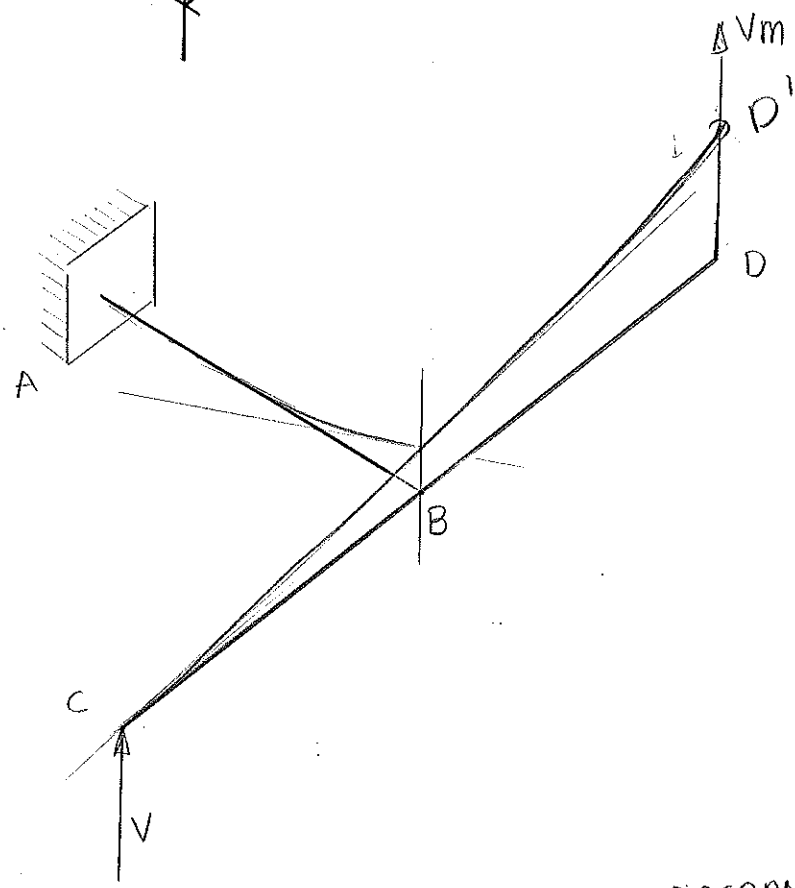


DIAGRAMA DE ESFUERZOS FLECTORES

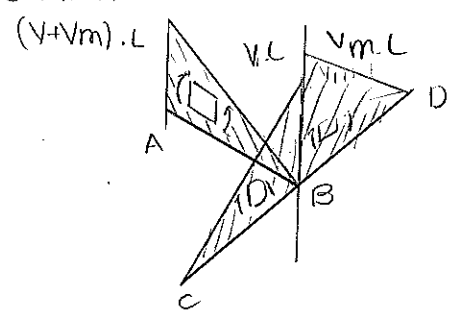
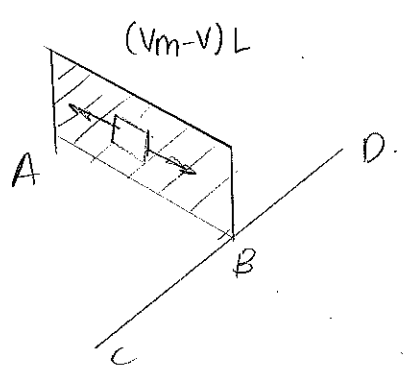
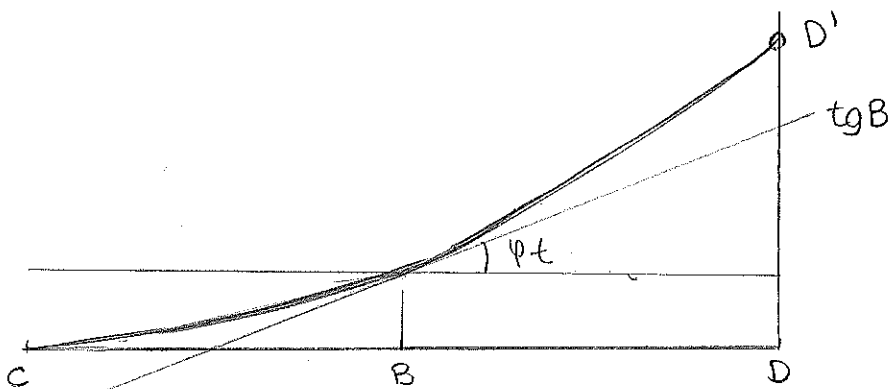


DIAGRAMA DE ESFUERZOS TORSORES.





$$\boxed{(1) \delta_c = 0}$$

$$\delta_c = -\delta(\text{flexión AB}) - \delta(\text{flexión BC}) + \delta(\text{torsión AB})$$

$$\delta_c(\text{flexión AB}) = \delta_{cA} = \frac{1}{EI} \left[\frac{1}{2} (V+V_m) L L + \frac{2}{3} L \right] = \frac{(V+V_m)L^3}{3EI}$$

$$\delta_c(\text{flexión BC}) = \delta_{cB} = \frac{1}{EI} \left[\frac{1}{2} V L \cdot L \cdot \frac{2}{3} L \right] = \frac{V L^3}{3EI}$$

$$\varphi_t = \varphi_{BA} = \frac{(V_m - V)L \cdot L}{GI_p} = \frac{(V_m - V)L^2}{2GI}$$

$$\delta_c(\text{torsión AB}) = \varphi_t \cdot L = \frac{(V_m - V)L^3}{2GI}$$

$$\frac{(V+V_m)L^3}{3EI} + \frac{V L^3}{3EI} = \frac{(V_m - V)L^3}{2GI}$$

$$4,766358973 \cdot 10^{-6} \cdot V = 2,279562987 \cdot 10^{-6} V_m$$

$$\boxed{V_m = 2,090909091 \cdot V}$$

$$(2) \delta_D = \Delta - \delta_m$$

$$\delta_m = \frac{V_m}{K}$$

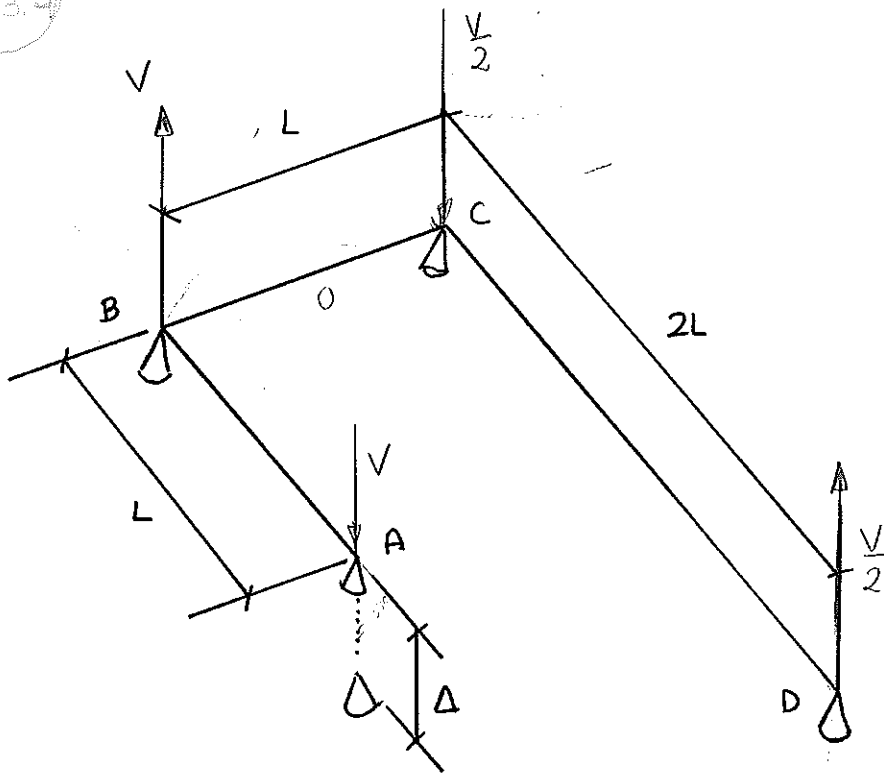
$$\delta_D = \delta_B(\text{flexión AB}) + \varphi_t \cdot L + \delta_{DB}$$

$$\delta_{DB} = \frac{1}{EI} \left[\frac{1}{2} V_m L \cdot L \cdot \frac{2}{3} L \right] = \frac{V_m L^3}{3EI}$$

$$\left. \begin{aligned} \frac{(V+V_m)L}{3EI} + \frac{(V_m - V)L^2}{2GI} \cdot L + \frac{V_m L^3}{3EI} &= \Delta - \frac{V_m}{K} \end{aligned} \right\}$$

$$\boxed{\begin{aligned} V &= 2933,30 \text{ N} \\ V_m &= 6133,26 \text{ N} \end{aligned}}$$

C-3.9



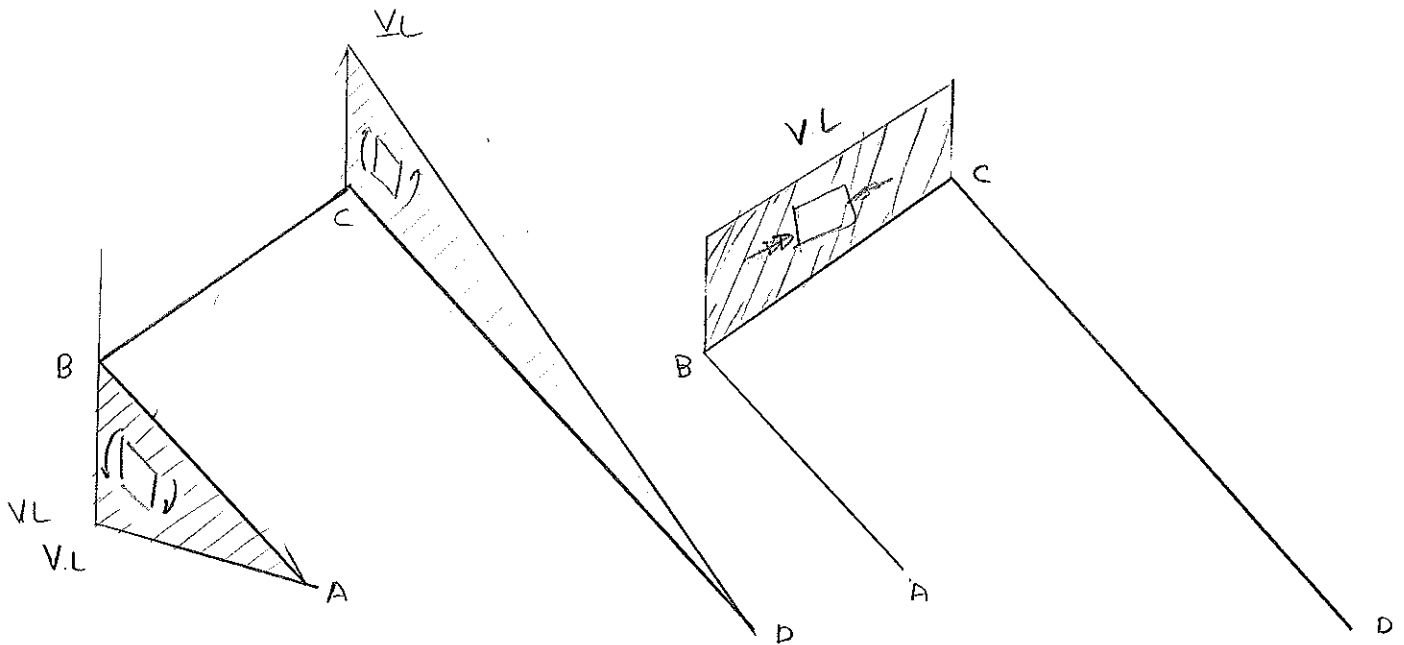
- ⊙ $d = 6\text{cm}$
- ⊙ $E = 200\text{GPa}$
- ⊙ $\sigma_f = 240\text{MPa}$
- ⊙ $G = \frac{2}{5}E$
- ⊙ $t = 50\text{cm}$
- ⊙ Tresca: $n = 1.5$

$$I = \frac{\pi r^4}{4} = 6,361725124 \cdot 10^{-7} \text{m}^4$$

Ec. compatibilidad: $\delta_A = \Delta$

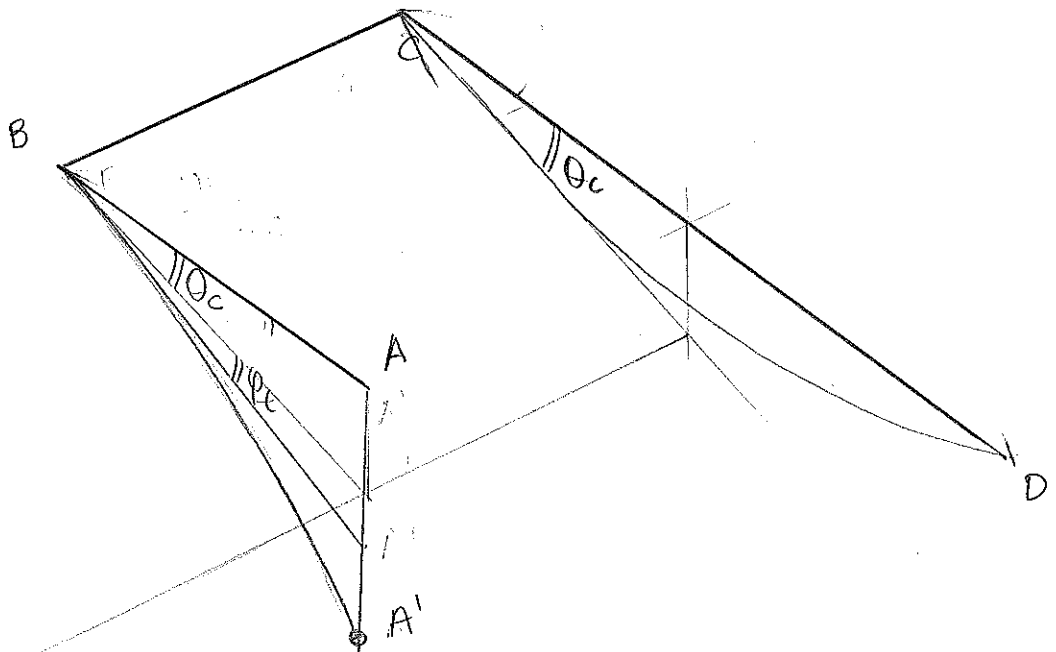
DIAGRAMA DE ESFUERZOS FLECTORES

DIAGRAMA DE ESFUERZOS TORSORES



SECCIÓN MÁS DESFAVORECIDA: (B)

$$\left. \begin{aligned} \sigma_{xx, \max} &= \frac{VL \cdot r}{I} \\ \tau_{xy, \max} &= \frac{VL \cdot r}{I_p} = \frac{VLr}{2I} \end{aligned} \right\} \begin{aligned} \sigma_{xx, \max} &= \frac{\sigma_f}{n} \Rightarrow V = 6785,840 \text{ N} \\ \sigma_{eq} = 2\tau &= \frac{VLr}{I} = \frac{\sigma_f}{n} \Rightarrow V = 6785,840 \text{ N} \end{aligned}$$



$$\Delta = +(\theta_c + \psi_{BC}) \cdot L + \delta_{AB}$$

$$\theta_c = \frac{\delta_{DC}}{2L}$$

$$\delta_{BC} = \frac{1}{EI} \left[\frac{1}{2} VL \cdot 2L \cdot \frac{2}{3} 2L \right] = \frac{4VL^3}{3EI} \Rightarrow \theta_c = \frac{2VL^2}{3EI}$$

$$\psi_{BC} = \frac{VL \cdot L}{GJ_p} = \frac{5VL^2}{4EI}$$

$$\delta_{AB} = \frac{1}{EI} \left[\frac{1}{2} VL \cdot L \cdot \frac{2}{3} L \right] = \frac{VL^3}{3EI}$$

$$\Delta = + \left(\frac{2VL^2}{3EI} + \frac{5VL^2}{4EI} \right) \cdot L + \frac{VL^3}{3EI} = 0.015 \text{ m}$$

$$\Delta_{\text{máx}} = 15 \text{ mm}$$

Diagrama de esfuerzos flexores.

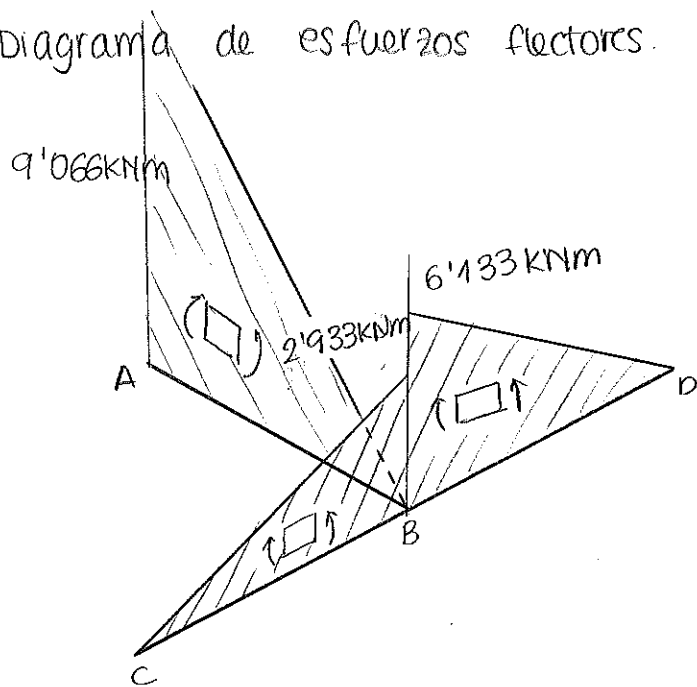
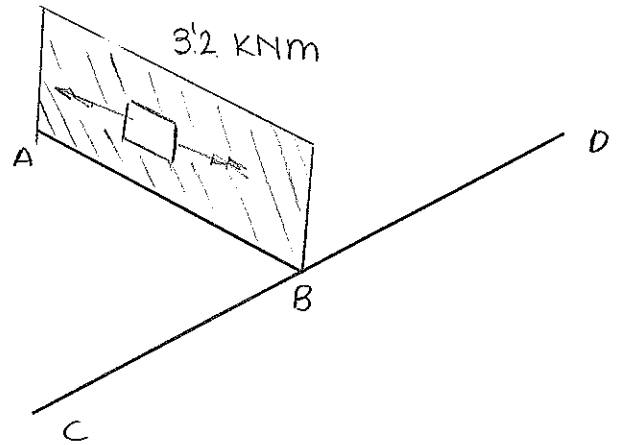
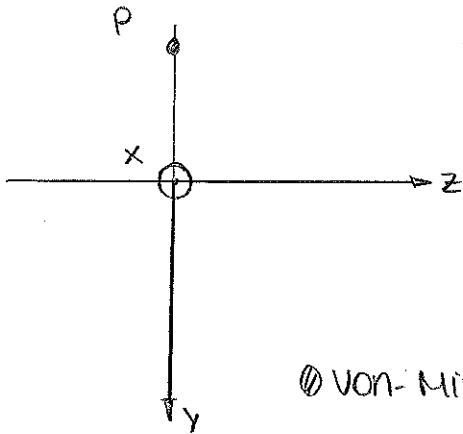


Diagrama de esfuerzos torsores



SECCIÓN MÁS DESFAVORECIDA: A.



Punto más desfavorecido: P

$$\sigma_{xx} = \frac{M_z \cdot y}{I} = 180,3623 \text{ MPa}$$

$$\tau_{xz} = \frac{M_x \cdot R}{I_p} = 31'8306 \text{ MPa}$$

$$\text{VON-MISES: } \sigma_{eq} = \sqrt{\sigma_{xx}^2 + 3\tau_{xz}^2} = 188,600 \text{ MPa}$$

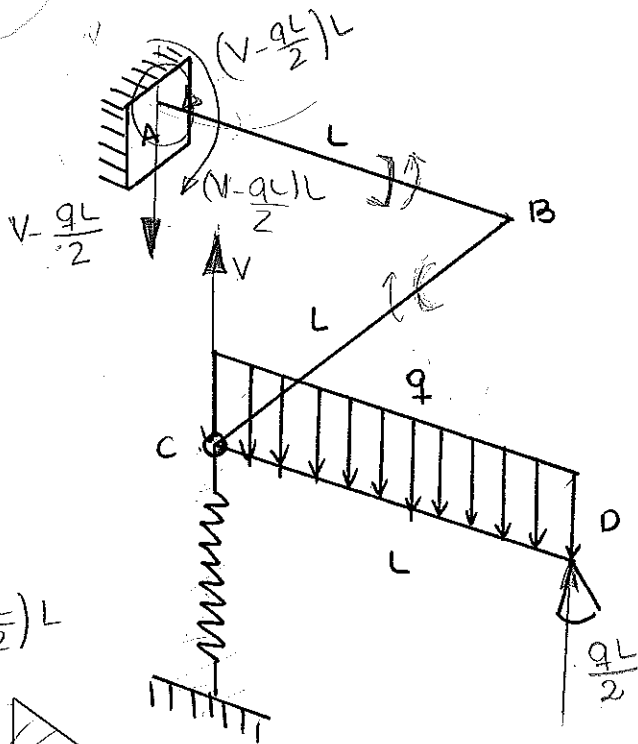
$$n = \frac{\sigma_f}{\sigma_{eq}} = 1,379$$

$$n = 1'379$$

EJE
ANTERIOR



C-3.12



EC. COMPATIBILIDAD

$$V = k \cdot \delta_c$$

⊗ $k = 400 \text{ kN/m}$

⊗ $E = 200 \text{ GPa}$

⊗ $G = 0.4 \cdot E$

⊗ $\sigma_f = 28 \text{ MPa}$

⊗ $L = 1 \text{ m}$

⊗ sec. circular : $d = 8 \text{ cm}$

⊗ $q = 50 \text{ kN/m}$

$$\left(V - \frac{qL}{2}\right)L$$

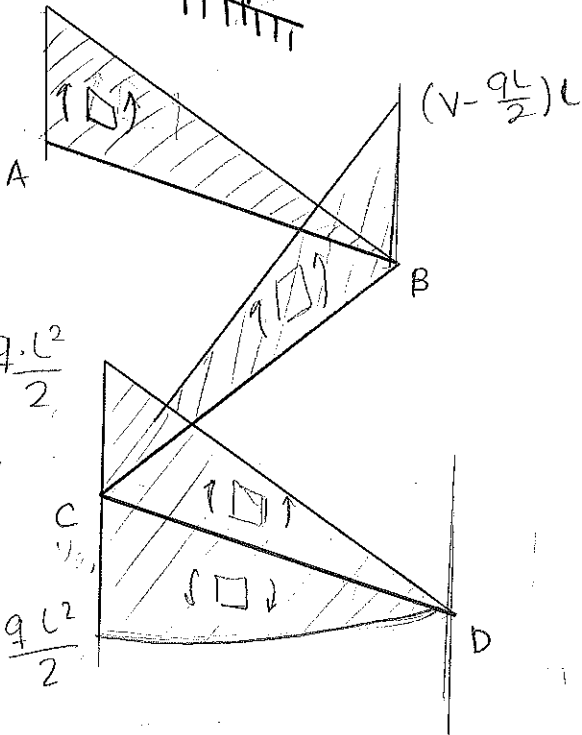


Diagrama de esfuerzos flectores

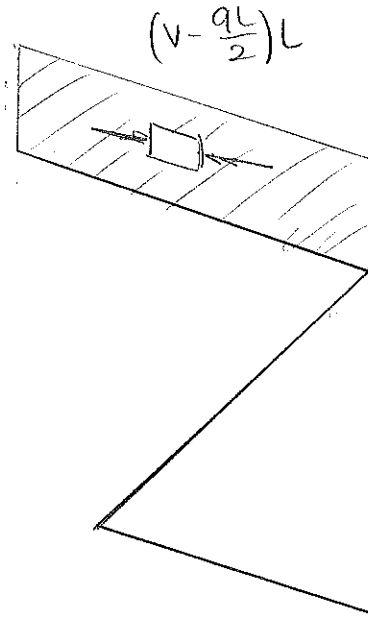


Diagrama de esfuerzos torsores.

$$Y_c = \delta(\text{flexión AB}) + \delta(\text{flexión BC}) + \delta(\text{torsión AB})$$

$$\delta(\text{flexión AB}) = \frac{1}{EI} \left(\frac{1}{2} \left(V - \frac{qL}{2} \right) L \cdot L \cdot \frac{2}{3} L \right) = \frac{L^3}{3EI} \left(V - \frac{qL}{2} \right)$$

$$\delta(\text{flexión BC}) = \frac{1}{EI} \left(\frac{1}{2} \left(V - \frac{qL}{2} \right) L \cdot L \cdot \frac{2}{3} L \right) = \frac{L^3}{3EI} \left(V - \frac{qL}{2} \right)$$

$$\psi_{AB} (\text{torsión}) = \frac{(V - \frac{qL}{2}) \cdot L^2}{2GI} = \frac{5}{4EI} \left(V - \frac{qL}{2} \right) L^2$$

$$\delta_A (\text{torsión AB}) = \frac{5L^3}{4EI} \left(V - \frac{qL}{2} \right)$$

$$\frac{V}{k} = - \frac{2L^3}{3EI} \left(V - \frac{qL}{2} \right) - \frac{5L^3}{4EI} \left(V - \frac{qL}{2} \right)$$

$$V = 16398,71836 \text{ N}$$

Sección más desfavorable: empotramiento A

$$\left. \begin{array}{l} \sigma_{xx} = 171,117 \text{ MPa} \\ \tau_{xz} = 85,559 \text{ MPa} \end{array} \right\} \sigma_{eq} = 226,367 \text{ MPa} \Rightarrow n = 1,237$$