

$$I_x \dot{\omega}_x + \omega_y \omega_z (I_z - I_x) = 0 \quad (1)$$

$$I_y \dot{\omega}_y + \omega_x \omega_z (I_x - I_z) = 0 \quad (2)$$

$$I_z \dot{\omega}_z + \omega_x \omega_y (I_z - I_x) = 0 \quad (3)$$

$$(1) \omega_x + (2) \omega_y + (3) \omega_z$$

$$I_x \omega_x \dot{\omega}_x + I_y \omega_y \dot{\omega}_y + I_z \omega_z \dot{\omega}_z = 0$$

$$\frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2 = \text{cte} \quad (1)$$

$$T = \frac{1}{2} \omega^T [I_0] \omega \Rightarrow T = \frac{1}{2} (\omega_x \ \omega_y \ \omega_z) \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$T = \text{cte}$$

$$\vec{H}_0 = \text{cte} \Rightarrow \vec{H}_0 \vec{H}_0 = \text{cte}$$

$$I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2 = \text{cte} = H_0^2 \quad (2)$$

$$I_z \dot{\omega}_z + \omega_x \omega_y (I_z - I_x) = 0 \quad (3)$$

$$(1) (2) \rightarrow \left. \begin{matrix} \omega_x = f(\omega_z) \\ \omega_y = f(\omega_z) \end{matrix} \right\} \rightarrow (3) \quad f(\omega_z, \dot{\omega}_z) = 0$$

Decoupled.

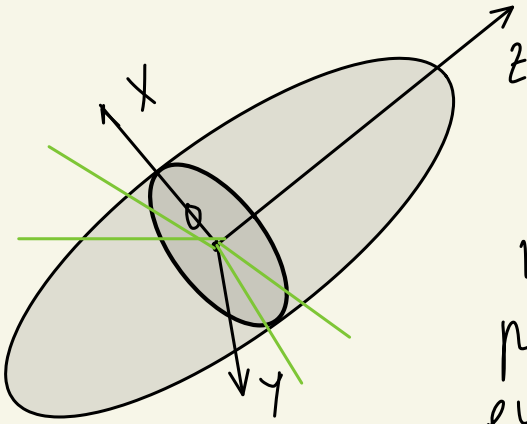
$$\omega_x, \omega_y$$

$$\omega_z$$

Ciróscopo E-P  $\rightarrow \vec{H}_0 = cte, T = cte$

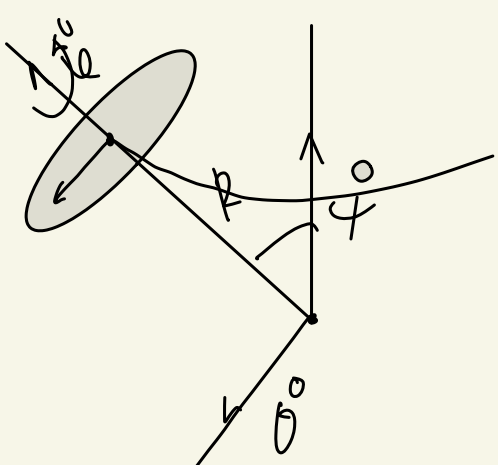
SÓLIDO CON PTO FIJO SIMÉTRICO

Sólido, elipsoide de revolución en el pto fijo O  $\rightarrow$

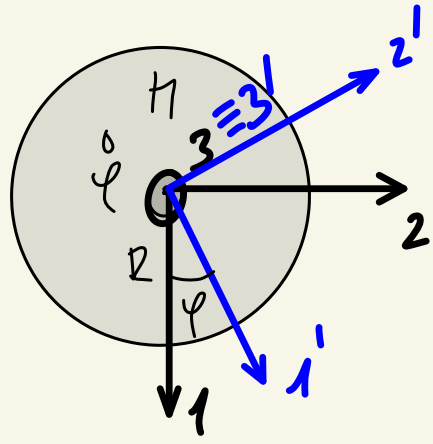


$I_x = I_y \rightarrow$  Circunferencia el  $(I_{x,y})$   
 momento de Inercia es el mismo  
 para cualquier par de ejes ortogonales  
 en esa sección

La exigencia para la elección de los ejes móviles en los que hacer la representación de las magnitudes es que no cambien en el tiempo las marcas.



$$\dot{\varphi}, \dot{\theta}$$



$$\left. \begin{aligned} & \rightarrow I_{x'} = \frac{1}{4} \pi R^2, \quad I_{y'} = \frac{1}{4} \pi R^2 \\ & \rightarrow I_x = \frac{1}{4} \pi R^2, \quad I_y = \frac{1}{4} \pi R^2 \end{aligned} \right\} \begin{array}{l} \text{tenga el triédro o} \\ \text{no tenga } \dot{\varphi} \\ \text{las } I_x \text{ e } I_y \text{ siguen cte} \end{array}$$

$\dot{\varphi}, \dot{\theta} \rightarrow$  no necesitan  $\dot{\varphi}$  aunque el sólido si lo tenga

$$\left. \begin{aligned} \vec{\omega}_S &= \dot{\theta} \vec{e}_z + \dot{\varphi} \vec{e}_z + \dot{\varphi} \vec{e}_z \\ \vec{\omega}_T &= \dot{\varphi} \vec{e}_z + \dot{\theta} \vec{e}_z \end{aligned} \right\}$$

Si el sólido con pto fijo es  
simétrico  $\rightarrow I_x = I_y \neq I_z$

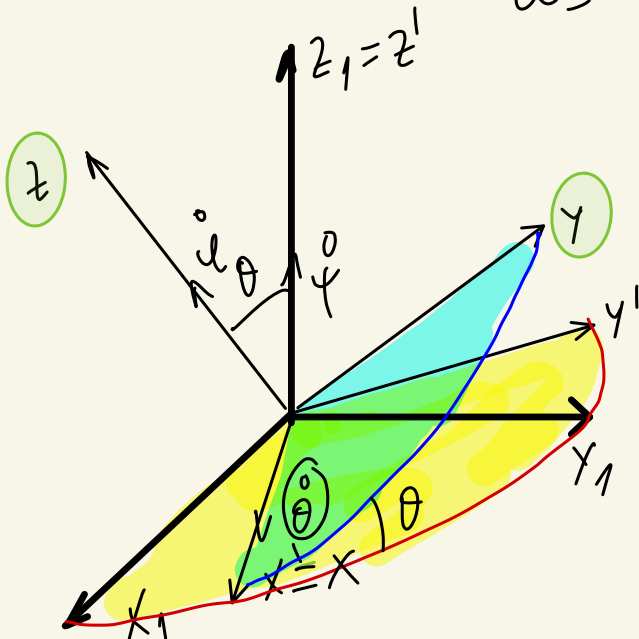
Utilizaremos un triédro xmiinulado

$\varphi \rightarrow$  No está definido, ~~el triédro no~~  
~~tiene  $\dot{\varphi}$~~   $\Rightarrow \varphi = 0$

$$\begin{aligned} \omega_x &= \dot{\varphi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi \\ \omega_y &= \dot{\varphi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi \\ \omega_z &= \dot{\varphi} + \dot{\theta} \cos \theta \end{aligned}$$

$$\left. \begin{aligned} \omega_x &= \dot{\theta} \\ \omega_y &= \dot{\varphi} \sin \theta \\ \omega_z &= \dot{\varphi} + \dot{\theta} \cos \theta \end{aligned} \right\} \vec{\omega}_S$$

$$\left. \begin{aligned} \omega_x &= \dot{\theta} \\ \omega_y &= \dot{\varphi} \sin \theta \\ \omega_z &= \dot{\varphi} \cos \theta \end{aligned} \right\} \vec{\omega}_T$$



# GIROSCOPO DE EULER-POINCARÉ SIMÉTRICO ( $I_x = I_y$ )

$$\frac{d\vec{H}_0}{dt} \Big|_F = I_x \overset{0}{\omega}_x \hat{l}_1 + I_x \overset{0}{\omega}_y \hat{l}_2 + I_z \overset{0}{\omega}_z \hat{l}_3 + \begin{pmatrix} \hat{l}_1 & \hat{l}_2 & \hat{l}_3 \\ \omega_x^T & \omega_y^T & \omega_z^T \\ I_x \omega_x^S & I_x \omega_y^S & I_z \omega_z^S \end{pmatrix}$$

$\omega_x^S = \omega_x^T$   
 $\omega_y^S = \omega_y^T$

$$\frac{d\vec{H}_0}{dt} \Big|_F = A_1 \omega_x \hat{l}_1 + A_1 \omega_y \hat{l}_2 + I_z \omega_z \hat{l}_3$$

$$\frac{d\vec{H}_0}{dt} = 0 \Rightarrow A_1 \omega_x \hat{l}_1 + A_1 \omega_y \hat{l}_2 + I_z \omega_z \hat{l}_3 = 0$$

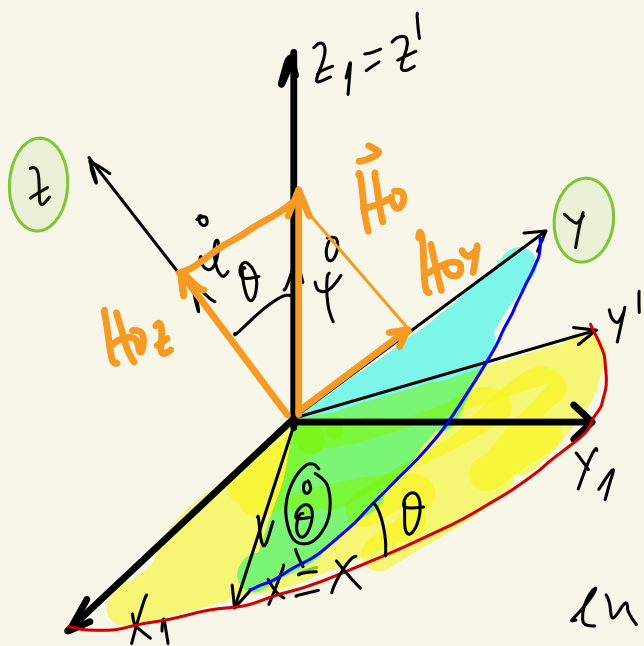
$$I_z \omega_z = 0 \Rightarrow \omega_z = cte$$

$$H_z = I_z \omega_z \Rightarrow H_z = cte$$

$\vec{H}_0 = cte$  en el espacio  
dirección fija

$\vec{H}_0$  tiene la dirección del eje  
cte fijo  $z_1$

$$H_z = H_0 \cos \theta \Rightarrow \theta = cte$$



en un G-E-P simétrico el ángulo  
de nutación es cte  $\Rightarrow \dot{\theta} = 0$

$$H_{Oy} = \cancel{H_O \sin \theta} \overset{cte}{\Rightarrow} H_{Oy} = cte \Rightarrow H_{Oy} = I_y \omega_y$$

$$\cancel{H_{Oy} = I_y \dot{\psi} \sin \theta} \overset{cte}{\Rightarrow} \dot{\psi} = cte$$

b E-P simétrico la velocidad de precesión es cte

$$\omega_z^s = cte \Rightarrow \omega_z^s = \dot{\psi} + \dot{\theta} \cos \theta \overset{cte}{\Rightarrow} \dot{\psi} = cte$$

b E-P asimétrico la velocidad de rotación propia es cte

Condiciones

b E-P

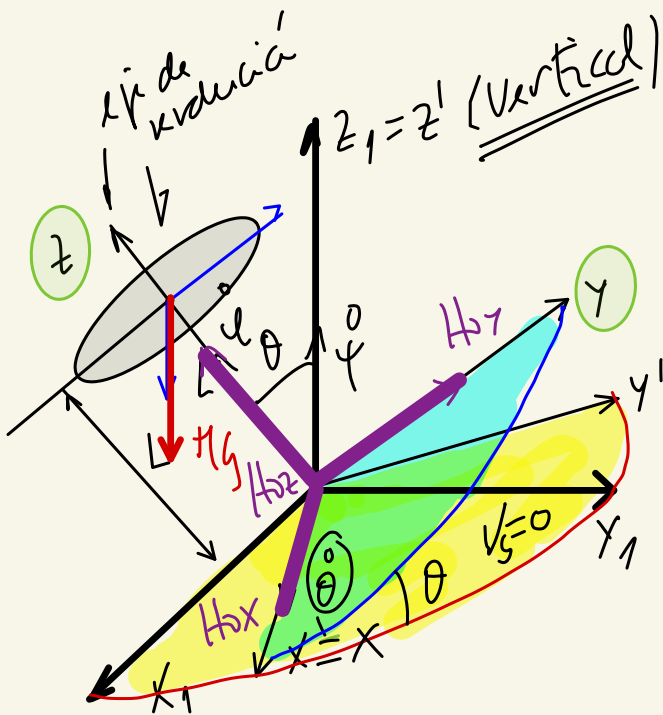
$$\vec{H}_O = cte, T = cte$$

b E-P asimétrico

$$\theta = cte, \dot{\theta} = 0, \dot{\psi} = cte, \dot{\phi} = cte$$

## 2- GIROSCOPO DE EULER LAGRANGE (simétrico)

Sólido con pto fijo O sometido solo a un momento peso y se supone que el CG no coincide con O.  $b \neq 0$



Ejes principales en O  
 $L = T - V \rightarrow E = cte$   $\left\{ \begin{array}{l} \text{Holónomo} \\ \text{Escalónomas} \\ \text{Perfectos, } \text{Fopd} \rightarrow \text{Per} \end{array} \right.$   
 $\psi \Rightarrow$  u' d'ica.

$$\frac{\partial L}{\partial \dot{\psi}} = (P_{\psi}) = I_x \dot{\psi} \sin^2 \theta + I_z (\dot{\psi} + \dot{\psi} \cos \theta) \cos \theta = A$$

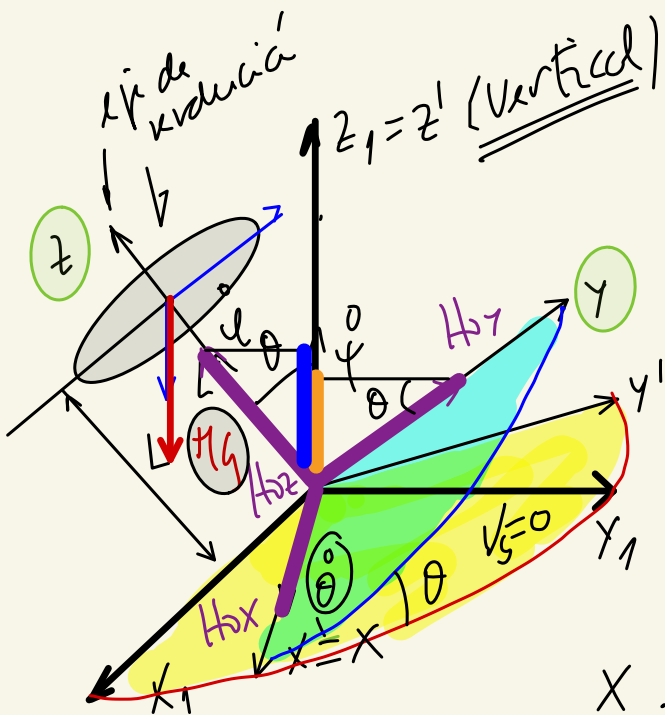
$\psi \Rightarrow$  u' d'ica.

$$\frac{\partial L}{\partial \dot{\varphi}} = (P_{\varphi}) = I_z (\dot{\varphi} + \dot{\psi} \cos \theta) = B$$

$\omega_z$

$$L = \frac{1}{2} I_x \dot{\theta}^2 + \frac{1}{2} I_x \dot{\psi}^2 \sin^2 \theta + \frac{1}{2} I_z (\dot{\psi} + \dot{\psi} \cos \theta)^2 - m g L \cos \theta$$

$B = I_z \omega_z = H_{bz} \Rightarrow B = H_{bz} \rightarrow$  La componente de  $\vec{H}_0$  según el eje  $z$  minimizado es cte



$$I_x \dot{\psi} \sin^2 \theta + I_z (\dot{\psi} + \dot{\psi} \cos \theta) \cos \theta = A$$

$$A = \underbrace{I_x \dot{\psi} \sin^2 \theta}_{H_{b\gamma}} + \underbrace{I_z (\dot{\psi} + \dot{\psi} \cos \theta) \cos \theta}_{H_{bz}}$$

$$A = H_{b\gamma} \sin \theta + H_{bz} \cos \theta = H_{bz1}$$

$H_{bx}$  no proyecta sobre el eje  $z_1$   
 $x$  está en el plano y  $z_1$  es vertical

$$|h_{L1}| = A = cte$$

la componente de  $H_0$  sobre el eje vertical es  $cte$

$$\left. \begin{aligned} A &= I_x \dot{\varphi} \sec^2 \theta + I_z (\dot{\varphi} + \dot{\varphi} \cos \theta) \cos \theta \\ B &= I_z (\dot{\varphi} + \dot{\varphi} \cos \theta) \end{aligned} \right\} \begin{aligned} A - B \cos \theta &= I_x \dot{\varphi} \sec^2 \theta \\ \dot{\varphi} &= \frac{A - B \cos \theta}{I_x \sec^2 \theta} \end{aligned}$$

si suponemos  $\theta = \theta(t) \rightarrow$  integrar  $\rightarrow \varphi = \varphi(t)$

$$\frac{B}{I_z} - \dot{\varphi} \cos \theta = \dot{\varphi} \Rightarrow \frac{B}{I_z} - \frac{A - B \cos \theta}{I_x \sec^2 \theta} \cos \theta = \dot{\varphi}$$

si suponemos  $\theta = \theta(t) \rightarrow$  integrar  $\rightarrow \varphi = \varphi(t)$  ✓

Energía es  $cte$

$$E = \frac{1}{2} I_x \dot{\theta}^2 + \frac{1}{2} I_x \dot{\varphi}^2 \sec^2 \theta + \frac{1}{2} I_z (\dot{\varphi} + \dot{\varphi} \cos \theta)^2 + m g L \cos \theta$$

$$E = \frac{1}{2} I_x \dot{\theta}^2 + \frac{1}{2} I_x \left( \frac{A - B \cos \theta}{I_x \sec^2 \theta} \right)^2 \sec^2 \theta + \frac{1}{2} I_z \frac{B^2}{I_z^2} + m g L \cos \theta$$

$$E = \frac{1}{2} I_x \dot{\theta}^2 + \frac{1}{2 I_x} \left( \frac{A - B \cos \theta}{\sec \theta} \right)^2 + \frac{B^2}{2 I_z} + m g L \cos \theta = cte$$

↑ si suponemos  $\theta = \theta(t)$

1- Equações dependentes b E-L número

$$I_x \dot{\psi} \sin^2 \theta + I_z (\dot{\psi} + \dot{\phi} \cos \theta) \cos \theta = A$$

$$I_z (\dot{\psi} + \dot{\phi} \cos \theta) = B$$

$$\rightarrow E = \frac{1}{2} I_x \dot{\theta}^2 + \frac{1}{2 I_x} \left( \frac{A - B \cos \theta}{\sin \theta} \right)^2 + \frac{B^2}{2 I_z} + \text{trg} L \cos \theta$$

A, B, E

se xalun.  
cu C.L.

2- Velocidades angulares

$$\dot{\psi} = \frac{A - B \cos \theta}{I_x \sin^2 \theta}, \quad \dot{\phi} = \frac{B}{I_z} - \frac{A - B \cos \theta}{I_x \sin^2 \theta} \cos \theta$$

para um ângulo  $\theta$  dado

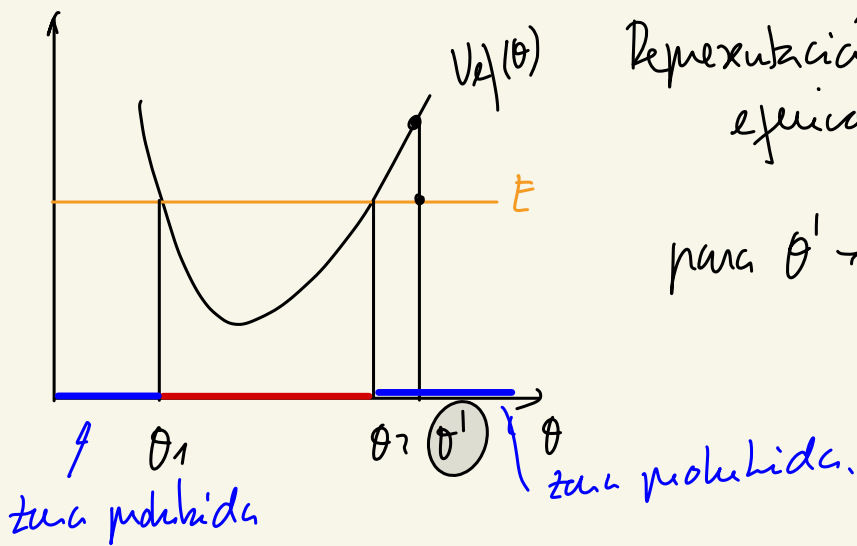
3-  $\dot{\theta} \sim$  para um  $\theta$  dado.

$$E = \underbrace{\frac{1}{2} I_x \dot{\theta}^2}_{T \text{ de rotação}} + \underbrace{\frac{1}{2 I_x} \left( \frac{A - B \cos \theta}{\sin \theta} \right)^2 + \frac{B^2}{2 I_z}}_{V_c \text{ (Potencial Centrifugo)}} + \underbrace{\text{trg} L \cos \theta}_{V_r \text{ potencial rot.}}$$

$V_{ef}(\theta)$

$$E = \frac{1}{2} I_x \dot{\theta}^2 + V_{ef}(\theta) \Rightarrow \frac{1}{2} I_x \dot{\theta}^2 = E - V_{ef}(\theta) \geq 0$$



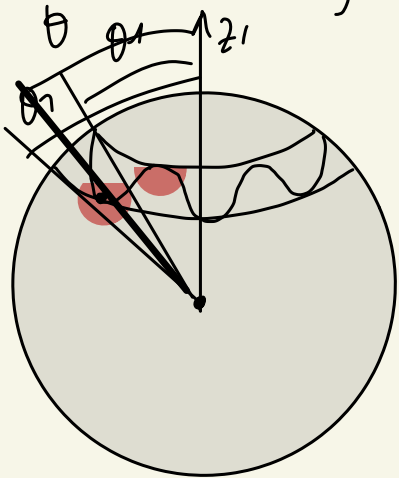


Representación unidimensional  
eficiente

para  $\theta' \rightarrow V_{ell}(\theta) > E$

$\theta_1 \leq \theta \leq \theta_2$   $\theta_1, \theta_2 \rightarrow$  Ángulos de viraje o de retroceso

$$E = V_{ell}(\theta) \Rightarrow \dot{\theta} = 0$$



$$\frac{1}{2} I_x \dot{\theta}^2 = E - V_{ell}(\theta)$$

$$\dot{\theta}^2 = \frac{2}{I_x} (E - V_{ell}(\theta))$$

$$\frac{d\theta}{dt} = \pm \sqrt{\frac{2}{I_x} (E - V_{ell}(\theta))}$$

$$\frac{d\theta}{\pm \sqrt{\frac{2}{I_x} (E - V_{ell}(\theta))}} = dt \quad \text{Integrando}$$

$$\theta = \theta(t)$$

PAR Girusrópico -  $\frac{d\vec{H}_b}{dt} = \vec{T}_{pero} \Rightarrow \vec{T}_b = -\frac{d\vec{H}_b}{dt}$

$$\vec{T}_b + \vec{T}_{pero} = 0.$$