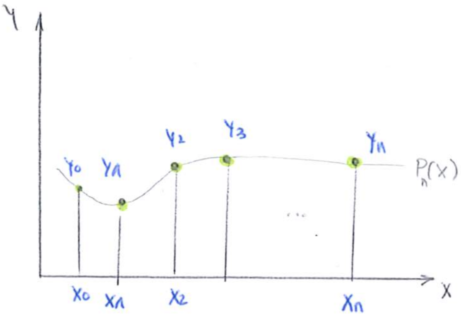
27/01/21

# CH. 1. INTERPOLATION

## LAGRANGE METHOD




* Find $P_n(x) \in \mathbb{P}_n$ / $P_n(x_i) = y_i$ $(i = 0, \dots, n)$

$$\boxed{P_n(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n}$$ $((x_i \neq x_j))$ (n+1 DEGREES OF FREEDOM)

(LAGRANGE POLYNOMIAL)

$$\begin{cases} x_i \equiv \text{nods} \\ (x_i, y_i) \equiv \text{interp. points / data} \end{cases}$$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ | & | & | & \ddots & | \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ | \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ | \\ y_n \end{pmatrix}$$

(ALSO USED TO PROVE UNIQUENESS)

└→ NEVER SINGULAR (ALWAY ≠ 0)

### LAGRANGE INTERPOLATION

- Non-oszillating
- Non-piecewise

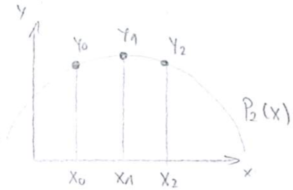
## EXISTENCE AND UNIQUENESS THEOREM (∃!)

* $\exists!$ $P_n(x) \in \mathbb{P}_n$ / $P_n(x_i) = y_i$ $(i = 0, 1, \dots, n)$

PROOF:

a) EXISTENCE

n=2

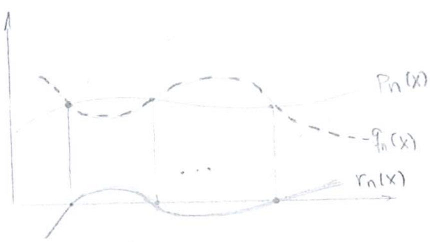



$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2 \in \mathbb{P}_2$$

$$\boxed{P_n(x) = \sum_{i=0}^{n} \Bigg( \underbrace{\prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}}_{L_i(x)} \Bigg) \cdot y_i}$$ (THE LAGRANGE REPRESENTATION OF A POLYNOMIAL)

PRODUCT └→ $L_i(x)$ (LAGRANGE BASED FUNCTION)

b) UNIQUENESS : Reduction to absurdity




* Assume $P_n(x)$, $q_n(x)$ 2 different solutions

$r_n(x) = P_n(x) - q_n(x) \in \mathbb{P}_n$

with n+1 roots against the fundamental th. of algebra

unless: $r_n(x) = 0 \longrightarrow q_n(x) = P_n(x)$

• EXAMPLE 2 $P_3(x)$ by $(1,6),(2,4),(3,2),(4,6)$ (GAUSS VS. LAGRANGE)

a) via linear system ("GAUSS") (of eqs.)

$* \; P_3(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

$$\begin{pmatrix} 1 & 1^1 & 1^2 & 1^3 \\ 1 & 2^1 & 2^2 & 2^3 \\ 1 & 3^1 & 3^2 & 3^3 \\ 1 & 4^1 & 4^2 & 4^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

$\rightarrow$ AUGMENTED MATRIX

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 4 \\ 1 & 3 & 9 & 27 & 2 \\ 1 & 4 & 16 & 64 & 6 \end{array}\right) \begin{array}{c} \langle R_2 - R_1 \rangle \\ \langle R_3 - R_1 \rangle \\ \langle R_4 - R_1 \rangle \end{array} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & \boxed{1} & 3 & 7 & -2 \\ 0 & 2 & 8 & 26 & -4 \\ 0 & 3 & 15 & 63 & 0 \end{array}\right) \begin{array}{c} \langle R_3 - 2R_2 \rangle \\ \langle R_4 - 3R_2 \rangle \end{array} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 7 & -2 \\ 0 & 0 & 2 & 12 & 0 \\ 0 & 0 & 6 & 42 & 6 \end{array}\right) \langle R_4 - 3R_3 \rangle$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 7 & -2 \\ 0 & 0 & 2 & 12 & 0 \\ 0 & 0 & 0 & 6 & 6 \end{array}\right) \qquad \begin{cases} c_0 + c_1 + c_2 + c_3 = 6 \longrightarrow \boxed{c_0 = 2} \\ c_1 + 3c_2 + 7c_3 = -2 \longrightarrow \boxed{c_1 = 9} \\ 2c_2 + 12c_3 = 0 \longrightarrow \boxed{c_2 = -6} \\ 6c_3 = 6 \longrightarrow \boxed{c_3 = 1} \end{cases}$$

$$\boxed{P_3(x) = 2 + 9x - 6x^2 + x^3}$$

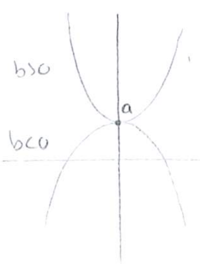
b) Via lagrange REPRESENTATION

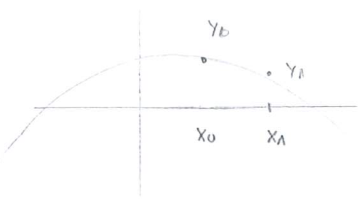
$$* \; P_3(x) = \sum_{i=0}^{3} \left( \prod_{\substack{j=0 \\ i \neq j}}^{3} \frac{(x - x_j)}{(x_i - x_j)} \right) y_i$$

$$P_3(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} \cdot 6 + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \cdot 4 + \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} \cdot 2 + \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \cdot 6$$

• EXERCISE 1

Study the existance and uniqueness of $\underbrace{p(x) = a + bx^2}_{\text{PARABOLA}}$ by $(x_0, y_0), (x_1, y_1)$

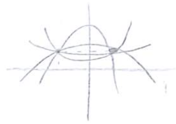



$$\begin{pmatrix} 1 & x_0^2 \\ 1 & x_1^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

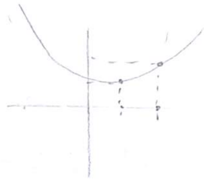
$$\left(\begin{array}{cc|c} 1 & x_0^2 & y_0 \\ 1 & x_1^2 & y_1 \end{array}\right) \langle R_2 - R_1 \rangle = \left(\begin{array}{cc|c} 1 & x_0^2 & y_0 \\ 0 & x_1^2 - y_0^2 & y_1 - y_0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & x_0^2 & y_0 \\ 0 & x_1^2 - x_0^2 & y_1 - y_0 \end{array}\right)$$



- CASE 1 : $x_1^2 - x_0^2 = 0$
  - $x_1 = x_0$
    - $y_0 = y_1 \longrightarrow \infty$ many solutions (compatible and determined system)
    - $y_0 \neq y_1 \longrightarrow$ NO solution
  - $x_1 = -x_0$

- CASE 2 : $x_1^2 - x_0^2 \neq 0$ — system with ONLY ONE solution
  ($x_1 \neq \pm x_0$) (compatible system)

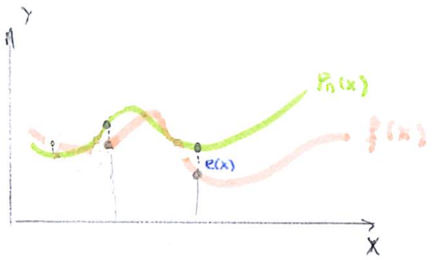


## ERRORS

General def: $\boxed{\text{error} = \text{exact} - \text{approx.}}$

- appr. > exact → error < 0 → ERROR IN EXCESS
- appr. < exact → error > 0 → ERROR IN DEFECT (POR DEFECTO)

TRUNCATION ERRORS : errors with exact arithmetic




* $e(x) = P_n(x) - f(x)$

* eventhough the arithmetic is exact we have an error, it will be because of the error of the method used.

ROUNDING ERRORS  Ex: 3,1416 ...

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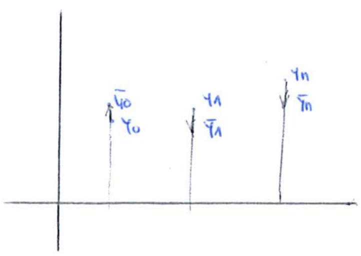
## ERROR PROPAGATION: AMPLIFICATION FACTOR (AF) AND INESTABILITY

Instead of $y_i$ we have $\bar{y}_i$ (perturbed balues) "all else exact".




$$\boxed{\varepsilon \geq |y_i - \bar{y}_i| \quad \forall i}$$

upper bound

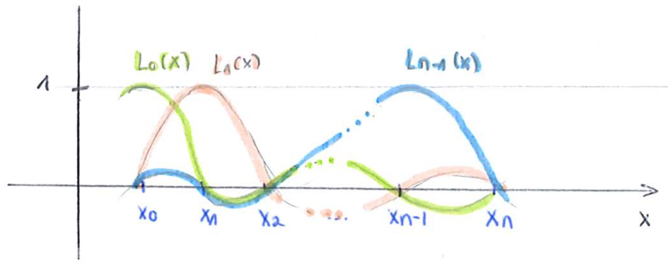
$$\begin{cases} p(x) = \sum_{i=0}^{n} L_i(x) \cdot y_i \\ \bar{p}(x) = \sum_{i=0}^{n} L_i(x) \cdot \bar{y}_i \end{cases}$$

$$\boxed{e_d(x) = p(x) - \bar{p}(x)} \longrightarrow e_d(x) = \sum_{i=0}^{n} L_i(x)(y_i - \bar{y}_i)$$

* applying absolute values :

$$\boxed{|e_d(x)| = \left|\sum_{i=0}^{n} L_i(x)\,(y_i - \bar{y}_i)\right| \le \sum_{i=0}^{n} |L_i(x)|\,|y_i - \bar{y}_i| \le \varepsilon \cdot \underbrace{\sum_{i=0}^{n} |L_i(x)|}_{A.F.}}$$

* $\boxed{AF = \sum_{i=0}^{n} |L_i(x)|} = \Sigma^+ L_i(x) + \Sigma^- |L_i(x)| = \boxed{\Sigma^+ L_i(x) - \Sigma^- L_i(x)}$ ( Amplification Factor )




$$P_n(x) = \sum_{i=0}^{n} L_i(x) \cdot 1 \equiv 1 \quad (f(x) \equiv 1)$$

$$\hookrightarrow \boxed{\Sigma^+ L_i(x) + \Sigma^- L_i(x) \equiv 1}$$

$$L_0(x) = \frac{(x - x_1) \cdots (x - x_n)}{(x_0 - x_1) \cdots (x_0 - x_n)} \longrightarrow L_i(x) = \delta_{ij} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

→ KRÖNELKER'S DELTA

so now we substitute : $\Sigma^+ L_i(x) + \Sigma^- L_i(x) = 1$ in our A.F

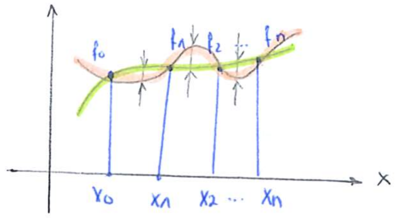
$$\boxed{AF = \Sigma^+ L_i(x) - \Sigma^- L_i(x) = (1 - \Sigma^- L_i(x)) - \Sigma^- L_i(x) = 1 - 2\Sigma^- L_i(x) = \boxed{1 + 2\Sigma^- |L_i(x)|}}$$

↳ >1 always

when $AF \gg 1 \rightarrow$ inestability

# FUNDAMENTAL THEOREM OF LAGRANGE TRUNCATION INTERPOLATION ERROR

↳ ABOUT LAGRANGE INTERPOLATION NOT REPRESENTATION




$\boxed{e(x) = f(x) - P_n(x)}$ Trunc. error.

## THEOREM:

$$\left.\begin{array}{l} x_i, x \in [a,b] \\ f \in C^{n+1}([a,b]) \end{array}\right\} \Longrightarrow \exists \xi \in [a,b] \;/\; \boxed{e(x) = \frac{f^{n+1)}(\xi)}{(n+1)!}\Pi(x)}$$

GREEK LETTER

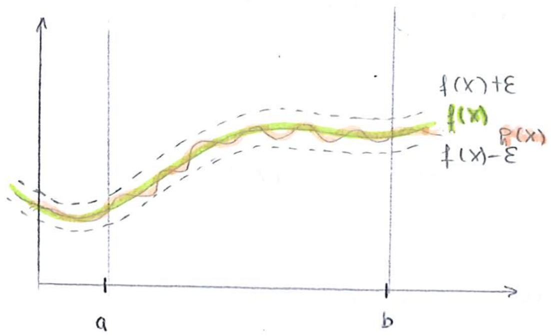
smoothness   where

continuous and smooth

$$\left\{\begin{array}{l} * \; e(x_i) = 0 \quad \forall \; (bc \;\; \Pi(x_i) = 0) \\ * \; f \in \mathbb{P}_n \Rightarrow e(x) \equiv 0 \end{array}\right.$$

$* \;\; \Pi(x) = (x-x_0)(x-x_1)\dots(x-x_n) = \prod_{i=0}^{n}(x-x_i)$

## WEIERSTRASS' UNIFORM APPROXIMATION THEOREM




UPPERBOUND (goiko muga)

$* \;\; \boxed{M_{n+1} = \max\left|f^{n+1)}(x)\right|} \quad x \in [a,b]$

$$|e(x)| = \left|\frac{f^{n+1)}(\xi)}{(n+1)!}\Pi(x)\right| \leq \frac{M_{n+1}}{(n+1)!}|\Pi(x)| \leq \frac{M_{n+1}}{(n+1)!}(b-a)^{n+1}$$

whe $n \uparrow$ $f^{n+1)}$ can increase to ∞ faster than $(n+1)!$ (creo) → no por meter mas va a ser mejor

**THEOREM:** $f \in C^0([a,b]) \Rightarrow \forall \;\; \varepsilon > 0 \;\; \exists \; p \in \mathbb{P} \;/\; |f(x) - p(x)| \leq \varepsilon \quad \forall x \in [a,b]$

uniform interpolation is possible thanks to Weierstrass

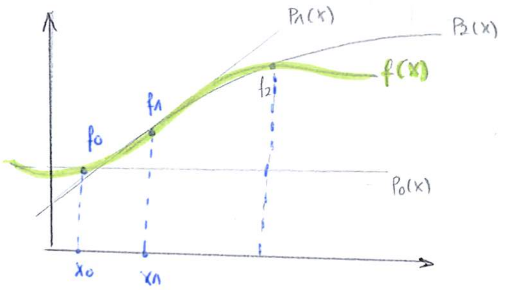
## NON - PERMANENCE OF THE NODES

$$P_n(x) = \frac{(x-x_1)(x-x_2)\dots}{(x_0-x_1)(\dots)(x_0-x_n)}\cdot f_0 + \frac{(\ )(\ )\dots}{(\ )(\ )\dots}\cdot f_1 + \dots + \frac{(\dots)(\ )}{(\dots)(\dots)}\cdot f_n$$

If we add now a node $n+1$, all the terms will change

Is a disadvantage of Lagrange when we add a node, because we have to rewrite it all over again.

## THE NEWTON REPRESENTATION OF LAGRANGE POLYNOMIAL




$P_0(x) = f_0 = f(x_0) \doteq a_0$

$P_1(x) = P_0(x) + h_1(x) = P_0(x) + a_1(x-x_0)$

...

$P_k(x) = P_{k-1}(x) + h_k(x) = P_{k-1}(x) + a_k(x-x_0)(\dots)(x-x_{k-1})$

...

$P_n(x) = P_{n-1}(x) + h_n(x) = a_0 + a_1(x-x_0)\dots + a_n(x-x_0)(x-x_1)(\dots)(x-x_{n-1})$

**DEFINITION:** $h_k(x) = \overbrace{\underbrace{P_k(x)}_{\in \mathbb{P}_k} - \underbrace{P_{k-1}(x)}_{\in \mathbb{P}_k}}^{\neq 0 \text{ in general}} \in \mathbb{P}_k$ $\qquad h_k(x_i) = 0 \quad (i = 0, \dots, k-1)$

* Whatever has to be added is a polynomial of degree $k-1$ or less.

$h_k(x) \in \mathbb{P}_k \longrightarrow h_k(x) = \underbrace{a_k}_{\text{LEADING COEFF}}(x-x_0)(x-x_1)\dots(x-x_{k-1})$

* Leading coef. of a polynomial = value of the coeff. that goes with the highest order.

$$\boxed{P_n(x) = \sum_{i=0}^{n} a_i \prod_{j=0}^{i-1}(x-x_j)}$$

; $a_k$ = Divided difference = Leading coeff of $P_k(x)$

## DIVIDED DIFFERENCES

03/02/21

Notations:

- $a_k = f[x_0, x_1, \dots, x_k] = f_{k,k}$ (Depends on $x_0, x_1, \dots, x_k$ ^ $f_0, f_1, \dots, f_k$)
- $f_{i,k} = f[x_{i-k}, x_{i-k+1}, \dots, x_k]$
  - $f_{i,0} = f[x_i] = f(x_i)$

## PROPERTIES

* Permutation or permanence of the div. coeff.

ORDER (k)

* $$\boxed{f_{i,k} = \frac{f_{i,k-1} - f_{i-1,k-1}}{x_i - x_{i-k}}}$$

we use the formula to fill the table

## TABLE OF DIVIDED DIFFERENCE (TDD)

| $x_i$ | $f(x_i) = f_{i,0}$ | $f_{i,1}$ | $f_{i,2}$ | ... | $f_{i,n}$ |
|---|---|---|---|---|---|
| $x_0$ | $f_{0,0} = \boxed{a_0}$ | — | — | ... | — |
| $x_1$ | $f_{1,0}$ | $f_{1,1} = \boxed{a_1}$ | — | ... | — |
| $x_2$ | $f_{2,0}$ | $f_{2,1} = x$ | $f_{2,2} = \boxed{a_2}$ | ... | — |
| $x_3$ | $f_{3,0}$ | $f_{3,1} = x$ | $f_{3,2} = x$ | ... | — |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{n-1}$ | $f_{n-1,0}$ | $f_{n-1,1} = x$ | $f_{n-1,2} = x$ | ... | — |
| $x_n$ | $f_{n,0}$ | $f_{n,1} = x$ | $f_{n,2} = x$ | ... | $f_{n,n} = \boxed{a_n}$ |

$a_0, a_1, \dots$ PRINCIPAL DIVIDED DIFFERENCES

### EXCERCISE 6

a) TDD of $f(x) = x^3$ on $x_i = \{0, 1, 3, 4\}$ → NEWTON FOR. $f_{i,k} = \frac{f_{i,k-1} - f_{i-1,k-1}}{x_i - x_{i-k}}$

| $x_i$ | $f_{i,0}$ | $f_{i,1}$ | $f_{i,2}$ | $f_{i,3}$ |
|---|---|---|---|---|
| 0 | $\boxed{0}$ | — | — | — |
| 1 | 1 | $\boxed{1}$ | — | — |
| 3 | 27 | 13 | $\boxed{4}$ | — |
| 4 | 64 | 37 | 8 | $\boxed{1}$ |

FOR LAGR.

$$f_{1,1} = \frac{f_{1,0} - f_{0,0}}{x_1 - x_0} = \frac{1-0}{1-0} = 1$$

$$f_{2,1} = \frac{f_{2,0} - f_{1,0}}{x_2 - x_1} = \frac{27-1}{2} = 13$$

$$f_{3,1} = \frac{f_{3,0} - f_{2,0}}{x_3 - x_2} = \frac{64-27}{1} = 37$$

$$f_{3,2} = \frac{f_{3,1} - f_{2,1}}{x_3 - x_1} = \frac{37-13}{3} = 8$$

$$\boxed{P_3(x) = 0 + 1(x-0) + 4(x-0)(x-1) + 1(x-0)(x-1)(x-3) = \dots = x^3}$$

b) Lagrange form of the pol.

$$P_3(x) = \frac{\qquad}{\qquad} \cdot 0 + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} \cdot 1 + \frac{(\quad)(\quad)}{(\quad)(\quad)} \cdot 27 + \frac{(\quad)(\quad)}{(\quad)(\quad)} \cdot 64 = x^3$$

ALGORITHM AND CODE

| X | F | F | F |
|---|---|---|---|
| $x_i$ | $f_{i,0}$ | $f_{i,1}$ | $f_{i,2}$ |
| | × | × | × |
| | × | □ | □ |
| | × | □ | Δ |
| | ⋮ | ⋮ | ⋮ |
| | × | □ | Δ |

we start substituting from the bottom

$$F = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} \rightarrow F = \begin{pmatrix} f_{0,0} = a_0 \\ f_{1,1} = a_1 \\ \vdots \\ \square \end{pmatrix} \rightarrow F = \begin{pmatrix} \quad \end{pmatrix}$$

(OVERWRITE OF VALUES FOR F)

```
X = [x0, x1, ..., xn]';
F = f(X) ;
```

PROGRAMMING THE CALC. OF F

```
X = [ x0, x1, x2, ..., xn ]';

F = [ f(x0), f(x1), f(x2), ..., f(xn)]';

nn = numel (X) ;  %(number of nodes)

n = nn - 1 ;  %(the "n" of the "theory")

for k = 1:n
    for row = nn : -1 : (k+1)      % 1ST VALUE : STEPSIDE : UNTIL
        F(row) = (F(row) - F(row-1))/(X(row) - X(row-k));
    end
end
```

## HORNER'S ALGORITHM TO EVALUATE A POLYNOMIAL

(FLOP → floating point operation)
↳ ⊕ or ⊙

$$P_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

↳ "DIRECTLY" → FLOPS: 14

$$P_4(x) = c_0 + x\left\{c_1 + x\left[c_2 + x\left(c_3 + x c_4\right)\right]\right\}$$

(nested product) → FLOPS: 8

$$P_4(x) = \underbrace{c_0 + \underbrace{x\{c_1 + \underbrace{x[c_2 + \underbrace{x(c_3 + \underbrace{x c_4}_{b_4})]}_{b_3}}_{b_2}\}}_{b_1}}_{b_0}$$

→ RUFFINI'S TABLE

| | $c_4$ | $c_3$ | $c_2$ | $c_1$ | $c_0$ |
|---|---|---|---|---|---|
| $x$ | | $xb_4$ | $xb_3$ | $xb_2$ | $xb_1$ |
| | $b_4$ | $b_3$ | $b_2$ | $b_1$ | $b_0 = P_4(x)$ |

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## HORNER-LIKE ALGORITHM (NESTED PRODUCTS) (OPTIMAL)

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_{n-1}(x-x_0)(x-x_1)(\cdots)(x-x_{n-2}) + a_n(x-x_0)(\cdots)(x-x_{n-2})(x-x_{n-1})$$

$$P_n(x) = a_0 + (x-x_0)\left\{a_1 + a_2(x-x_1) + \dots + a_{n-1}(x-x_1)(\cdots)(x-x_{n-2}) + a_n(x-x_1)(\cdots)(x-x_{n-2})(x-x_{n-1})\right\}$$

$$P_n(x) = \underbrace{a_0 + (x-x_0)\left\{\underbrace{a_1 + (x-x_1)\left[\underbrace{a_2 + (x-x_2)\left(\cdots\left(\underbrace{a_{n-2} + (x-x_{n-2})\left(\underbrace{a_{n-1} + (x-x_{n-1})\underbrace{a_n}_{b_n}}_{b_{n-1}}\right)}_{b_{n-2}}\right)\right)}_{b_2}\right]}_{b_1}\right\}}_{b_0}$$

...

***PROGRAMMING***

* $i \neq n$ → because $n$ goes to 0 and in OCTAVE we can't
↳ si queremos $n = nn-1$ ; for $i = n-1:-1:0$ ... then $(i+1)$
(MORE LITERAL)

```
% X = nodes , F = principal div diffs
bi = F(end);
nn = length or numel (X) ; % no of nodes
for  i = (nn-1):-1:1
  bi = bi * (x - X(i)) + F(i);
end
Pnofx = bi;
```

EVALUATING POINT

# ROUNDING OF NUMBERS

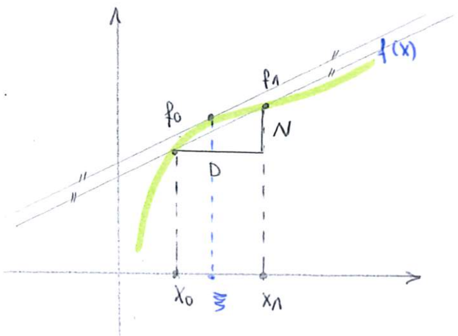
## SIGNIFICANT DIGITS VS. DECIMALS + SYMMETRIC AND TRUNCATION ROUNDING

(Ex) $\pi \to 3'14159 2654 \ldots$

- 5 SIGNIF. DIGITS
  - SYMMETRIC ROUNDING $\to \overbrace{3'1416}^{\text{5 DIGITS}}$
  - TRUNCATION ROUNDING $\to$ 3'1415
- 4 DECIMALS
  - SYMMETRIC ROUNDING $\to 3'\overbrace{1416}^{\text{4 DECIMALS}}$
  - TRUNCATION ROUNDING $\to$ 3'1415

* If we are said to immitate a computer $\to$ we round in every step

## RELATION BETWEEN DIVIDED DIFFERENCES (DD) AND DERIVATIVES




$\begin{cases} N = \text{Numerator} \\ D = \text{Denominator} \end{cases}$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{N}{D} = \text{slope}$$

$$* \; f \in C^1([x_0, x_1]) \Rightarrow \exists \; \xi \in [x_0, x_1] \; / \; f'(\xi) = \frac{f_1 - f_0}{x_1 - x_0}$$

$$* \; f \in C^{1}([x_0, x_1]) \Rightarrow \exists \; \xi \in [x_0, x_1] \; / \; f[x_0, x_1] = \frac{f^{1)}(\xi)}{1!}$$

THEOREM:

$$\left.\begin{array}{l} f \in C^n([a,b]) \\ x_0, x_1, \ldots, x_n \in [a,b] \end{array}\right\} \Rightarrow \exists \; \xi \in [a,b] \; / \; f[x_0, x_1, \ldots, x_n] = \frac{f^{n)}(\xi)}{n!}$$

## TABLES OF DIV. DIFF WITH A CONSTANT (ZERO) COLUMN

$f \in \mathbb{P}_k$

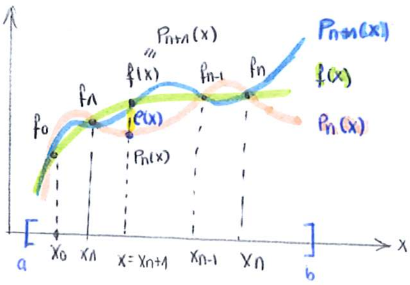
| $x_i$ | $f_{i,0}$ | $f_{i,1}$ | ... | $f_{i,k}$ | $f_{i,k+1}$ |
|---|---|---|---|---|---|
| $x_0$ | $f_{0,0}$ | — | ... | — | — |
| $x_1$ | $f_{1,0}$ | $f_{1,1}$ | ... | — | — |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{n-1}$ | $f_{n-1,0}$ | $f_{n-1,1}$ | ... | C | 0 |
| $x_n$ | $f_{n,0}$ | $f_{n,1}$ | ... | C | 0 |

$f_{i,k}$: ↓ KTE; $f_{i,k+1}$: → ∅

$f \in \mathbb{P}_k \underset{\text{SUFFICIENT CONDITION}}{\overset{\not\Leftarrow}{\Longrightarrow}}$ col. $f_{i,k} \equiv$ kte

## TRUNCATION ERROR BY LAGRANGE DIFF. EXPRESSED WITH D.D.

$$\boxed{e(x) = f[x_0, x_1, \dots, x_n, x]\, \Pi(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\, \Pi(x)}$$




PROOF:

$*$ $e(x) = f(x) - P_n(x) = P_{n+1}(x) - P_n(x) = h_{n+1}(x) =$

$= a_{n+1}(x-x_0)(x-x_1)\cdots(x-x_n) = a_{n+1}\, \Pi(x)$

$\hookrightarrow \boxed{e(x)} \underset{\text{EXACT EQ.}}{=} f[x_0, x_1, \dots, x_n, \underset{= x}{x_{n+1}}]\, \Pi(x) = \underbrace{f[x_0, x_1, \dots, x_n, x]}_{\text{DD OF ORDER } n+1}\, \Pi(x)$

So if $e(x) = f[x_0, x_1, \dots, x_n, x] \cdot \Pi(x) \overset{\text{then}}{=} \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\, \Pi(x)$ f.s (for some) $\xi \in [a,b] \,/\, x_i \in [a,b]$ $(i = 0, \dots, n)$

best est. error = best est. exact − approx.

10/02/21

**EXCERCISE 10**

| $x_i$ | 0 | 0'2 | 0'3 | 0'4 | 0'7 | 0'9 |
|---|---|---|---|---|---|---|
| $f(x_i)$ | 132'651 | 148'877 | 157'464 | 166'375 | 195'112 | 216 |

a) What is the degree of the polynomial?

| i | $x_i$ | $f_{i,0}$ | $f_{i,1}$ | $f_{i,2}$ | $f_{i,3}$ | $f_{i,4}$ | $f_{i,5}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 132'651 | — | — | — | — | — |
| 1 | 0'2 | 148'877 | 81'13 | — | — | — | — |
| 2 | 0'3 | 157'464 | 85'87 | 15'8 | — | — | — |
| 3 | 0'4 | 166'375 | 89'11 | 16'2 | 1 | — | — |
| 4 | 0'7 | 195'112 | 95'79 | 16'7 | 1 | 0 | — |
| 5 | 0'9 | 216 | 104'44 | 17'3 | 1 | 0 | 0 |

↳ DEGREE 3 OR ↑

b) Obtain it using as few data as possible, and evaluate it (optimally) at $x=0'25$

HORNER

$$\boxed{P_3(x) = 132'651 + 81'13x + 15'8\,x(x-0'2) + 1\cdot x(x-0'2)(x-0'3) =}$$

$$= 132'651 + x\left\{81'13 + (x-0'2)\left[15'8 + \underbrace{(x-0'3)}_{-0'05}\right]\right\}$$ (now we substitute $x=0'25$)

$15'75$

$0'7875$

$81'9175$

$20'479375$

$153'130375$

$$\boxed{P_3(0'25) = 153'13}$$

EXCERCISE 13

| $x_i$ | $1'1$ | $1'3$ | $1'4$ | $1'6$ | $1'7$ |
|---|---|---|---|---|---|
| $f(x_i)$ | $1'669$ | $1'971$ | $2'151$ | $2'577$ | $2'828$ |

Approx $f(1'35)$ by evaluating the interpol. pol of degree $\leq 3$ optimally. Estimate the Error.
(4 or more signf. digits and symm. rounding)

*Hacemos la tabla

$$P_3(x) = 1'669 + 1'51(x-1'1) + 0'9667(x-1'1)(x-1'3) + 0'2667(x-1'1)(x-1'3)(x-1'4) =$$

$$= 1'669 + (x-1'1)\left\{1'51 + (x-1'3)\left[0'9667 + \underbrace{(x-1'4)\cdot 0'2667}_{-0'01334}\right]\right\}$$ (now $x=1'35$)

$0'04767$

$1'5577$

$0'389417$

$2'0584$

$p(1'35) = 2'0584$

$e(x) \simeq h_4(x) = 0'25(x-1'1)(x-1'3)(x-1'4)(x-1'6) = \Pi(x)\cdot 0'25$

$\hookrightarrow e(x) = f(x) - P_3(x) \longrightarrow e(1'35) = f(1'35) - P_3(1'35) \simeq P_4(1'35) - P_3(1'35)$

# FINITE DIFFERENCES

**DEFINITION:** (FORWARD FIN. DIFF) (+ h)

$$\boxed{\begin{aligned} &\Delta^0 f(x) = f(x) \\ &\Delta^1 f(x) = \Delta f(x) = f(x+h) - f(x) \\ &\Delta^k f(x) = \Delta(\Delta^{k-1} f(x)) \quad (k = 2, 3, 4, \dots) \end{aligned}}$$

(k=2) $\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+h) - f(x)) = f(x+h+h) - f(x+h) -$
$- [f(x+h) - f(x)] = f(2h+x) - 2f(x+h) + f(x)$

(we do the same that in the TABLE of DD but without dividing)

**TABLE:**

| | $\Delta^0 f(x_i)$ | $\Delta^1 f(x_i)$ | $\Delta^2 f(x_i)$ | ... |
|---|---|---|---|---|
| $x_0 = x$ | $f(x)$ | — | — | |
| $x_1 = x+h$ | $f(x+h)$ | $f(x+h) - f(x)$ | — | |
| $x_2 = x+2h$ | $f(x+2h)$ | $f(x+2h) - f(x+h)$ | $f(x+2h) - 2f(x+h) + f(x)$ | |
| ⋮ | ⋮ | ⋮ | ⋮ | |

$\Delta^n f(x) = \Delta^{n-1}(\Delta f(x)) = \Delta^{n-2}(\Delta(\Delta f(x))) = \Delta^{n-3}(\Delta(\Delta(\Delta f(x))))$

$\Delta^2(\Delta f(x)) = \Delta^2(f(x+h) - f(x)) = \Delta^3 f(x)$ → which means: $\boxed{\Delta^{n-1}(\Delta f(x)) = \Delta^n f(x)}$

## RELATIONSHIP BETWEEN: FINITE DIFF ⟷ DIV. DIFF ⟷ DERIVS

* $f[x_0, x_1, \dots, x_n] = f[x_0, x_0+h, x_0+2h, \dots, x_0+nh] = \dfrac{\Delta^n f(x_0)}{n!\, h^n} = \dfrac{f^{n)}(\xi)}{n!}$ f.s. $\xi$ btw.t.n

for some — between the nodes

DIV. DIFF → HAS TO BE DIVIDED MANY TIMES

→ $\boxed{\Delta^n f(x_0) = h^n \cdot f^{n)}(\xi)}$

## BINOMIAL COEFFICIENTS (NEWTON COEFF)

**DEFINITION:** $\boxed{\binom{m}{i} = \dfrac{m!}{i!\,(m-i)!}}$ → m CHOOSE i

Ex $\binom{5}{3} = \dfrac{5!}{3!\,2!} = \dfrac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot 2 \cdot 1} = 10$ → 10 ways to divide in subsets of 3

123, 124, 125, 134, 135, 145, 234, 235, 245, 345 } 10

* $\binom{t}{n} = \dfrac{t!}{n!\,(m-n)!} = \dfrac{t(t-1)(t-2)\dots(t-n+1)}{n!}$ $\forall t \in \mathbb{R}, n \in \mathbb{N}$

→ EVENTHOUGH t IS NOT AN INTEGER

$\binom{0}{0} = \dfrac{0!}{0!\,0!} = 1$ → PASCAL'S TRIANGLE

$\binom{0}{0} = 1$

$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$

$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

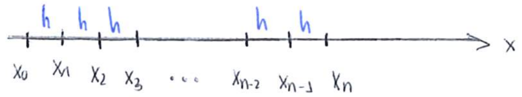
$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$

1 4 6 4 1

# NEWTON POLYNOMIAL WITH EVENLY-SPACED NODES

EQUALLY (=)

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_{n-1}(x-x_0)(x-x_1)(\cdots)(x-x_{n-2}) + a_n(x-x_0)(x-x_1)(\cdots)(x-x_{n-2})(x-x_{n-1})$$




$$\{x_i = x_0 + ih \quad (i = 0, 1, \dots, n)$$

$$\longmapsto x = x_0 + h\cdot t \qquad t = \frac{x-x_0}{h}$$

$$P_n(x(t)) = q(t) = f[x_0] + f[x_0, x_1]\,h\cdot t + f[x_0, x_1, x_2]\,ht\cdot h(t-1) + f[x_0, x_1, x_2, x_3]\,ht\cdot h(t-1)h(t-2) + \dots$$

(DIV. DIFF.)

$$\dots + f[x_0, x_1, \dots, x_{n-1}]\,ht\cdot h(t-1)\cdots h(t-(n-2)) + f[x_0, x_1, \dots, x_n]\,ht\cdot h(t-1)\cdots h(t-(n-2))\cdot h(t-(n-1))$$

(FINITE DIFF.)

$$= \Delta^0 f(x_0) + \frac{\Delta^1 f(x_0)}{1!\,\cancel{h^1}}\cancel{h}t + \frac{\Delta^2 f(x_0)}{2!\,\cancel{h^2}}\cancel{h}t\cancel{h}(t-1) + \dots + \frac{\Delta^n f(x_0)}{n!\,\cancel{h^n}}\cancel{h}t\cancel{h}(t-1)(\cdots)\cancel{h}(t-(n-1))$$

(WITH BINOMIAL COEFF.)

$$= \Delta^0 f(x_0)\binom{t}{0} + \Delta^1 f(x_0)\binom{t}{1} + \Delta^2 f(x_0)\binom{t}{2} + \dots + \Delta^n f(x_0)\binom{t}{n} = \boxed{\sum_{i=0}^{n} \Delta^i f(x_0)\binom{t}{i} = q(t) = P_n(x(t))}$$

## HORNER-LIKE (NESTED-PRODUCT) EVALUATION OF $q(t)$

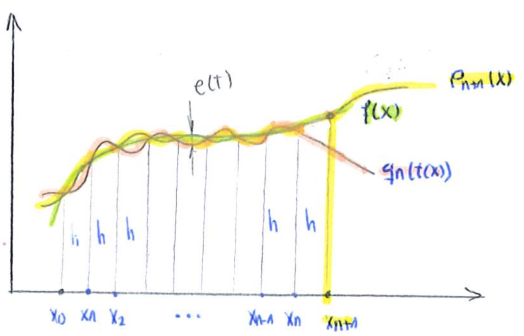
$$q(t) = \Delta^0 f(x_0)\cdot 1 + \Delta^1 f(x_0)\cdot\frac{t}{1} + \Delta^2 f(x_0)\,\frac{t\cdot(t-1)}{2\cdot 1} + \dots + \Delta^n f(x_0)\,\frac{t(t-1)(t-2)(\cdots)(t-(n-2))(t-(n-1))}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

FROM THE FINITE DIFF. TABLE

(TAKING COMMON FACTORS)

$$q(t) = \Delta^0 f(x_0) + \underbrace{\frac{t}{1}\Big\{\Delta^1 f(x_0) + \underbrace{\frac{(t-1)}{2}\Big[\Delta^2 f(x_0) + \frac{(t-2)}{3}\Big(\Delta^3 f(x_0) + \dots \cdot \Big(\dots + \underbrace{\frac{t-(n-2)}{n-1}\Big(\Delta^{n-1} f(x_0) + \underbrace{\frac{t-(n-1)}{n}\,\underbrace{\Delta^n f(x_0)}_{b_n}}_{b_{n-1}}\Big)\Big)}_{\dots}\Big)\Big]}_{b_1}\Big\}}_{b_0 = q(t) = P_n(x(t))}$$

## ESTIMATION OF TRUNCATION ERROR




$$\begin{cases} e(t) = f(x(t)) - P_n(x(t)) = f(x(t)) - q_n(t) \\ e(t) \approx \Delta^{n+1} f(x_0)\binom{t}{n+1} \end{cases}$$

Adding an extra NODE

# PROPAGATION OF A PERTURBATION $\delta$ IN A TABLE OF FINITE DIFF. (TFD)

| $\Delta^0 f(x_0)$ | $\Delta^1 f(x_0)$ | $\Delta^2 f(x_0)$ | $\Delta^3 f(x_0)$ | $\Delta^4 f(x_0)$ | $\Delta^5 f(x_0)$ | ... |
|---|---|---|---|---|---|---|
| $f_0$ | — | — | — | — | — | |
| $f_1$ | x | — | — | — | — | |
| $f_2$ | x | x | — | — | — | |
| $f_3$ | x | x | x | — | — | |
| $f_4 + \delta$ | $x+\delta$ | $x+\delta$ | $x+\delta$ | $x+\delta$ | — | |
| $f_5$ | $x-\delta$ | $x-2\delta$ | $x-3\delta$ | $x-4\delta$ | $x-5\delta$ | |
| $f_6$ | x | $x+\delta$ | $x+3\delta$ | $x+6\delta$ | $x+10\delta$ | |
| $f_7$ | x | x | $x-\delta$ | $x-4\delta$ | $x-10\delta$ | |
| $f_8$ | x | x | x | $x+\delta$ | $x+5\delta$ | |

WE WILL ALWAYS HAVE $\delta$

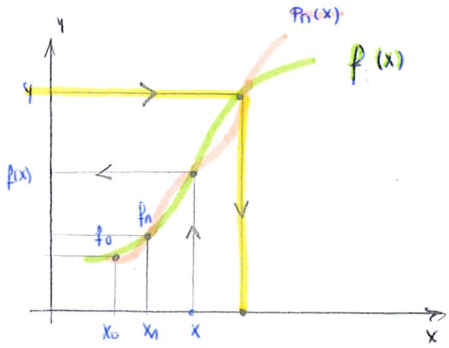
## CONCLUSION

- $\delta$ propagates in a triangle
- propagates with alternating signs (+/-)
- abs. values ↑ (binomial coeffs)
- $f \in \mathbb{P}_k \rightarrow$ k-th column $\equiv$ kte $\rightarrow$ then all 0
- f "well behaved", typically get $\simeq 0$ (small values)

COEFFS. OF THE PASCAL TRIANG.

* WELL BEHAVED FUNCTION: smooth, $\infty$ derivatives

## INVERSE INTERPOLATION




(IS GOING TO BE APPROX. BECAUSE WE ARE USING $P_n(x)$ THAT IS APPROX.)

our data now is y

$x = f^{-1}(y)$? → we want to know the "x" for which we obtain "y"

$P_n(x) = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)\dots(x-x_{n-1})$

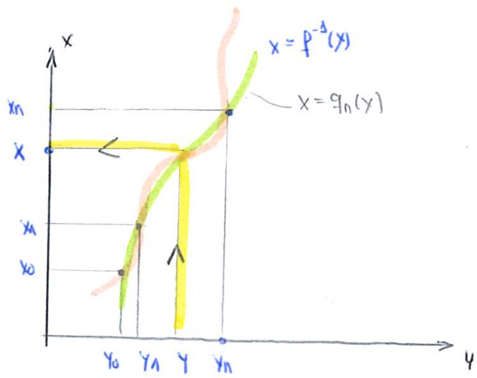
$\underbrace{P_n(x)}_{\text{UNKNOWN}} = \overbrace{y}^{\text{DATA}}$

↓

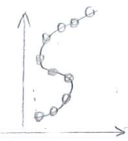
suppose we have:

| $x_i$ | $x_0$ | $x_1$ | $x_2$ | ... | $x_n$ |
|---|---|---|---|---|---|
| $y_i$ | $y_0$ | $y_1$ | $y_2$ | ... | $y_n$ |

→ we are changing the axes →




* SI LA FUNCIÓN ES "INEVALUABLE"



→ YOU CHOSE THE NODES YOU NEED, NOT THE WHOLE FD.

↳ WE TAKE A MONOTONIC PART

↳ ONLY INCREASING / DECREASING

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## EXCERCISE 14

| $x_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $y_i$ | 1 | 2 | 4 | 8 | 15 | 26 | ? | ? |

⟶ EQUALLY SPACED ⟶ FINITE DIFF

a) From a polyn. of what degree?

| $x_i$ | $\Delta^0 f(x_i)$ | $\Delta^1 f(x_i)$ | $\Delta^2 f(x_i)$ | $\Delta^3 f(x_i)$ | $\Delta^4 f(x_i)$ |
|---|---|---|---|---|---|
| 0 | 1 | — | — | — | |
| 1 | 2 | 1 | — | — | |
| 2 | 4 | 2 | 1 | — | |
| 3 | 8 | 4 | 2 | 1 | |
| 4 | 15 | 7 | 3 | 1 | |
| 5 | 26 | 11 | 4 | 1 | |
| 6 | (42) | (16) | (5) | (1) | |
| 7 | (64) | (22) | (6) | (1) | |

⟶ DEGREE 3 OR MORE

b) Find "??" without constructing the polynomial

we are going to assume our polynomial is of degree 3

$\Delta^3 f(x_7) = 1 = \Delta^3 f(x_8)$

$\rightarrow \Delta^3 f(x_7) = \Delta^2 f(x_7) - 4 = 1 \rightarrow \Delta^2 f(x_7) = 5 \rightarrow \Delta^3 f(x_8) = \Delta^2 f(x_8) - 5 \rightarrow \Delta^2 f(x_8) = 6$

$\rightarrow \Delta^2 f(x_7) = \Delta^1 f(x_7) - 11 \rightarrow \Delta^1 f(x_7) = 16 \rightarrow \Delta^1 f(x_8) = 22$

$\rightarrow \boxed{\Delta^0 f(x_7) = 26 + 16 = 42} \rightarrow \boxed{\Delta^0 f(x_8) = 42 + 22 = 64}$

## EXCERCISE 15

h=2 h=2 h=2 h=2 h=2 h=2

| $x_i$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| $y_i$ | ? | 93 | 259 | 569 | 1071 | 1813 | ? |

a) Guess "??" (without the polyn.)

$\boxed{y_6 = 2843}$
$\boxed{y_0 = 23}$
Assuming degree = 3

| $x_i$ | $\Delta^0 f(x_i)$ | $\Delta^1 f(x_i)$ | $\Delta^2 f(x_i)$ | $\Delta^3 f(x_i)$ | $\Delta^4 f(x_i)$ |
|---|---|---|---|---|---|
| 2 | (23) | — | — | — | — |
| 4 | 93 | (70) | — | — | — |
| 6 | 259 | 166 | (96) | — | — |
| 8 | 569 | 310 | 144 | (48) | — |
| 10 | 1071 | 502 | 192 | 48 | ( ) |
| 12 | 1813 | 742 | 240 | 48 | 0 |
| 14 | (2843) | (1030) | (288) | (48) | ( ) |

⟶ DEGREE ≥ 3

EXCERCISE 15

b) write the polynomial → CHANGE OF VARIABLE $x = x_0 + ht$ $\left| \begin{array}{l} x = 2+2t \\ \hookrightarrow t = \frac{x-2}{2} \end{array} \right.$

$$P_3(x) \longrightarrow P_3(x(t)) = q_3(t) = \underbrace{23}_{y_{0,0}} + \underbrace{70}_{y_{1,1}} \binom{t}{1} + \underbrace{96}_{y_{2,2}} \binom{t}{2} + \underbrace{48}_{y_{3,3}} \binom{t}{3}$$

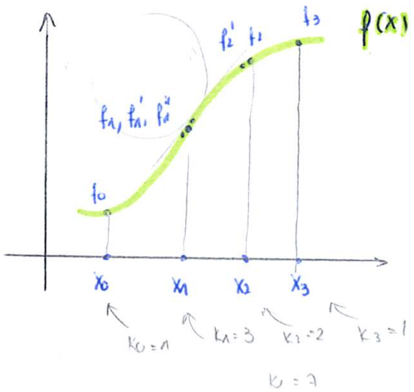
$$\boxed{q_3(t) = 23 + 70t + 96\,\frac{t(t-1)}{2} + 48\,\frac{t(t-1)(t-2)}{3\cdot 2\cdot 1} = 23 + t\left\{70 + \frac{(t-1)}{2}\left[96 + \frac{(t-2)}{3}\,48\right]\right\}}$$

c) Evaluate it at $x=5$ (optimally)

$$x = 5 \longrightarrow \left[ t = \frac{5-2}{2} = 1'5 \right. \longrightarrow q_3(1'5) = 23 + 1'5\left\{70 + \frac{0'5}{2}\left[96 - \frac{0'5}{3}\,48\right]\right\} = \boxed{161}$$

↳ que se pega, muy cerca

## OSCULATING POLYNOMIALS




$$f^{m)}(x_i) = f_i^{m}$$

we want a polynomial such that:

$$\left\{ \begin{array}{l} P(x_0) = f(x_0)\ ;\ P'(x_0) = f'(x_0)\ ;\ \dots\ ;\ P^{m_0)}(x_0) = f^{m_0)}(x_0) \\ P(x_1) = f(x_1)\ ;\ P'(x_1) = f'(x_1)\ ;\ \dots\ ;\ P^{m_1)}(x_1) = f^{m_1)}(x_1) \\ \dots \\ P(x_n) = f(x_n)\ ;\ P'(x_n) = f'(x_n)\ ;\ \dots\ ;\ P^{m_n)}(x_n) = f^{m_n)}(x_n) \end{array} \right.$$

$m_0, m_1, \dots, m_n$ don't have to be equal

CONDITIONS:

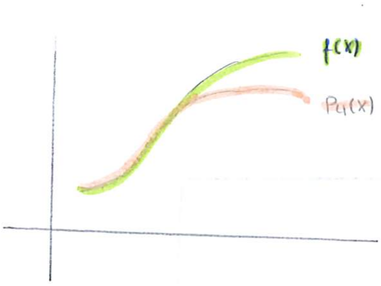
$$\boxed{P(x_i) = f(x_i)\ ;\ P'(x_i) = f'(x_i)\ ;\ \dots\ ;\ P^{m_i)}(x_i) = f^{m_i)}(x_i) \qquad (i = 0, 1, \dots, n)}$$

$k_i$ = (number of conditions at $x_i$) $= m_i + 1$ → $\boxed{k_i = m_i + 1}$

$\boxed{k = \sum_{i=0}^{n} k_i}$ NUMBER OF CONDITIONS FOR THE POLYNOMIAL

THEOREM: $\exists!\ P_{k-1}(x) \in \mathbb{P}_{k-1}$ that satisfies the osculation interpolation conditions.

## TAYLOR POLYNOMIAL




$$f(x) = \overbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \dots + \frac{f^{m)}(x_0)(x-x_0)^m}{m!}}^{\text{EXACT}} + \overbrace{\frac{f^{m+1)}(\xi)(x-x_0)^{m+1}}{(m+1)!}}^{\text{APPROX}}$$

* If $f \in C^{m+1}(\underbrace{[x_0, x] \cup [x, x_0]})$ then $\exists\, \xi \in [x_0, x] \cup [x, x_0]$ /

and $f^{m+1)}$ is defined and bounded in

$$\begin{cases} \text{EXACT} \rightarrow P_m(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \dfrac{f^{m)}(x_0)(x-x_0)^m}{m!} \\ \text{APPROX} \rightarrow e(x) = \dfrac{f^{m+1)}(\xi)(x-x_0)^{m+1}}{(m+1)!} \end{cases}$$

Taylor : $n = 0$ ; $m_0 = m$

## HERMITE POLYNOMIAL (NOT ORTHOGONAL)

* Any $n$, $m_i = 1$ $\forall i$ ($k_i = 2$ $\forall i$)

## ERROR FOR AN OSCULATING POLINOMIAL

$$\boxed{e(x) = \frac{f^{k)}(\xi)}{k!}(x-x_0)^{k_0}(x-x_1)^{k_1}\dots(x-x_n)^{k_n}}$$

↳ GENERALIZATION OF LAGR. / TAYLOR

/ Taylor : $\begin{cases} k = m+1 = k_0 \\ n = 0 \quad (1 \text{ node}) \end{cases}$

\ Lagrange : $\begin{cases} n = \text{any} \\ k = n+1 \\ k_i = 1 \end{cases}$

$e(x)$ of a Hermite polynomial → ($k = 2(n+1) = 2n+2$)

↳ $$\boxed{e(x) = \frac{f^{2n+2)}(\xi)}{(2n+2)!}(x-x_0)^{2}(x-x_1)^2\dots(x-x_n)^2}$$

## TABLE OF DIVIDED DIFFERENCES WITH REPETITIONS

Ex

| | $x_i$ | $f_{i,0}$ | $f_{i,1}$ | $f_{i,2}$ | $f_{i,3}$ | $f_{i,4}$ | $f_{i,5}$ | ... |
|---|---|---|---|---|---|---|---|---|
| $z_0 =$ | $x_0$ | $\boxed{f(x_0)} = a_0$ | — | — | — | — | — | |
| $z_1 =$ | $x_1$ (2 cond) | $f(x_1)$ | $\boxed{x} = a_1$ | — | — | — | — | |
| $z_2 =$ | $x_1$ | $f(x_1)$ | $f'(x_1)/1!$ | $\boxed{x} = a_2$ | — | — | — | |
| $z_3 =$ | $x_2$ | $f(x_2)$ | x | x | $\boxed{x} = a_3$ | — | — | |
| $z_4 =$ | $x_3$ | $f(x_3)$ | x | x | x | $\boxed{x} = a_4$ | — | |
| $z_5 =$ | $x_3$ (4 cond) | $f(x_3)$ | $f'(x_3)/1!$ | x | x | x | $a_5 = \boxed{x}$ | $\boxed{\dots}$ |
| $z_6 =$ | $x_3$ | $f(x_3)$ | $f'(x_3)/1!$ | $f''(x_3)/2!$ | x | x | x | |
| $z_7 =$ | $x_3$ | $f(x_3)$ | $f'(x_3)/1!$ | $f''(x_3)/2!$ | $f'''(x_3)/3!$ | x | x | |

$$\frac{f^{n)}(\xi)}{n!}$$

17/02/24

So the polynomial will be:

$$P_7(x) = a_0 + a_1(x-z_0) + a_2(x-z_0)(x-z_1) + \dots + a_7(x-z_0)(x-z_1)(\dots)(x-z_6)$$

23/02/24

EXCERCISE 26

a) oscullating pol. / $f(1)=2$ / $f'(1)=3$ / $f(2)=6$ / $f'(2)=7$ / $f''(2)=8$

| $z_i$ | $f_{i,0}$ | $f_{i,1}$ | $f_{i,2}$ | $f_{i,3}$ | $f_{i,4}$ |
|---|---|---|---|---|---|
| 1 | $\boxed{2}$ | — | — | — | — |
| 1 | 2 | $\boxed{3}$ | — | — | — |
| 2 | 6 | 4 | $\boxed{1}$ | — | — |
| 2 | 6 | 7 | 3 | $\boxed{2}$ | — |
| 2 | 6 | 7 | $8/2!$ | 1 | $\boxed{-1}$ |

$$\boxed{P_4(x) = 2 + 3(x-1) + 1(x-1)^2 + 2(x-1)^2(x-2) - 1(x-1)^2(x-2)^2}$$

b) $e(x)$

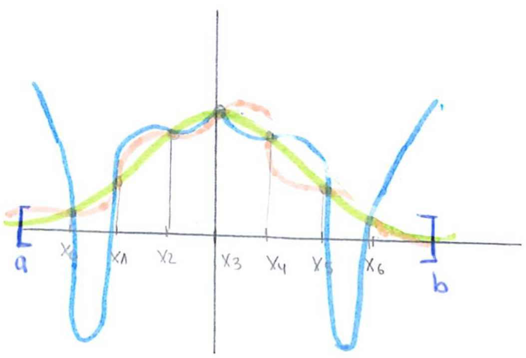
$$e(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi(x) \quad \text{f.s. } \xi \text{ between } x \text{ if } f \in C^5$$

$$\hookrightarrow e(x) = \frac{f^{5)}(\xi)}{5!}\Pi(x) = f[1,1,2,2,2,x]\,\Pi(x) = \frac{f^{5)}(\xi)}{5!}(x-1)^2(x-2)^3$$

RUNGE EFFECT

Oscillations that come "out of nowhere"

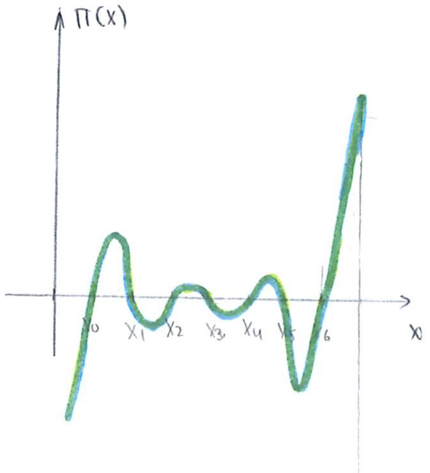
SPURIOUS → not in the nature of $f(x)$




The pol. we would expect

Actual pol.

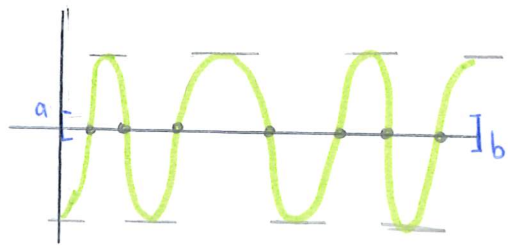
$$e(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\underbrace{(x-x_0)(x-x_1)\cdots(x-x_{n-1})(x-x_n)}_{\Pi(x)}$$




* The large oscillations in $\Pi(x)$ translates into errors in our pol. when is equally spaced.

* Near the center less oscill.

we will have a max and min and we will try to reach them as many times as possible.
For that, we need to know in which nodes this happens.
↳ Chebyshev
(WE'LL SEE IT LATER)




## $\|\cdot\|_\infty$ DEFINITION (INFINITY NORM)

In $\mathbb{R}^n$ ; $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$\|\bar{x}\|_2 = +\sqrt[2]{x_1^2 + x_2^2 + \dots + x_{n-1}^2 + x_n^2}$$

$$\|\bar{x}\|_n = \sqrt[n]{|x_1|^n + |x_2|^n + \dots + |x_n|^n}$$

...

$$\boxed{\|\bar{x}\|_\infty = \max_{i=1,2,\dots,n} |x_i|}$$

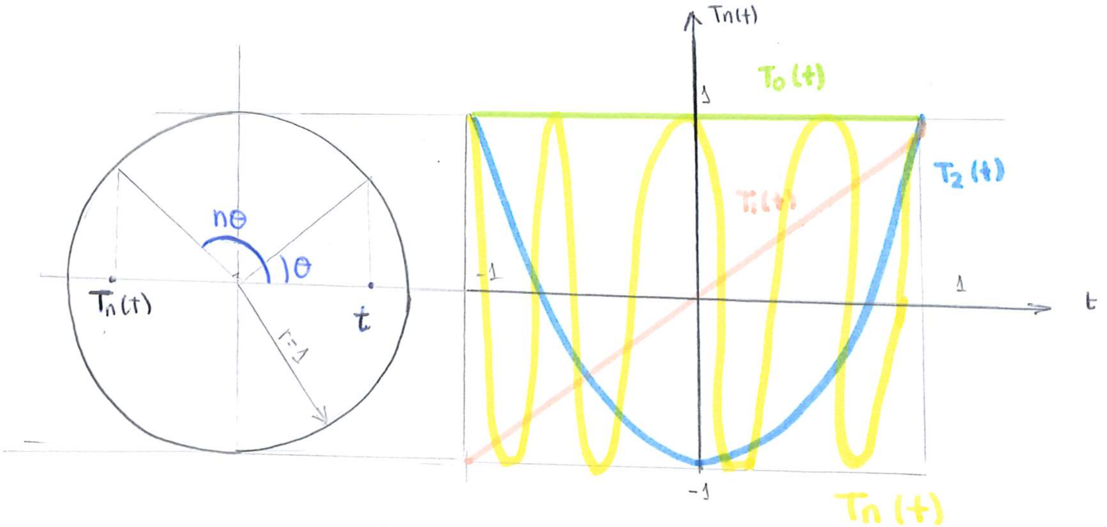
→ INFINITY NORM ≡ NORM OF THE MAXIMUM

$$\boxed{\|f\|_\infty = \max(f(x)) \quad x \in [a,b] \longrightarrow \text{minimize } \|\Pi\|_\infty}$$

(maximizing the norm minimizes the error)
↳ the norm of $\Pi(x)$

## CHEBYSHEV POLYNOMIALS OF THE 1ST KIND

**DEFINITION**: $T_n(t) = \cos(n \underbrace{\arccos(t)}_{\theta})$ $\quad t \in [-1, 1] \quad (n = 0, 1, \dots)$




$T_0(t) = \cos(0 \cdot a\cos t) = 1$

$T_1(t) = \cos(a\cos t) = t$

$T_2(t) = \cos(2a\cos t) = 2t^2 - 1$ $\longrightarrow$ RECURSIVE LAW $\longrightarrow$ $\boxed{T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t)}$ $(n = 1, 2, \dots)$

$T_3(t) = 2t(2t^2-1) - t = 4t^3 - 3t$

$T_4(t) = 2t(4t^3-3t) - (2t^2-1) = 8t^4 - 8t^2 + 1$

* PROOF OF THE RECUR. LAW: $T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t)$

- $\cos((n+1)a\cos(t)) = \cos((n+1)\theta) = \cos(\theta n + \theta) = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta$
- $2\cos\theta \cos(n\theta) - \cos[n\theta - \theta] \dots$

* LEADING COEFF $= 2^{n-1}$ $\quad (n = 1, 2, \dots)$

ROOTS OF $T_n(t)$ = "CHEBYSHEV NODES in $[-1, 1]$ → IN FACT -1 AND 1 WILL NEVER BE A NODE

- $T_n(t) = 0 \longrightarrow \cos(n\theta) = 0 \longrightarrow n\theta = \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z}) \longrightarrow \theta = \frac{\pi/2 + k\pi}{n}$

- $\boxed{t_i = \cos\theta = \cos\left(\frac{\pi/2 + k\pi}{n}\right) \quad \underbrace{(i = 0, 1, 2, \dots, n-1)}_{n \text{ roots}}}$

$= \cos\left(\frac{(1+2k)\pi}{2n}\right)$

## LINEAR CHANGE OF VARIABLE $[a,b] \longleftrightarrow [-1,1]$

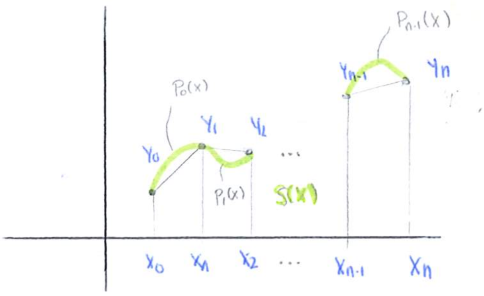
$$\boxed{x = \underbrace{\frac{a+b}{2}}_{M} + \underbrace{\frac{(b-a)}{2}}_{J = dx/dt} t}$$

M = midpoint
J = jacobian

24/02/21

## SPLINES : PIECEWISE POLYNOMIAL FUNCTIONS

* solution for the Rounge effect.




- If using pol. of degree 1 we won't have a continious 1st derivative.
- our approx $\longrightarrow$ pol of degree 3 $\longrightarrow$ MOST OF THE TIMES IN THE EXCERCISES (3 or ODD)

$$\boxed{P_i(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

$\hookrightarrow a_0, a_1, a_2, a_3 \Rightarrow 4$ dof

$(x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n) \longrightarrow$ INTERPOL. POINTS : SHORTED AND DIFFERENT

* A spline of order $l$ is a piecewise pol. function $S(x)$ that satisfies:

- $\boxed{S(x_i) = f(x_i) \quad (i = 0, 1, \ldots, n)}$ PIECW. APPROX = ACTUAL POL.
- $\boxed{S(x)\big|_{[x_i, x_{i+1}]} = P_i(x) \in \mathbb{P}_l}$
- $\boxed{S \in C^{l-1}(\mathbb{R})}$ $\rightarrow$ SMOOTHNESS

* if $l = 3$ (CUBIC) $\begin{cases} S(x)\big|_{[x_i, x_{i+1}]} = P_i(x) \in \mathbb{P}_3 \\ S \in C^2(\mathbb{R}) \quad \text{(quite smooth)} \end{cases}$

## NUMBER OF DEGREES OF FREEDOM VS. CONDITION

n dof = n

n conds :

| | |
|---|---|
| $2n$ | (interpolate) |
| $n-1$ | $(C^1)$ |
| $n-1$ | $(C^2)$ |
| $4n - 2$ | |

$4n - \underbrace{2}_{\text{2dof that we can use freely}}$

## CUBIC SPLINE WITH BOUNDARY CONDITIONS

TO SATISFY THE 2 SPACE [illegible]

$$\begin{cases} P_0'(x_0) = f'(x_0) \\ P_{n-1}'(x_n) = f'(x_n) \end{cases}$$

* NOT - a - knot condition: 3rd der. is continious (we are not doing that)

## NATURAL CUBIC SPLINE

$$\begin{cases} P_0''(x_0) = 0 \\ P_{n-1}''(x_n) = 0 \end{cases}$$

## PERIODIC CUBIC SPLINE

* Only if $f(x)$ is PERIODIC

Not for the excercises

$$\begin{cases} P_0'(x_0) = P_{n-1}'(x_n) \\ P_0''(x_0) = P_0''(x_n) \end{cases}$$

## OPTIMAL POL. // THEOREM OF OPTIMALLITY

* $\forall g \in C^2([x_0, x_n]) \,/\, g(x_i) = f(x_i) \quad (i = 0, 1, \dots, n)$, $g'(x_0) = f'(x_0)$, $g'(x_n) = f'(x_n)$ $\quad \int_{x_0}^{x_n} g''(x)^2\,dx \geq \int_{x_0}^{x_n} s''(x)^2\,dx$

  CUBIC SPLINE WITH BOUNDARY COND

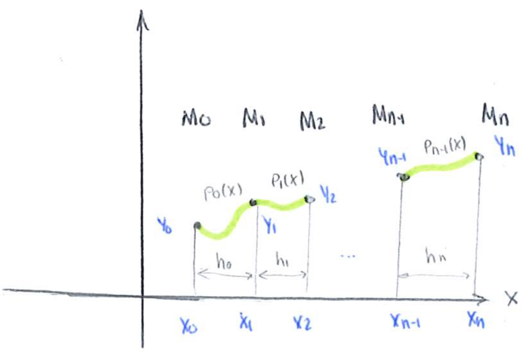
* $\forall g \in C^2([x_0, x_n]) \,/\, g'(x_i) = f(x_i) \quad (i = 0, 1, \dots, n)$, $g''(x_0) = 0$, $g''(x_n) = 0$ $\quad \int_{x_0}^{x_n} g''(x)^2\,dx \geq \int_{x_0}^{x_n} s''(x)^2\,dx$ } The one with the least amount of oscillations

  NATURAL CUBIC SPLINE

* $\forall g \in C^2([x_0, x_n]) \,/\, g'(x_i) = f(x_i) \quad (i = 0, 1, \dots, n)$, $g'(x_0) = g'(x_n)$, $g''(x_0) = g''(x_n)$ $\quad \int_{x_0}^{x_n} g''(x)^2\,dx \geq \int_{x_0}^{x_n} s''(x)^2\,dx$

  PERIODIC CUBIC SPLINE

## MOMENTS




* $\boxed{M_i = S''(x)}$ and find them 1ST

$(n-1)\cdot(n-1)$

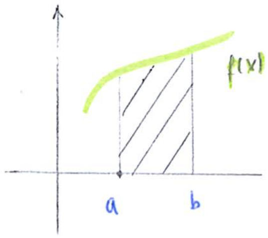
$(n+1)\cdot(n+1)$

* the matrix will be tridiagonal and strictly diagonally dominant.

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & a_{23} \\ & & \end{pmatrix} \qquad \begin{matrix} a_{11} > a_{12} \\ a_{22} > a_{21} + a_{23} \\ \dots \end{matrix}$$

$$\underbrace{\begin{pmatrix} 2h_0 & h_0 & \cdots & 0 & 0 \\ h_0 & 2(h_1+h_2) & h_2 \cdots & 0 & 0 \\ 0 & h_1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & \cdots & h_{(n-1)} & 2h_{n-1} \end{pmatrix}}_{(T)} \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}$$

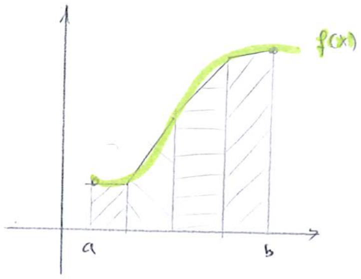
# CH. 2. INTEGRATION




$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{if } F' = f$$

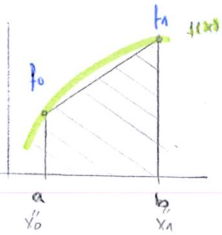
BURROW'S LAW

## COMPOUND TRAPEZOIDAL RULE




Addition of the areas (trapeziums)

* If we don't divide it into intervals:



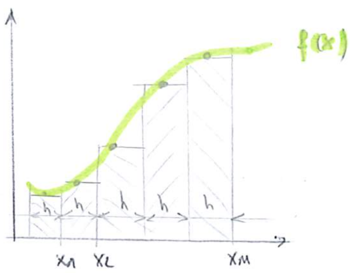

$$\underbrace{(b-a)}_{h}\left(\frac{f_0+f_1}{2}\right) = \underbrace{\frac{h}{2}}_{w_0} f_0 + \underbrace{\frac{h}{2}}_{w_1} f_1$$

WEIGHT FUNCTIONS

$$\boxed{Q = \frac{h}{2}(f_0+f_1)}$$

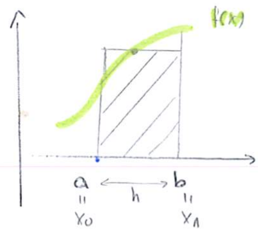
03/03/21

## COMPOUND MIDPOINT RULE




we take the M.P. of each interval and obtain areas (square (rectang)) that we will add.

* If we don't divide it into intervals:

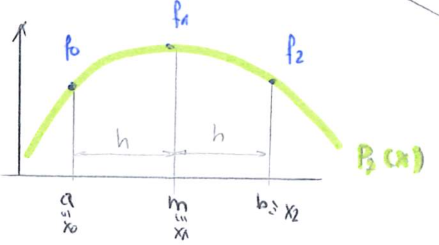



$$\boxed{Q = h \cdot f_0}$$

$Q \simeq I = \int_a^b f(x)dx$ (QUADRATURE FUNC)

## QUADRATURE RULE

$$\boxed{I = \int_a^b f(x)\,dx \simeq Q}$$ ⟶ Instead of integrating our function, we will integrate our Lagrange interpolation polynomial.




$$I = \int_a^b P_2(x)\,dx = Q$$

EXAMPLE

(Qss)

## SIMPSON'S RULE VIA INTEGRATION OF LAGRANGE BASED FUNCTIONS

$$\boxed{I = \int_a^b f(x)\,dx \simeq \int_a^b P_2\,dx = Q}$$ (QUAD. RULE)

- $$P_2(x) = \underbrace{\frac{(x-x_1)(x-x_2)}{\underbrace{(x_0-x_1)(x_0-x_2)}_{2h^2}}}_{L_0(x)} f_0 + \underbrace{\frac{(x-x_0)(x-x_2)}{\underbrace{(x_1-x_0)(x_1-x_2)}_{-h^2}}}_{L_1(x)} f_1 + \underbrace{\frac{(x-x_0)(x-x_1)}{\underbrace{(x_2-x_0)(x_2-x_1)}_{2h^2}}}_{L_2(x)} \cdot f_2 \quad \Big\}\ \text{LAGR. POL.}$$

$$* \quad Q = \int_a^b P_2\,dx = \int_a^b \left(L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2\right) dx = \underbrace{\left(\int_a^b L_0(x)\,dx\right)}_{W_0} f_0 + \underbrace{\left(\int_a^b L_1(x)\,dx\right)}_{W_1} f_1 + \underbrace{\left(\int_a^b L_2(x)\,dx\right)}_{W_1} f_2$$

### THEOREM:

- $P_n(x) = \sum_{i=0}^{n} L_i(x) f_i$
- $$\boxed{Q = \sum_{i=0}^{n} W_i f_i \quad \text{where} \quad W_i = \int L_i(x)\,dx} \quad (i=0,1,\dots,n)$$

## CHANGE OF VARIABLE

$x = x_0 + ht$ for $W_1 = \int_a^b L_1(x)\,dx = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{h}\,dx$

* Applying the change of variable: $x = x_1 + ht \rightarrow x - x_2 = \underbrace{x_1 - x_2}_{-h} + ht$

$$W_1 = \int_{-1}^{1} \frac{\cancel{h}(t+1)\,\cancel{h}(t-1)}{\cancel{h}(-\cancel{h})} \cdot h\,dt = \frac{4h}{3} \rightarrow \boxed{W_1 = \frac{4h}{3}}$$

$\boxed{W_0 = h/3}$ —— TYPICAL VALUES WE ARE GOING TO USE

## $Q_{SS}$ VIA NEWTON POLYNOMIAL (same as before but using Newton interp. pol.)

$$Q_{SS} = \int_{x_0}^{x_2} P_2(x)\,dx = \int_0^2 \underbrace{P_2(x(t))}_{q_2(t)}\,h\,dt$$

FROM FIN. DIFF. TABLE

* $q_2(t) = f(x_0) + \Delta f(x_0)\binom{t}{1} + \Delta^2 f(x_0)\binom{t}{2} = f_0 + (f_1 + f_0)t + (f_2 - 2f_1 + f_0)\frac{t(t-1)}{2}$ } NEWTON POL

$$Q_{SS} = \int_0^2 \left[f_0 + (f_1 - f_0)t + (f_2 - 2f_1 + f_0)\frac{t^2 - t}{2}\right]h\,dt = h\left(2f_1 + \frac{f_2 - 2f_1 + f_0}{3}\right) = \left(\frac{f_0}{3} + \frac{4}{3}f_1 + \frac{f_2}{3}\right)h$$

$\hookrightarrow \underbrace{f_0\underbrace{\left(\frac{h}{3}\right)}_{W_0} + f_1\underbrace{\left(\frac{4h}{3}\right)}_{W_1} + \ldots}_{\text{SAME AS BEFORE}}$

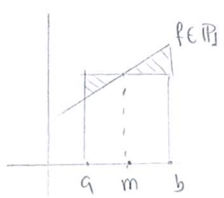
$\hookrightarrow \boxed{Q_{SS_2} = \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)}$

## POLYNOMIAL DEGREE OF A QUADRATURE RULE

Polynomial degree (of exactitude) or "order" of a quadrature rule

$$\boxed{N \geq n = n_{n-1}}$$

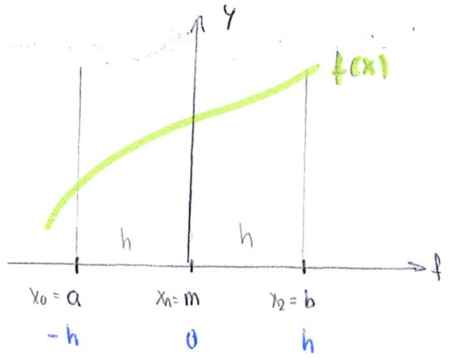
* In the compound trapezoidal rule: we integrate polynom. of degree 0 and 1 so: $\boxed{N=1}$
* In the compound MIDPOINT RULE: we integr. pol of degree 0 and luckily also of degree 1: $\boxed{N=1}$




* In the Simpson's RULE $\rightarrow \boxed{N=3}$

## INDETERMINATE COEFFICIENTS. (Q's VIA INDET. COEF).

1) AXE "X" + FUNCTION + NODES
2) PLACE AXE "Y"
3) IMPOSE THE EXACT INTEGR.




$*$ EXACT INTEGRATION OF:

$f(x) \equiv 1 \longrightarrow \int_{-h}^{h} 1\,dx = 2h \overset{N \geq 2}{=} W_0 \cdot 1 + W_1 \cdot 1 + W_2 \cdot 1$

$f(x) \equiv x \longrightarrow \int_{-h}^{h} x\,dx = 0 = W_0(-h) + W_1(0) + W_2(h) = (W_2 - W_0)\cancel{h} = 0$

$f(x) \equiv x^2 \longrightarrow \int_{-h}^{h} x^2 dx = \frac{2h^3}{3} = W_0 \cdot h^2 + W_1(0) + W_2 h^2 = (W_0 + W_2)\cancel{h^2} = \frac{2h^{\cancel{3}}}{3}$

So we have:

$$\begin{cases} \cdot\ W_0 + W_1 + W_2 = 2h \\ \cdot\ W_2 = W_0 \\ \cdot\ W_0 + W_2 = 2h/3 \end{cases}$$

$\longrightarrow W_1 = 2h - 2h/3 = 4h/3$

$\longrightarrow W_0 = W_2 = h/3$ ) WHAT WE EXPECTED

## TRUNCATION ERROR (FOR INT.)

$$\boxed{\underbrace{\int_a^b f(x)\,dx}_{I} = \underbrace{\int_a^b P_n(x)\,dx}_{Q} + E} \longrightarrow \boxed{E = \overset{\text{EXACT}}{I} - \underset{\text{APPROX}}{Q}}$$

The error in the integral is the same as the int. of the error.

$$* \ E = \int_a^b f(x)\,dx - \int_a^b P_n(x)\,dx = \int_a^b (f(x) - P_n(x))\,dx = \int_a^b e(x)\,dx$$

$$\int_a^b e(x)\,dx = \int_a^b f[x_0, x_1, \ldots, x_n, x]\,\pi(x)\,dx = \int_a^b \frac{f^{n+1}(\xi(x))}{(n+1)!}\,\pi(x)\,dx$$

* case : $f \in \mathbb{P}_{n+1}$ WITH CENTRAL SYMM. of NODES AND ODD NUMBER OF NODES

• $f^{(n+1)}(\xi) = \text{kte}$ → $\boxed{E = f^{(n+1)}(\xi) \int_a^b \pi(x)\,dx = 0}$


all of them cancel out → $\int_a^b \pi(x)\,dx = 0$

09/03/21

$f \in \mathbb{P}_n$ , $E = 0$ $(N \geq n)$

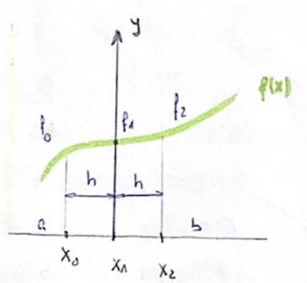
$f \in \mathbb{P}_{n+1}$ , $E = 0$ $(N \geq n+1)$ (odd ∧ central symm)

POL DEGREE

$\boxed{E = k \cdot f^{(N+1)}(\xi) \quad \text{f.s. } \xi \in [a,b]}$ GENERAL EXPRESSION

- $f \in \mathbb{P}_N \rightarrow f^{(N+1)}(\xi) = 0 \rightarrow E = 0$
- $f \in \mathbb{P}$ w/ $\deg(f) = N+1 \longrightarrow f^{(N+1)}(\xi) = \text{kte} \rightarrow E \neq 0$

TRUNCATION ERROR OF SIMPLE SIMPSON'S RULE ( $E_{SS}$ )




$$\begin{cases} \int_{-h}^{h} 1\,dx = Q \\ \int_{-h}^{h} x\,dx = Q \\ \int_{-h}^{h} x^2\,dx = Q \end{cases} \longrightarrow Q = \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$$

INDET. COEFF

$N \geq 2$
$N = 3$

$$\int_{-h}^{h} x^3 dx = Q + \overset{\nearrow 0}{\cancel{E}} = Q + I \longrightarrow 0 = \frac{h}{3}\left((-h)^3 + 4\cdot 0^3 + h^3\right) = E \rightarrow E = 0$$

$$f(x) = x^{N+1} = x^4 \longrightarrow \int_{-h}^{h} x^4 dx = Q + E \longrightarrow \frac{2h^5}{5} = \frac{h}{3}\underbrace{\left((-h)^4 + 4\cdot 0^4 + h^4\right)}_{2h^4} + \underbrace{k f^{(4)}(\xi)}_{E} \quad \text{f.s. } \xi \in [a,b]$$

$f = x^4 \Rightarrow f^{(4)} = 4! = 4 \cdot 3 \cdot 2 = 24$

$f' = 4x^3$

$f'' = 12x^2$

$f''' = 24x$

$f^{(4)} = 24$ KTE

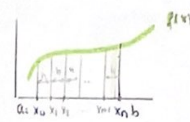
$$\frac{2h^5}{5} = \frac{2h^5}{3} + 24k \rightarrow k = \frac{h^5}{12}\left(\frac{1}{5} - \frac{1}{3}\right) = \frac{-h^5}{6 \cdot 15} = \frac{-h^5}{90}$$

$$\boxed{E_{SS} = -\frac{f^{(4)}(\xi)}{90} \cdot h^5}$$

## INTERPOLATORY QUADRATURE RULE CLASSIFICATION

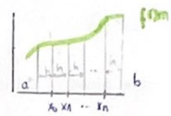
**NEWTON-COTES RULES** (Equally spaced nodes)

- CLOSED RULES (CNC) →




  - $h = (b-a)/n$
  - $x_i = x_0 + ih$ $(i = 0, \dots, n)$

- OPEN RULES (ONC) →




  - $h = (b-a)/(n+2)$
  - $x_i = a + (i+1)h$ $(i = 0, \dots, n)$

**GAUSS RULES** (Unequally spaced nodes)

- G-LEGENDRE: $\int_{-1}^{1} f(x)\,dx$
- G-CHEBYSHEV: $\int_{-1}^{1} f(x)\frac{1}{\sqrt{1-x^2}}\,dx$
- G-LAGUERRE: $\int_{0}^{\infty} f(x)e^{-x}\,dx$
- G-HERMITE: $\int_{-\infty}^{\infty} f(x)e^{-x^2}\,dx$

## TABLE OF NEWTON-COTES (NC)

$N = 2n+1$

$N = nn$ (impar)

(la bereccia de h)

ORDEN DE CONVG.

f.s $\xi \in [a,b]$

| NAME | CLASIF | NODES $nn$ | Q | $(nn-1) = n$ | $N \geq n$ | Es | $N+2 =$ OCS | Ec | OCC |
|---|---|---|---|---|---|---|---|---|---|
| Mid Point | ONC | 1 | $2hf_0$ | 0 | 1 | $h^3 f''(\xi)/3$ | 3 | $h^2(b-a)f''(\xi)/6$ | 2 |
| Trap. | CNC | 2 | $\frac{h}{2}(f_0 + f_1)$ | 1 | 1 | $-h^3 f''(\xi)/12$ | 3 | $-h^2(b-a)f''(\xi)/12$ | 2 |
| Simpson | CNC | 3 | $\frac{h}{3}(f_0 + 4f_1 + f_2)$ | 2 | 3 | $-f^{(4)}(\xi)h^5/90$ | 5 | $-h^4(b-a)f^{(4)}(\xi)/180$ | 4 |
| 2nd Simpson | CNC | 4 | $\frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$ | 3 | 3 | $-3h^5 f^{(4)}(\xi)/80$ | 5 | $-h^4(b-a)f^{(4)}(\xi)/80$ | 4 |
| Milne/Boole | CNC | 5 | $\frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)$ | 4 | 5 | $-8h^7 f^{(6)}(\xi)/945$ | 7 | $-2h^6(b-a)f^{(6)}(\xi)/945$ | 6 |
| | ONC | 2 | $\frac{3h}{2}(f_0 + f_1)$ | 1 | 1 | $3h^3 f''(\xi)/4$ | 3 | $h^2(b-a)f''(\xi)/4$ | 2 |
| | ONC | 3 | $\frac{4h}{3}(2f_0 - f_1 + 2f_2)$ | 2 | 3 | $14h^5 f^{(4)}(\xi)/45$ | 5 | $7h^4(b-a)f^{(4)}(\xi)/90$ | 4 |
| | CNC | 9 | $\frac{h}{14175}(3956f_0 + \dots - 3\,2f_2 + \dots)$ | 8 | 9 | ... | 10 | ... | 9 |

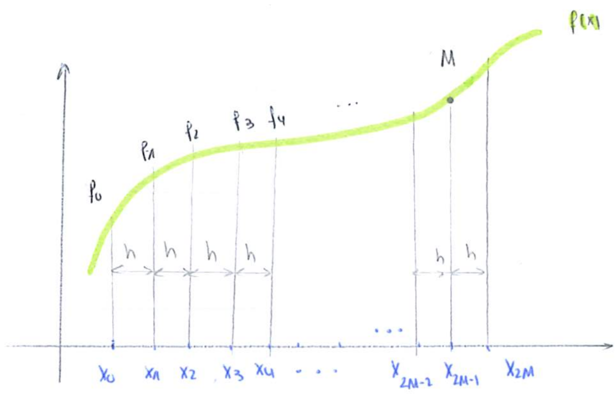
NEGATIVE WEIGHT (w(x))

Es COEFF: ONC (+), CNC (−)

BIG O = OCS → ex $\frac{14h^5 f^{(4)}(\xi)}{45} = O(h^5)$ ?

09/03/21

## COMPOUND RULES  Ex. $Q_{CS}$.




- M equally-wide subintervals

$$(b-a) = M \cdot 2 \longrightarrow h = \frac{b-a}{2M}$$

$$Q_{CS} = \frac{h}{3}(f_0 + 4f_1 + f_2) + \frac{h}{3}(f_2 + 4f_3 + f_4) + \frac{h}{3}(f_4 + 4f_5 + f_6) + \dots$$

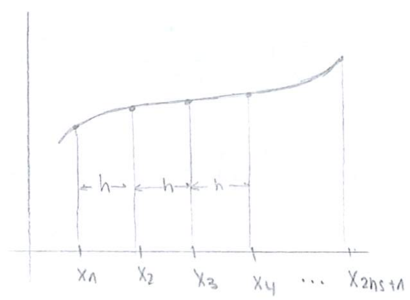
10/03/21

### CODING

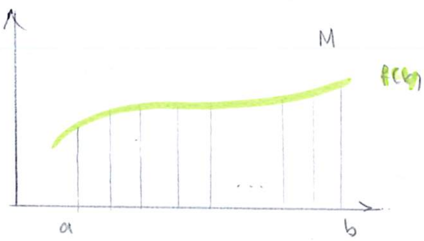
function handed

```
function   Q = compound simpson (f, a, b, ns)

h = (b-a) / ns / 2;

xi = linspace (a, b, 2* ns +1);

Q = h/3 * ( f(a) + f(b)) ...

   + 4*h/3 *sum ( f (xi (2:2: end-1))) ...

   + 2*h/3* sum ( f (xi( 3:2: end-2)))
```

INTERIOR EVEN NODES (IN OCTAVE)




### TRUNCATION ERROR OF THE COMPOUND SIMPSON'S RULE ($E_{CS}$)




(Algebraic sum)

- $$E_{CS} = \sum_{i=1}^{M} \underbrace{-\frac{h^5}{90} f^{4)}(\xi_i)}_{E_{SS,i}} = \quad (\text{f.s. } \xi_i \in \text{ ith-subinterval if } f \in C^4([a,b]))$$

$$= \frac{-h^5}{90} M \frac{\sum_{i=1}^{M} f^{4)}(\xi_i)}{M} = \quad (\exists\ \xi \in [a,b] \ / \ f^{4)}(\xi) = \overline{f^{4)}})$$

$$= -\frac{h^{\cancel{5}\,4}}{90} \cdot \frac{(b-a)}{2K} \cdot \overline{f^{4)}} = \longleftarrow \boxed{\text{WEIERSTRASS'S INTERMEDIATE VALUE}}$$

$$= \underbrace{\frac{-h^{4}}{180}(b-a)\, f^{4)}(\xi)}_{O(h^4)}$$

occ

$$|E_{CS}| = \left| -\frac{h^4}{180}(b-a)\, \cancel{f^{4)}}^{\,MAX}(\xi) \right| \ \dots$$

## ERROR PROPAGATION. AMPLIFICATION FACTOR (AF)

- $Q = \sum_{i=0}^{n} w_i f(x_i) = \sum_{i=0}^{n} w_i f_i$

* Instead of $f_i$ we have perturbed values $\bar{f}_i$ ("all else equal / exact")

- $\bar{Q} = \sum_{i=0}^{n} w_i \bar{f}_i$ $\qquad |f_i - \bar{f}_i| \leq \varepsilon \ \forall i$

$$|E_r| = |Q - \bar{Q}| = \left|\sum_{i=0}^{n} w_i f_i - \sum_{i=0}^{n} w_i \bar{f}_i\right| = \left|\sum_{i=0}^{n} w_i (f_i - \bar{f}_i)\right| \leq \sum_{i=0}^{n} |w_i| \, |(f_i - \bar{f}_i)| \leq \boxed{|E_r| \leq \varepsilon \underbrace{\left(\sum_{i=0}^{n} |w_i|\right)}_{AF}}$$

- $\int_a^b f(x)\,dx = \int_a^b 1\,dx = b-a = Q = \sum_{i=0}^{n} w_i f(x_i) = \sum_{i=0}^{n} w_i \cdot 1 = \underline{\sum_{i=0}^{n} w_i = (b-a)}$

$$(b-a) = \sum_{i=0}^{n} w_i = \sum_{i=0}^{n+} w_i + \sum_{i=0}^{n-} w_i = \sum^{+} |w_i| - \sum^{-} |w_i|$$

- $AF = \sum |w_i| = \sum^{+} |w_i| + \sum^{-} |w_i| = (b-a) + \sum^{-} |w_i| + \sum^{-} |w_i| = \boxed{(b-a) + 2\sum^{-} |w_i| = AF}$

* If all $w_i > 0 \longrightarrow AF = b-a$ and $Q$ is said to be "STABLE".

## THEOREM ABOUT CONVERGENCE OF QUADR. RULES WITH POSITIVE COEFFICIENTS

* Assume there is an algorithm generating sets of 1, 2, 3, ..., nodes.

$$\begin{cases} \{x_0^0\} \\ \{x_0^1, x_1^1\} \\ \{x_0^2, x_1^2, x_2^2\} \\ \dots \\ \{x_0^n, x_1^n, x_2^n, \dots, x_n^n\} \\ \dots \end{cases}$$

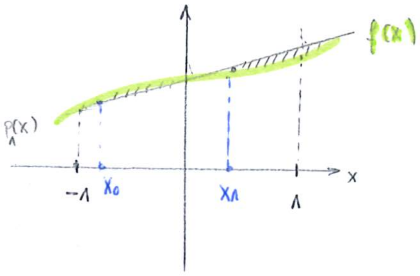
- $f \in C([a,b])$
- And also assume all corresponding interpolatory weights are $>0 \rightarrow w_i > 0$
- Call $Q^n = \sum_{i=0}^{n} w_i^n f(x_i^n)$

↳ Then: $\boxed{\lim_{n\to\infty} Q^n = I = \int_a^b f(x)\,dx}$ (converges to $I$)

### SIMILAR THEOREM WITH COMPOUNDS RULES

If $f \in C([a,b])$, calling $Q_c^M$ = the value given by a compound rule with positive coefficients and $M$ subintervals, then: $\boxed{\lim_{M\to\infty} Q_c^M = I = \int_a^b f(x)\,dx}$

## GAUSS – LEGENDRE RULE OF 2 NODES (GAUSS QUADRATURE)




• Nodes: not necessarily equally spaced.

$\int_{-1}^{1} f(x)\,dx$ ;

(EXACT INT)

EXACT INT. of $f(x)=1$ : $\int_{-1}^{1} \underbrace{1}_{f(x)} dx = 2 = \underline{W_0 \cdot 1 + W_1 \cdot 1} + \cancel{E}^{\,0}$

EXACT INT. OF $f(x)=x$ : $\int_{-1}^{1} x\,dx = 0 = \underline{W_0 \cdot x_0 + W_1 \cdot x_1}$

EXACT INT. OF $f(x)=x^2$ : $\int_{-1}^{1} x^2 dx = \frac{2}{3} = \underline{W_0 x_0^2 + W_1 x_1^2}$

EXACT INT OF $f(x)=x^3$ : $\int_{-1}^{1} x^3 dx = 0 = \underline{W_0 x_0^3 + W_1 x_1^3}$

$*\ \boxed{Q_{GL2} = Q} = \sum_{i=0}^{1} w_i f(x_i) = \boxed{W_0 f(x_0) + W_1 f(x_1)}$

$\begin{cases} n=1 \\ N \geq 3 \end{cases}$ ; $\boxed{N = 3 = 2n+1} = \underset{2\text{nodes}}{2\underline{n}-1}$

SOL: $W_0 = W_1 = 1$ / $x_0 = -1/\sqrt{3}$ / $x_1 = 1/\sqrt{3}$
$W_0 = W_1 = 1$ / $x_0 = 1/\sqrt{3}$ / $x_1 = -1/\sqrt{3}$
BOTH PRODUCE THE EXACT SAME QUADR. RULE

( 4 degrees of freedom ↗ $W_0, W_1$ ↘ $x_0, x_1$ )

$Q = 1 \cdot f(-1/\sqrt{3}) + 1 \cdot f(1/\sqrt{3})$

→ FORMALLY : 2 diff. sol
→ 1 only Q

## GAUSS - LEGENDRE RULE OF 3 NODES

(6 degrees of freedom ↗ $W_0, W_1, W_2$ ↘ $x_0, x_1, x_2$ )

EXACT INT OF :

$f(x)=1$ : $\int_{-1}^{1} 1\,dx = 2 = W_0 \cdot 1 + W_1 \cdot 1 + W_2 \cdot 1$

$f(x)=x$ = $\int_{-1}^{1} x\,dx = 0 = W_0 x_0 + W_1 x_1 + W_2 x_2$

$f(x)=x^2$ = $\int_{-1}^{1} x^2 dx = 2/3 = W_0 x_0^2 + W_1 x_1^2 + W_2 x_2^2$

$f(x)=x^3$ = $\int_{-1}^{1} x^3 dx = 0 = W_0 x_0^3 + W_1 x_1^3 + W_2 x_2^3$

$f(x)=x^4$ = $\int_{-1}^{1} x^4 dx = \frac{2}{5} = W_0 x_0^4 + W_1 x_1^4 + W_2 x_2^4$

$f(x)=x^5$ = $\int_{-1}^{1} x^5 dx = 0 = W_0 x_0^5 + W_1 x_1^5 + W_2 x_2^5$

SOL: WE WOULD OBTAIN 6 DIFF. SOLUTIONS, FORMALLY ⟶ SAME Q FOR ALL

$\begin{cases} n=2 \\ N = 5 = \underset{3\text{nodes}}{2\underline{n}-1} \end{cases}$

$*$ $n+1$ nodes $\{x_0, \dots, x_n\}$ ; $n$ dif $= 2(n+1) = \underline{2n+2}$ CONDITIONS and $\underline{N = 2n+1}$

• REAL, DISTINCT AND IN THE INTERVAL a-b

## SCALAR PRODUCT AND ORTHOGONAL POLYNOMIALS OF FUNCTIONS

- SCALAR PRODUCT: $\langle f, g \rangle = \int_a^b f(x)\, g(x)\, w(x)\, dx$ w/ $w(x) \geq 0 \quad \forall x \in (a,b)$

- ORTHOGONALITY: f, g orthogonal $(f \perp g) \iff \langle f, g \rangle = 0$

- Legendre pols $\{\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x), \dots\}$ are orthogonal in $[-1,1]$ WR to $w(x) \equiv 1$

$$\int_{-1}^{1} \varphi_i(x)\varphi_j(x)\, dx = 0 \quad \forall i,j \;/\; i \neq j$$

DON'T NEED TO KNOW HOW TO APPLY IT ON EXCERCISES

* we are going to orthogonalize using Grand Schmidt method (ALGEBRA)

$\{1, x, x^2, x^3, \dots, \dots\}$ ORTHOGONALIZE

$\{1, \varphi_1(x), \varphi_2(x), \dots\}$ NORMALIZE → most usual normalization: $\underline{\varphi_i(1) = 1}$

$\{1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x, \dots\}$

→ THE MOST IMPORTANT THING IS THAT THEY ARE ORTHOGONAL

- ROOTS: $x = \pm\sqrt{\frac{1}{3}}$ ; $x = \pm\sqrt{3/5}$ ... → same as GAUSS-LEGENDRE RULE !!

Ex)

$\{1, x, \frac{3}{2}x^2 - \frac{1}{2}\} \equiv$ Base of $\mathbb{P}_2$ ORTHOGONAL TO all polynomials of degree equal or less than itself.

- ORTHOGONAL POL.
- FAMILY OF ORTHOG.

## FUNDAMENTAL THEOREM OF GAUSS-LEGENDRE QUADRATURE

If $f \in \mathbb{P}_{2n+1}$ and $Q_{GL}$ is the interpolatory quadrature rule with nodes equal to the Legendre polynomial $\varphi_{n+1}(x)$ of degree $n+1$, then $f$ is integrated exactly by $Q_{GL}$ (that rule):

$$\boxed{\int_{-1}^{1} f(x)\,dx = \sum_{i=0}^{n} w_i f(x_i) + \cancel{E}}$$

PROOF:

Let $f$ be any polynomial of degree $\leq 2n+1$.

Divide it by $\varphi_{n+1}(x)$ (Leg. pol of degree $n+1$)

$P_{2n+1}(x)$ $\quad\Big|\ \varphi_{n+1}(x)$

$\psi_n(x)$ $\quad a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = q_n(x) \in \mathbb{P}_n$

EVERY TIME ADDING A MONOMIAL,

WE REDUCE DE DEGREE OF THE REMAINDER

$$\boxed{P_{2n+1}(x) = q_n(x) \cdot \varphi_{n+1}(x) + \psi_n(x)}$$

IT CAN BE 0

$$\boxed{\int_{-1}^{1} P_{2n+1}(x)\,dx} = \cancel{\int_{-1}^{1} q_n(x)\,\varphi_{n+1}(x)}^{\,0} + \int_{-1}^{1} \psi_n(x) = \boxed{\int_{-1}^{1} \psi_n(x) = \underbrace{\sum_{i=0}^{n} w_i \cdot \psi(x_i)}_{\text{FOR EVERY INTERP. Q OF } \underbrace{n+1 \text{ NODES}}_{\text{FOR ANY}}}}$$

Choose the nodes : $x_i = \xi_i =$ roots of $\varphi_{n+1}(x)$

$$P_{2n+1}(\xi_i) = q_n(\xi_i)\,\cancel{\underbrace{\varphi_{n+1}(\xi_i)}_{\text{IS A ROOT}}}^{\,0} + \psi_n(\xi_i) \longrightarrow P_{2n+1}(\xi_i) = \psi_n(\xi_i)$$

$$\boxed{\int_{-1}^{1} P_{2n+1}(x)\,dx = \sum_{i=0}^{n} w_i\, P_{n+1}(\xi_i)}$$

## $Q_{GL2}$ VIA LEGENDRE POLYNOMIAL $\varphi_2(x)$

17/03/21

$$\varphi_2(x) = \frac{3x^2}{2} - \frac{1}{2}$$

1ST) ROOTS OF $\varphi_2(x)$ = NODES

$$\frac{3x^2}{\cancel{2}} - \frac{1}{\cancel{2}} = 0 \rightarrow x = \pm\sqrt{1/3}$$

2ND) WEIGHTS $\begin{cases} \text{Integrate Lagrange based function} \\ \text{Indet. Coe.} \end{cases}$

· EXAT INT. OF

$x^0 = 1 \qquad \int_{-1}^{1} 1\,dx = 2 = W_0 \cdot 1 + W_1 \cdot 1$

$x^1 = x \qquad \int_{-1}^{1} x\,dx = 0 = W_0\left(-1/\sqrt{3}\right) + W_1\left(1/\sqrt{3}\right)$

WE ONLY NEED THIS 2 IN THIS CASE

$\vdots \qquad \vdots$

## $E_{GL2}$ (ERROR TERM OF GAUSS-LEGENDRE RULE OF 2 NODES)

* $E = k \cdot f^{(N+1)}(\xi)$ f.s. $\xi \in (-1,1)$ if $f \in C^{N+1}([-1,1])$

$N = 2n+1 = 3 = 2n_n - 1$

$f(x) = x^4 \longrightarrow f^{(4)} = 4! = 24$

EXACT INT. OF:

$x^0 = 1 \to \int_{-1}^{1} 1\,dx = 2 = W_0 + W_1$

$x^1 : x \to \int_{-1}^{1} x\,dx = 0 = -\frac{1}{\sqrt{3}} W_0 + \frac{1}{\sqrt{3}} W_1$

$x^2 \to \int_{-1}^{1} x^2 dx = Q$

$x^3 \to \int_{-1}^{1} x^3 dx = \frac{2}{5} = Q + E = \left(\frac{-1}{\sqrt{3}}\right)^4 \cdot 1 + 1 \cdot \left(\frac{1}{\sqrt{3}}\right)^4 + k \cdot \underbrace{24}_{f^{(N+1)}(\xi)}$

## $[-1,1] \to [a,b]$

$x = \underbrace{\frac{a+b}{2}}_{m} + \underbrace{\frac{b-a}{2}}_{J = dx/d\xi} \xi$

* $\int_a^b f(x)\,dx = \int_{-1}^{1} \underbrace{f\left(\frac{a+b}{2} + \frac{b-a}{2}\xi\right)\left(\frac{b-a}{2}\right)}_{g(x)} d\xi \simeq Q = \frac{5}{9} g(-\sqrt{3/5}) + \frac{8}{9} g(0) + \frac{5}{9} g(\sqrt{3/5}) =$

$= \underbrace{\boxed{\frac{5}{9} \cdot \frac{b-a}{2}}}_{W_0} f\left(\underbrace{\boxed{\frac{a+b}{2} + \frac{b-a}{2}(-\sqrt{3/5})}}_{X_0}\right) + \underbrace{\boxed{\frac{8}{9} \cdot \frac{b-a}{2}}}_{W_1} \cdot f\left(\underbrace{\boxed{\frac{a+b}{2}}}_{X_1}\right) + \underbrace{\boxed{\frac{5}{9}\frac{b-a}{2}}}_{W_2} f\left(\underbrace{\boxed{\frac{a+b}{2} + \frac{b-a}{2}\sqrt{\frac{3}{5}}}}_{X_2}\right)$

## COMPOUND GAUSS (IN THE EXCERCISES)

## TERMINATION CRITERIUM

" Find $\int_a^b f(x)\,dx$ with "error" less than ... "

↳ TERMINATION CRITERIUM

## GAUSS - CHEBYSHEF FORMULAS OF THE 1ST KIND (RULES)

$[a,b] = [-1,1]$ w $w(x) = 1/\sqrt{1-x^2}$

$\int_{-1}^{1} f(x) \cdot \frac{1}{\sqrt{1-x^2}} dx$ → nodes : the same as the ones of CH1.

$T_{n+1}(x)$ $\underbrace{\{x_0, x_1, x_2, \dots, x_n\}}_{n+1 \text{ NODES}}$

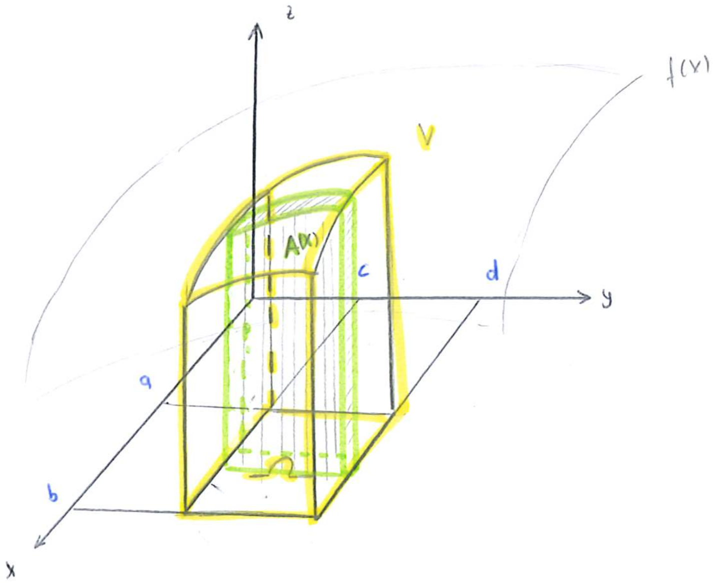
↳ weights : they are equal and their sum is $\pi$ ; $W_i = \frac{\pi}{n+1} \to \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \pi = \Sigma W_i$

## GAUSS - LAGUERRE RULES

$$\int_0^\infty f(x)\, e^{-x}\, dx$$

| n | $x_i$ | $w_i$ |
|---|---|---|
| 0 | x | x |
| 1 | x, x | x, x |
| 2 | x, x, x | x, x, x |

## NUMERICAL APPROX. OF SOME DOUBLE INT.



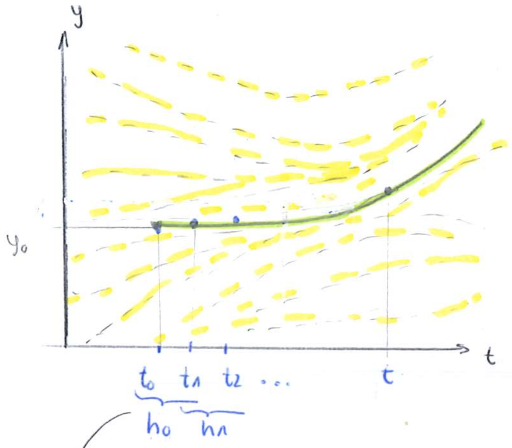

$$I \equiv V = \iint_\Omega f(x,y)\, dx\,dy = \int_a^b A(x)\, dx$$

$$A(x) = \int_c^d f(x,y)\, dy$$

EXACT WHEN : N of $f(x,y)$ $\leq$ N of the RULE we are using

# CH. 4. INITIAL VALUE PROBLEMS

- $t \equiv$ independent variable (time)
- $y' = f(t,y)$ (simplest $\underbrace{\overset{\text{ORD. DIFF. EQ}}{\text{ODE}}}_{\text{EDA}}$)
- slope field (campo de pendientes) →




- IVP (INIT. VAL. PROB) $\begin{cases} \text{ODE} & y' = f(t,y) \quad \text{(ORD. DIFF. EQ)} \\ \text{IC} & y(t_0) = y_0 \quad \text{(INIT. COND.)} \end{cases}$

* $\dfrac{y_1 - y_0}{h_0} = f(t_0, y_0)$ ; $h_0 = t_1 - t_0$

$$\boxed{y_{k+1} = y_k + f(t_k, y_k)\, h_k}$$ EULER'S METHOD (E)

* $y^{n)} = f(t, \underset{u_1}{y}, \underset{u_2}{y'}, \underset{u_3}{y''}, \dots, \underset{u_n}{y^{n-1)}})$

$$\begin{cases} u_1' = y' = u_2 \\ u_2' = u_3 = y'' \\ \vdots \\ u_{n-1}' = u_n = y^{n-1)} \\ u_n' = f(t, u_1, u_2, \dots, u_n) = y^{n)} \end{cases}$$



## NOTATION FOR SYSTEMS OF ODE

S. ODE-s $\begin{cases} y_1' = f_1(t, y_1, y_2, \dots, y_n) \\ y_2' = f_2(t, y_1, y_2, \dots, y_n) \\ \vdots \\ y_n' = f_n(t, y_1, y_2, \dots, y_n) \end{cases}$

ICs (INIT. COND) $\begin{cases} y_1(t_0) = y_{10} \\ y_2(t_0) = y_{20} \\ \vdots \\ y_n(t_0) = y_{n0} \end{cases}$

* UNKNOWNS: $y_1(t), y_2(t), \dots, y_n(t)$

$y' = f(t, y) \longrightarrow \begin{cases} y_1' = f_1(t, y) \\ y_2' = f_2(t, y) \\ \vdots \quad\quad \vdots \\ y_n' = f_n(t, y) \end{cases} \xrightarrow{\text{(more compact)}} \boxed{y_n' = \underset{\sim}{f}(t, y)}$ S. ODE-s (COMPACT WAY)

$y = (y_1, y_2, \ldots, y_n)$

IC-s : $\boxed{\underset{\sim}{y}(t_0) = y_0}$ (COMPACT WAY)

## LINEAR SYSTEM OF ODE-S (WITH CONSTANT COEFF.)

$$\begin{cases} y_1' = C_{11}(t)\, y_1 + C_{12}(t)\, y_2 + \ldots + C_{1n}(t)\, y_n + g_1(t) = f_1(t, y) \\ y_2' = C_{21}(t)\, y_1 + C_{22}(t)\, y_2 + \ldots + C_{2n}(t)\, y_n + g_2(t) = f_2(t, y) \\ \vdots \\ y_n' = C_{n1}(t)\, y_1 + C_{2n}(t)\, y_2 + \ldots + C_{nn}(t)\, y_n + g_n(t) = f_n(t, y) \end{cases}$$

MATRIX FORM :

$$\boxed{y' = \underbrace{\begin{pmatrix} C_{11}(t) & C_{12}(t) & \ldots & C_{1n}(t) \\ C_{21}(t) & C_{22}(t) & \ldots & C_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}(t) & C_{n2}(t) & \ldots & C_{nn}(t) \end{pmatrix}}_{J(t)\ \text{(JACOBIAN MATRIX)}} y + g(t)} \rightarrow \boxed{y' = J(t)\, y + g(t)}$$

COEFFS DEPEND ON $t$

* Linear system with $\underbrace{\text{constant}}_{J \text{ DOESN'T DEPEND ON } t}$ coeffs : $\boxed{y' = J \cdot y + g(t)}$

## JACOBIAN MATRIX OF A SYSTEM OF ODE-S

* DEFINITION : $J = \left( \dfrac{\partial f_i}{\partial y_j} \right)$ at $(i, j)$

* For a lin. system of ODE-s $J(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) & \ldots & C_{1n}(t) \\ C_{21}(t) & C_{22}(t) & \ldots & C_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}(t) & C_{n2}(t) & \ldots & C_{nn}(t) \end{pmatrix}$

# PICARD'S THEOREM OF ∃! SOLUTION TO INIT. VALUE PROB. (IVP)

* The IVP $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ has a UNIQUE solution if $f$, $\frac{\partial f}{\partial y} = f_y$ exist in

$R = \{(t,y) \,/\, (t_0 < t < t_f) \wedge (-\infty < y < \infty)\}$ and $\exists L \in \mathbb{R}^+ \,/\, \forall (t,y_1),(t,y_2) \in R$, $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$

(REGION)

some ⊕ constant

L = LIPSCHITZ → f is Lipschitian (si cumple lo de arriba)

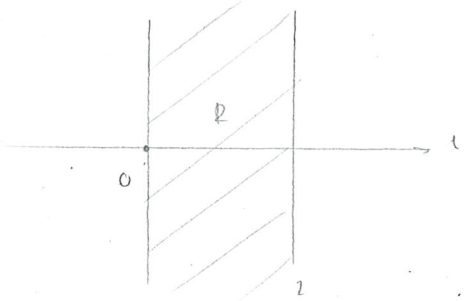
* If $f$, $f_y$ ∃ and $f_y$ is continuous in $R$ ⟹ $f$ is Lipschitian

31/03/21

EXC. 1

∃! sol of $\begin{cases} y' = 1 + t\sin(ty) \\ y(0) = y_0 \end{cases}$ $t \in [0,2]$

$1 + t\sin(ty) \to f(t,y)$




$f(t,y) = 1 + t\sin(ty) \in C(R)$

$\frac{\partial f}{\partial y} = f_y(t,y) = t^2\cos(ty) \in C(R)$

$t^2 \in [0,4]$, $\cos(ty) \in [-1,1]$

$f_y \in [-4,4]$ bounded ⟹ $f$ is Lipschitian $(L=4)$

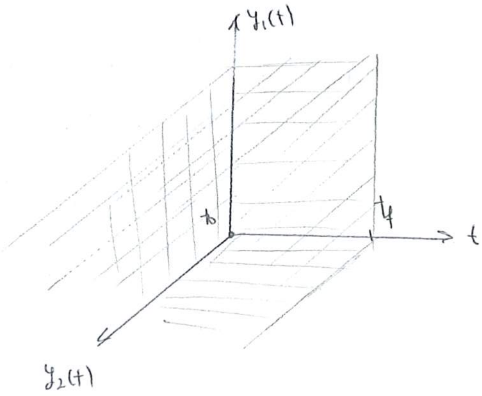
## THEOREM OF ∃! OF SOL. TO IVP (SYSTEMS) (Picard)

THE REGION IS A VOLUME

* Let IVP be $\underline{y}' = \underline{f}(t, \underline{y})$ with $\underline{y}(t_0) = \underline{y}_0$ $(t_0 \le t \le t_f)$.

* Let $R = \{(t, \underline{y}) / \ t_0 \le t \le t_f, \ \underline{y} \in \mathbb{R}^n\} \in \mathbb{R}^{n+1}$

$$\underline{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$




* If $\underline{f} \in C(R)$ and $\exists L \in \mathbb{R}^+ /$

$\forall (t, \underline{y}_1), (t, \underline{y}_2)$, $\| \underline{f}(t, \underline{y}_1) - \underline{f}(t, \underline{y}_2) \| \le L \| (t, \underline{y}_1) - (t, \underline{y}_2) \|$ (with $\| \ \|$ any norm in $\mathbb{R}^{n+1}$),

then IVP has a UNIQUE SOLUTION. $\longrightarrow$ $\underline{f}(t, \underline{y})$ is Lipschitian.

THEOREM: If $f_{y_1}, f_{y_2}, \dots, f_{y_n}$ are continious and bounded in $R$, then $\underline{f}(t, \underline{y})$ is Lipschitian.

### EXC. 3

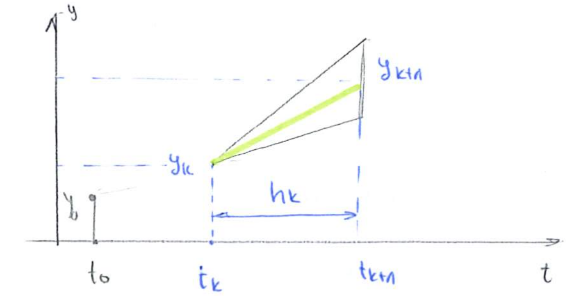
APPLY EULER'S METHOD, 5 STEPS WITH h=0'1

to SOLVE $\begin{cases} y' = (1+t)\, y^2/2 = f(t, y) \\ y(0) = 1 \end{cases}$

```
f = @(t,y) (1+t)* y^2 /2 ;

tø = 0 ; yø = 1 ; h = 0.1 ;

% y_{k+1} = y_k + f(t_k, y_k)*h_k
```

```
y1 = yø + f(tø, yø) * h        (= 1.05)

t1 = tø + h;

y2 = y1 + f(t1, y1) * h        (= 1.1106375)

t2 = t1 + h; t3 = t2 + h; t4 = t3 + h; t5 = t4 + h

y3 = y2 + f(t2, y2) * h;

y4 = y3 + f(t3, Y3) * h;

y5 = y4 + f(t4, Y4) * h;
```

## ENHANCED EULER'S (HENN'S) METHOD (E)




$$\boxed{y_{k+1} = y_k + f(t_k, y_k)\cdot h_k}$$ (EULER)

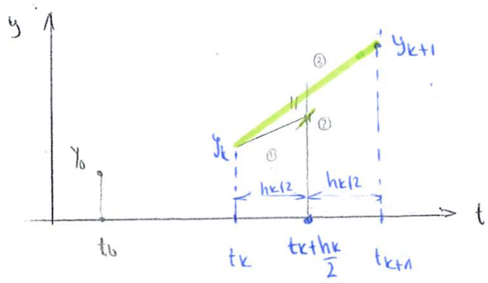
AVERAGE SLOPE

$$\boxed{y_{k+1} = y_k + \frac{f(t_k, y_k) + f(t_k + h_k, y_k + f(t_k, y_k)\cdot h_k)}{2}\cdot h_k}$$ (ENHANCED EULER)

$$\begin{cases} \boxed{k_1 = f(t_k, y_k)h_k} \\ \boxed{k_2 = f(t_k + h_k, y_k + k_1)} h_k \end{cases} \longrightarrow \boxed{y_{k+1} = y_k + \frac{k_1 + k_2}{2}}$$

(RUNGE-KUTTA STYLE)
(RK 2)

## MODIFIED EULER METHOD (MIDPOINT) (E)



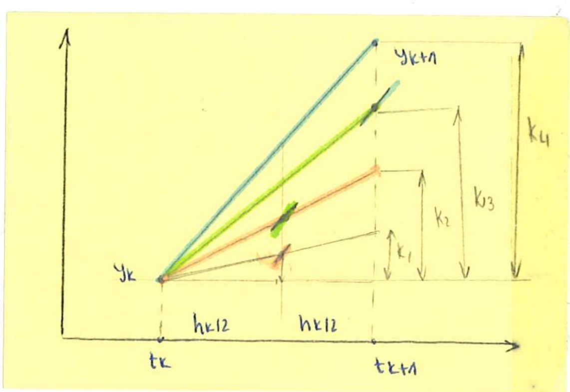

$$\boxed{y_{k+1} = y_k + f(t_k + h_k/2, y_k + f(t_k, y_k) h_k/2) h_k}$$

(ADVANCED FORMULA FOR MIDPOINT METHOD)

$$\begin{cases} \boxed{k_1 = f(t_k, y_k) h_k} \\ \boxed{k_2 = f(t_k + h_k/2, y_k + \frac{k_1}{2})} h_k \end{cases} \longrightarrow \boxed{y_{k+1} = y_k + k_2}$$

(RUNGE-KUTTA STYLE)

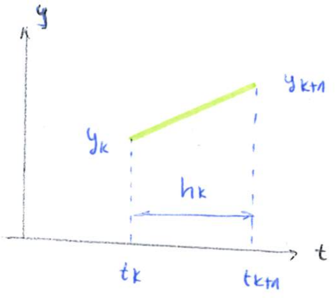
## RUNGE-KUTTA METHOD OF ORDER 4 (RK4)




$$\begin{cases} k_1 = f(t_k, y_k) h_k \\ k_2 = f(t_k + h_k/2, y_k + k_1/2) h_k \\ k_3 = f(t_k + h_k/2, y_k + k_2/2) h_k \\ k_4 = f(t_k + h_k, y_k + k_3) h_k \end{cases}$$

$$\boxed{y_{k+1} = y_k + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}}$$

## BACKWARD EULER METHOD (I)




$$\boxed{y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\,h_k}$$

$y_{k+1}^{(1)}$

unknow on both sides → IMPLICIT

* unconditionally stable (ADVANTAGE)

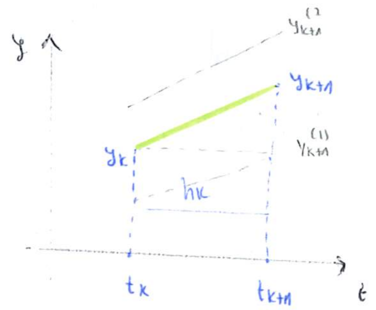
* computational cost (DISADVANTAGE)

## IMPLICIT VS. EXPLICIT METHOD

(I) · unknow in both sides → IMPLICIT : we will iterate (↑↑ COMP. COST BUT ↑↑ STABILITY)

(E) we can obtain it directly with the data we have → EXPLICIT (EULER)

## TRAPEZOIDAL METHOD (I)




$$\boxed{y_{k+1} = y_k + \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}\,h_k}$$

{ ∞ COMPUT. COST
{ OC = 2 (ORDER OF CONV.) → OPTIMAL

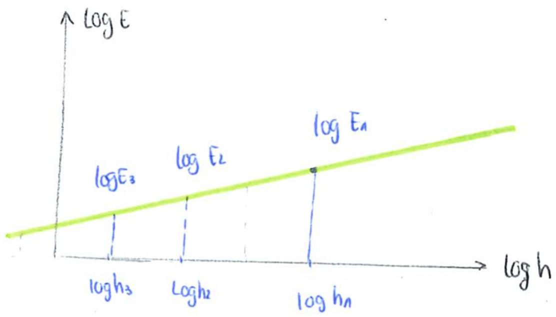
## ORDER OF CONVERGENCE / ORDER OF PRECISION : O

* COMPUTER EXAM

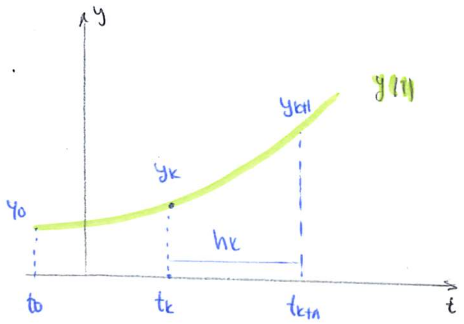
$E \simeq k \cdot h^{O}$ (ERROR)

ON OCTAVE : $\log = \ln$ ($\log_{10} = \log$)

$$\log(E) = \log(k\,h^{O}) = \log(k) + O\log(h)$$



## TAYLOR BASED METHODS (E)




$$\boxed{y(t_{k+1}) = y(t_k) + y'(t_k)h_k + \frac{y''(t_k)h_k^2}{2!} + \frac{y'''(t_k)h_k^3}{3!} + \dots}$$ (TAYLOR Σ'S EXPANSION)

$$y' = f < \begin{matrix} t \\ y \end{matrix}$$

$$y'' = \frac{\partial^2 y}{\partial t^2} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = f_t + f_y \cdot f$$

...

(NOT ALWAYS USEFUL) (OC=2)

$$\boxed{y_{k+1} = Y_k + f(t_k, y_k)h_k + \frac{f_t(t_k,y_k) + f_y(t_k,y_k)\cdot f(t_k,y_k)}{2!}h_k^2 + \dots}$$

## CHARACTERISTICS OF RUNGE-KUTTA METHOD

* Explicit
* Orders: 1, 2, 3, ..., 10, ..., 20, ...
* computational cost: number of evaluations of f per step = order until order 4 (n.e.f/st)
  for ↑↑ orders: n.e.f/st > ORDER
* "popular"
* Not too good for "stiff" (RÍGIDO) systems of ODE's

**EXC. 6** APPLY THE MODIFIED EULER METHOD: h=0'1
$$\begin{cases} y' = -y + t + 1 \\ y(0) = 1 \end{cases} \quad t \in [0,1]$$

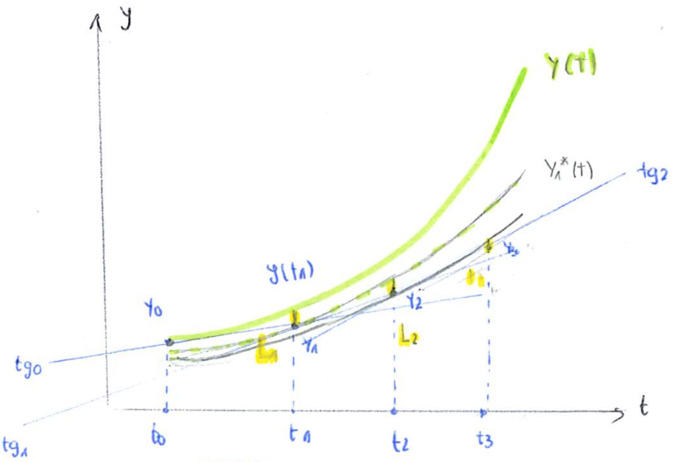
```
f = @(t,y) - y + t + 1;
tØ = Ø ; h = 0.1 ; t1 = h ; t2 = 2*h; t3 = 3*h;
yØ = 1

y1 = y0 + f(t0 + h/2 , y0 + f(t0,y0) * h/2) * h
y2 = y1 + f(t1 + h/2 , y1 + f(t1,y1 * h/2) * h
...
```

EXC. 7 RK4 : $\begin{cases} y' = t+y \\ y(0) = 0 \end{cases}$ $t \in [0,1]$ $h = 0.1$

```
f = @(t,y) t + y ; h = 0.1;

t0 = 0 ; t1 = h ; t2 = 2*h ; ... ; t10 = 10*h ;

y0 = 0 ;

k1 = f(t0, y0)*h ;

k2 = f(t0 + h/2 , y0 + k1/2)*h ;

k3 = f(t0 + h/2 , y0 + k2/2)*h ;

k4 = f(t0 + h/2 , y0 + k3/2)*h ;

y1 = y0 + (k1 + 2*k2 + 2*k3 + k4)/6 ;
```

↳ OTRA VEZ PARA (1) : $k_1 = f(t_1, y_1)$ ...

## STABILITY OF ODES



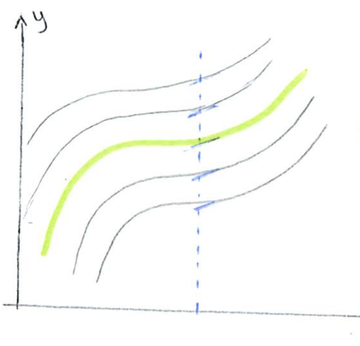

* $L_i$ ≡ LOCAL ERRROR : attributable to step i

$E > L_1 + L_2 + L_3 + \dots$ (IN OUR DRAWING)

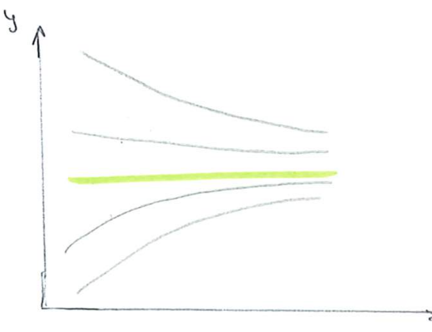
$\boxed{E_k \gtreqless \sum_{i=1}^{k} L_i}$

GREATER, EQUAL OR LOWER

$E > \Sigma L_i$ (HACIENDO tg A LA CURVA POR $Y_0$ ...)

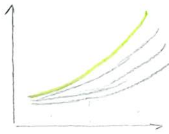



$E = \Sigma L_i$



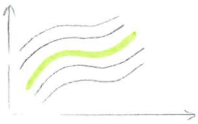

$E < \Sigma L_i$

UNSTABLE ODE : $E > \sum L_i \longrightarrow \dfrac{\partial f}{\partial y} > 0$



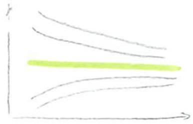
$J = \left( \dfrac{\partial f_i}{\partial y_i} \right)$

NEUTRAL ODE : $E = \sum L_i \longrightarrow \dfrac{\partial f}{\partial y} = 0$



* SINGLE ODE : $J = \dfrac{\partial f}{\partial y} = f_y$

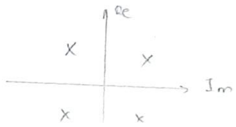
STABLE ODE : $E < \sum L_i \longrightarrow \dfrac{\partial f}{\partial y} < 0$



* THE STABILITY DEPENDS ON THE ODE ITSELF, NOT IN THE SOLVING METHOD

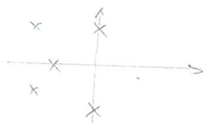
## STABILITY OF SYSTEMS OF ODE'S

UNSTABLE S.ODEs :

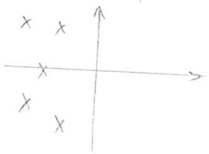



X = EIGENVALUES OF THE S ODE

NEUTRAL S.ODEs :



STABLE S.ODEs

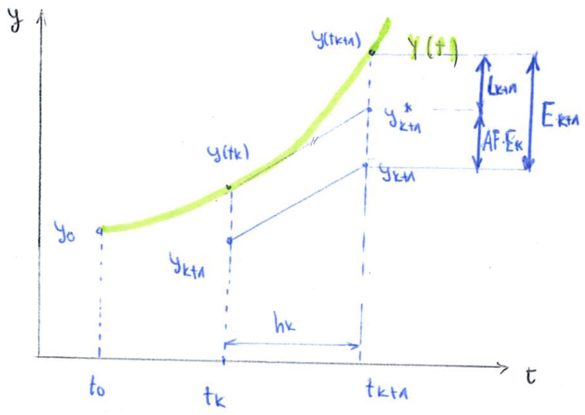


RÍGIDO

* A stm of ODEs is stiff if it is a system of ODEs and it is stable and the real and/or imaginary parts of the eig (J) are "very" disproportionate

↳ WE CALCULATE THE EIG(J) AND CHECK THE STIFFNESS

## TRUNCATION ERRORS (EULER'S METHOD)




$y^*_{k+1}$ = point of arrival with Euler's method if you start from $(t_k, y(t_k))$

· EULER: $y_{k+1} = y_k + f(t_k, y_k) h_k$

$y^*_{k+1} = y(t_k) + f(t_k, y(t_k)) h_k$

$f \in C^2$

· LOCAL ERROR: $\boxed{L_{k+1} = \left[ y(t_{k+1}) \right] - \left[ y^*_{k+1} \right]} = \left[ \cancel{y(t_k)} + \cancel{y'(t_k) h_k} + \dfrac{y''(\xi)}{2!} h_k^2 \right] - \left[ \cancel{y(t_k)} + \cancel{f(t_k, y(t_k)) h_k} \right] = \boxed{\dfrac{y''(\xi) h_k^2}{2!}}$

↳ $\boxed{L_{k+1} = O(h_k^2)}$

$\boxed{OC = LOC - 1}$ (ORD. OF CONV. = LOCAL ORD. CONV. − 1)

Euler method is of order 1

## AMPLIFICATION FACTOR OF EULER'S METHOD (AF)

WE ADD (+) ^ (−)

· GLOBAL ERROR :

$$\boxed{E_{k+1}} = y(t_{k+1}) - y_{k+1} = \underbrace{y(t_{k+1}) - y^*_{k+1}}_{L_{k+1}} + (y^*_{k+1}) - (y_{k+1}) =$$

$$= L_{k+1} + (y(t_k)) + f(t_k, y(t_k))h_k) - (y_k + f(t_k, y_k)h_k) = L_{k+1} + E_k + (f(t_k, y(t_k)) - f(t_k, y_k))h_k =$$

$$= L_{k+1} + E_k + f_y(t_k, \xi) E_k \cdot h_k = \boxed{L_{k+1} + E_k \underbrace{(1 + J(t_k, \xi)h_k)}_{AF}}$$

* $E_{k+1} = L_{k+1} + AF \cdot E_k \longrightarrow \boxed{AF = (1 + J(t_k, \xi)h_k)}$

$|AF| = 1 \longrightarrow E = \sum L_i$ (NEUTRAL)

$|AF| < 1 \longrightarrow E < \sum L_i$ (STABLE)

$|AF| > 1 \longrightarrow E > \sum L_i$ (UNSTABLE)

20/04/21

* Absolute stability condition of Euler method: $|AF| < 1$

$-1 < AF < 1$

$-1 < 1 + J \cdot h < 1$

$|AF| < 1 \longleftrightarrow$ METHOD STABLE

$-2 < Jh < 0$

$$\boxed{Jh \in (-2, 0)}$$

STABILITY INTERVAL

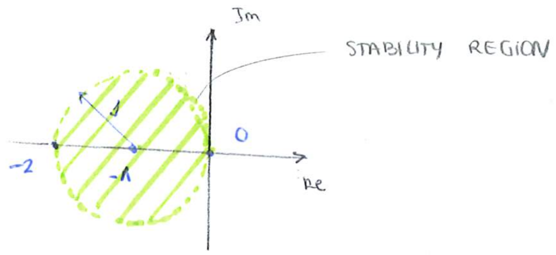
If ODE is unstable : $J > 0 \longrightarrow J \cdot h > 0 \longrightarrow Jh \notin (-2, 0) \longrightarrow$ METHOD unstable

If ODE is stable : $J < 0 \longrightarrow -2 < Jh < 0 \longrightarrow \boxed{h < \frac{-2}{J} = h_t}$ : threshold step size for absolute stability of the method.

## STABILITY OF EULER'S METHOD WITH S ODES

Euler's method will be stable if and only if : $\boxed{eigs(Jh) = h \cdot eigs(J)}$

(AS IN LIN. ALGEBRA)

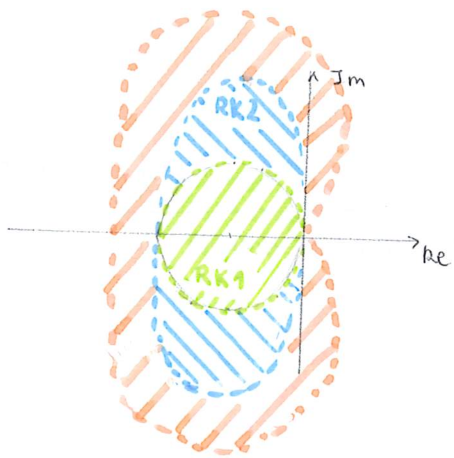



## ABSOLUTE STABILITY INTERVAL OF ANY METHOD

By definition, it is the interval where J·h must lie for the method to be stable solving an ODE.

## ABS. STAB. REGION OF ANY METHOD

Def: The region of the complex plane where eigs(J h) must lie for the method to be stable solving a SODEs.




* They are symmetric with respect to the Re axis

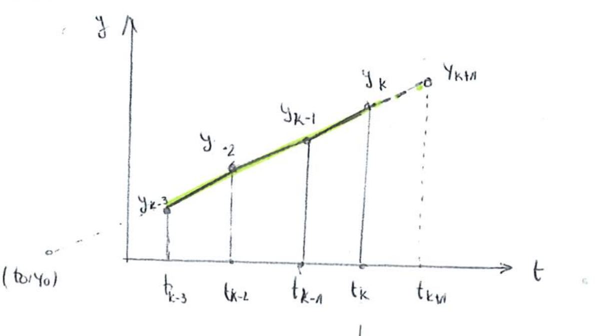
* solving system of real ODEs

$J = (f_y)$ ; $eigs(J) = f_y$

$eigs(Jh) = h \cdot f_y$

## Ⓔ MULTISTEP METHODS (LMM) (LINEAR MULTISTEP METHOD)

AB4

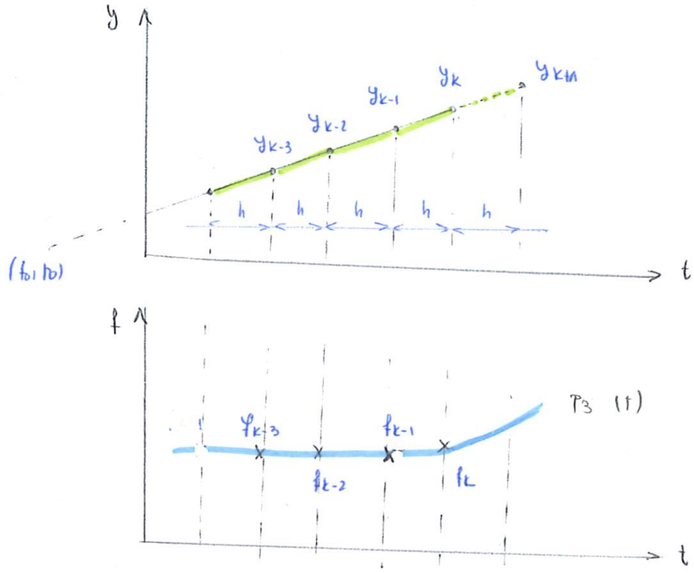



LAST VALUE OBTAINED (LAST STEP)

$y' = f'(t, y)$

- ABn = Adams Bashforth methods are EXPLICIT multistep methods

Multistep methods = methods "with memory"
single-step methods = methods "without memory"




$$\begin{cases} y' = f(t,y) \\ f_i = f(t_i, y_i) \end{cases}$$

$$\boxed{y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} y'(t)\,dt}$$

(BARROW'S LAW)
(EXACT)

DISCRETIZE

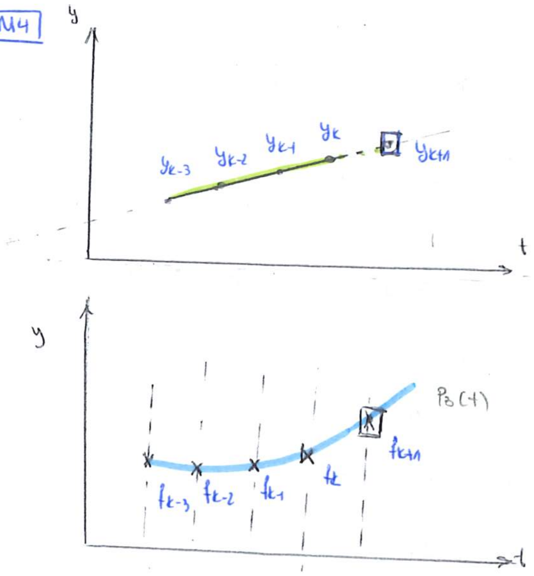
$$\boxed{y_{k+1} = y_k = \int_{t_k}^{t_{k+1}} f_3(t)\,dt}$$

$$\boxed{y_{k+1} = y_k + \frac{h}{24}\left(55 f_k - 59 f_{k-1} + 37 f_{k-2} - 9 f_{k-3}\right)}$$

(AB4 advanced formula)

* To calculate $y_{k+1}$ we are not increasing the computational cost
↳ not extra

AM4



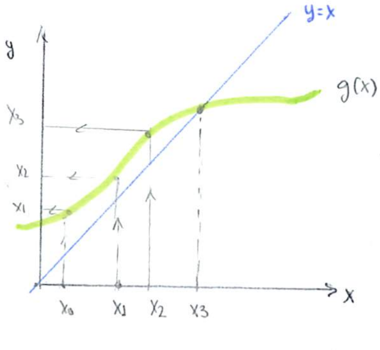

• AM: Adams-Moulton method of order n (AMn) is an IMPLICIT multistep method.

$$\boxed{y_{k+1} = y_k + \frac{h}{24}\left(9 f(t_{k+1}, y_{k+1}) + 19 f_k - 5 f_{k-1} + f_{k-2}\right)}$$

(AM4 advanced formula)

* comp. cost: $\infty$

## FIXED POINTS ITER. (rev)




$$\begin{cases} y_{k+1} = g(y_{k+1}) \\ x = g(x) \end{cases}$$

→ cond. for convergence

$|g'(x)| < 1$ (and $x \in$ to that neighbourhood)

we suppose: $x_0 \rightarrow x_1 = g(x_0)$
$x_2 = g(x_1)$
$x_3 = g(x_2)$
$\vdots$
$x_{i+1} = g(x_i)$

IT CONVERGES

EXC. 14

| $x_i$ | $y_i$ | $f_i$ |
|---|---|---|
| 0 | 0 | 0 |
| 0'2 | 0'02103 | 0'22103 |
| 0'4 | 0'09137 | 0'49137 |
| 0'6 | 0'22156 | 0'82156 |

4 POINTS → METHOD OF ORDER 4

Apply Adams PC method of order 4 for 2 more points of: $\begin{cases} y' = x+y \\ y(0) = 0 \end{cases}$ with $h = 0'2$, Scheme $PC(2\cdot 10^{-5})$ (Rounding 5 dec.)

NB AB4: $y_{n+1} = y_n + \frac{h}{24}(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3})$ pred.

AM4: $y_{n+1} = y_n + \frac{h}{24}(9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2})$ "corrected"

WITH OCTAVE NOTATION:

f = @(t, y) t+y;

y0 = 0;

h = 0.2;

t0 = 0; t1 = h; t2 = 2*h; t3 = 3*h; $\overbrace{\text{t4 = 4*h; t5 = 5*h;}}^{\text{2 extra points (asked)}}$

y1 = 0'02103; y2 = 0'09137; y3 = 0'22156;

(f0 = ...; f1 = ...) → we don't really need it, we already have f(t,y), t and y

y4p = y3 + h/24 *(55* f(t3,y3) - 59* f(t2,y2) + 37* f(t1,y1) - 9* f(t0,y0)); → $\boxed{0'42467}$

y4c1 = y3 + h/24 * (9 * f(t4, y4p) - 19 * f(t3,y3) - 5* f(t2,y2) + f(t1,y1)); → $\boxed{0'42486}$

tc1 = y4c1 - y4p; → $\boxed{0'00019 > 2\cdot 10^{-5}}$ we have to keep on iterating

y4c2 = y3 + h/24 *(9* f(t4, y4c1) - 19 * f(t3,y3) - 5* f(t2,y2) + f(t1,y1)); → $\boxed{0'42487}$

tc2 = y4c2 - y4c2; → $\boxed{0'00001 < 2\cdot 10^{-5}}$ stop iterating

y4 = y4c2;

(same for y5)

EXC-13

solve $\begin{cases} y' = (1+x)\cdot y^2/2 \\ y(0) = 1 \end{cases}$ $x \in [0, 0'7]$ $h = 0'1$; $\longrightarrow$ Milne's PC method with RK4

$\begin{cases} y(0'1) \simeq y_1 = 1'055409 \\ y(0'2) \simeq y_2 = 1'123596 \\ y(0'3) \simeq y_3 = 1'208459 \end{cases}$

$\begin{cases} y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_n - f_{n-1} + 2f_{n-2}) & \text{MILNE} \\ y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}) & \text{SIMPSON} \end{cases}$

both order 4 with scheme P(EC)E $PC^A$

coding:

```
f = @(t,y) (1+t)* y^2/2;
h = 0.1;
t0 = 0; t1 = h; t2 = 2*h; t3 = 3*h; ... ; t7 = 7*h;
Y0 = 1; Y1 = 1'05549; Y2 = 1'123596; Y3 = 1'208459;
Y4P = Y0 + 4*h/3*(2*f(t3,Y3) - f(t2,Y2) + 2*f(t1,Y1));
Y4C1 = Y2 + h/3*(f(t4,Y4P) + 4*f(t3,Y3) + f(t2,Y2));
tc = Y4C1 - Y4P; (if tc < 1) → Y4 = Y4C1
⋮
```

## CONVERGENCE AND STABILITY OF LINEAR MULTISTEP METHODS

0) Identify the number of steps of the method, k.

1) Identify advance formula with:

$$\boxed{y_{n+k} = -\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\sum_{j=0}^{k} \beta_j f_{n+j}}$$

$\alpha_k = 1$ ALWAYS

2) Write characteristic polinomials: $\boxed{\rho(z) = \sum_{j=0}^{k} \alpha_j z^j}$ $\boxed{\sigma(z) = \sum_{j=0}^{k} \beta_j \cdot z^j}$

3) Method is consistent iff: $\boxed{\rho(1) = 0 \; ; \; \rho'(1) = \sigma(1)}$
(CONSISTENCY)

↳ consistent means: $\lim_{h\to 0} \frac{L_{k+1}}{h} = 0$ EULER

4) stable if: |roots of $\rho(z)$| $\leq 1$ all (and if 1, simple roots)

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5) convergence of order p if:

$$\boxed{\frac{1}{m}\sum_{i=0}^{k} \alpha_i\, j^{m} = \sum_{j=0}^{k} \beta_j\, j^{m-1}}$$

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**Exc. 10**

a) study conv. and order of conv. of:

MILNE'S METHOD: $y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_n - f_{n-1} + 2f_{n-2})$

0) NUMBER OF STEPS : $\boxed{k = (n+1) - (n-3) = 4}$

1) IDENTIFY ADV. FORMULA : $y_{n+4} = \sum_{i=0}^{3} \alpha_i y_{n+i} + h\sum_{n=0}^{4} \beta_j f_{n+j}$

$$\begin{cases} y_{n+4} = -\alpha_0 y_n - \alpha_1 y_{n+1} - \alpha_2 y_{n+2} - \alpha_3 y_{n+3} + h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4}) \\ y_{n+4} = y_n + \frac{4h}{3}(2f_{n+3} - f_{n+2} + 2f_{n+1}) \end{cases}$$

(adaptation of Milne's method (+3))

$$\begin{cases} \alpha_0 = -1 \\ \alpha_1 = \alpha_2 = \alpha_3 = 0 \\ \alpha_4 = 1 \end{cases} \qquad \begin{cases} \beta_0 = \beta_4 = 0 \\ \beta_1 = \beta_3 = 8/3 \\ \beta_2 = -4/3 \end{cases}$$

WHEN THE LAST $\beta = 0 \rightarrow$ EXPLICIT METHOD

2) CHARACT. POL : $\boxed{\rho(z) = -1 + z^4}$ $\boxed{\sigma(z) = \frac{8}{3}z - \frac{4}{3}z^2 + \frac{8}{3}z^3}$

3.) CONSISTENCY : $\rho(1) = -1 + 1 = 0$ ✓ $\rho'(1) = 3\cdot 1^3 = \frac{8}{3}\cdot 1 - \frac{4}{3}\cdot 1 + \frac{8}{3}\cdot 1^3 = 3 = \sigma(1)$ ✓

4) STABILITY : $\rho(z) = 0 \rightarrow z^4 = 1 = (z^2)^2 = 1$ $\begin{cases} z^2 = +1 \\ z^2 = -1 \end{cases}$ $\begin{cases} \boxed{z = \pm 1} \\ \boxed{z = \pm i} \end{cases}$

modulus of the roots $\leq 1 = 1 \rightarrow$ all simple ✓

5) CONV. OF ORDER p.: $\frac{1}{m}\sum_{i=0}^{4} \alpha_i j^m = \sum_{j=0}^{4} \beta_j j^{m-1}$ $(m = 1, 2, \dots)$

$m=1 \longrightarrow \frac{1}{1}(-1\cdot\cancel{0}^1 + \underbrace{1\cdot 4^1}_{4}) = \frac{8}{3}\cdot 1^0 + \frac{-4}{3}\cdot 2^0 + \frac{8}{3}\cdot 3^0 \longrightarrow 4 = 4$

$m=2 \rightarrow \frac{1}{2}(1\cdot 4^2) \dots \longrightarrow 8 = 8$

$m=3 \rightarrow \frac{1}{3}(1\cdot 4^3) = \frac{8}{3}\cdot 1^2 - \frac{4}{3}2^2 + \frac{8}{3}\cdot 3^2 \longrightarrow \dots$

$m=4 \rightsquigarrow \dots$

$m=5 \longrightarrow$ DEGREE OF CONV : $\boxed{p=4}$

c) AM 4 : $y_{n+1} = y_n + \frac{h}{24}(9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2})$

0) $k = (n+1) - (n-2) = 3$

1) $y_{n+3} = \sum_{j=0}^{2} \alpha_j y_{n+j} + h \sum_{j=0}^{3} \beta_j f_{n+j} = -(\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2}) + h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3})$

(+2)

$y_{n+3} = y_{n+2} + \frac{h}{24}(9 f_{n+3} + 19 f_{n+2} - 5 f_{n+1} + f_n)$

$\begin{cases} \alpha_0 = \alpha_1 = 0 \\ \alpha_2 = -1 \\ \alpha_3 = 1 \end{cases}$ $\begin{cases} \beta_0 = 1/24 & \beta_2 = 19/24 \\ \beta_1 = -5/24 & \beta_3 = 9/24 \end{cases}$

2) $\boxed{\rho(z) = -z^2 + z^3}$ $\boxed{\sigma(z) = \frac{z}{24} - \frac{5z^2}{24} + \frac{19}{24} z^3 + \frac{9}{24} z^4}$

3) $\rho(1) = -1 + 1 = 0$ ✓

$\rho'(1) = -2 + 3 = 1$

$\sigma(1) = \frac{1}{24} - \frac{5}{24} + \frac{19}{24} + \frac{9}{24} = \frac{24}{24} = 1$

✓ CONSISTENT

4) $\rho(z) = 0 \longrightarrow z^2(z-1) = 0$ $\begin{cases} z = 0 \text{ (DOUBLE)} \\ z = 1 \end{cases} \longrightarrow |z| \le 1$ ; 1 SIMPLE ✓ $\longrightarrow$ STABLE

5) $\frac{1}{m} \sum_{j=0}^{3} \alpha_j j^m = \sum_{j=0}^{3} \beta_j j^{m-1} \rightarrow \frac{1}{m}(-2^m + 3^m) = -\frac{5}{24} \cdot 2^{m-1} + \frac{19}{24} 3^{m-1} + \frac{9}{24} 4^{m-1} + \frac{1}{24} \cdot 0^{m-1}$

$m=1 \rightarrow -2+3 = 1 = -\frac{5}{24} + \frac{19}{24} + \frac{9}{24} + \frac{1}{24} = 1$ ✓ $(1=1)$

$m=2 \rightarrow \frac{1}{2}(-4+9) = \frac{5}{2} = -\frac{5}{12} + \frac{19}{8} + \frac{9}{6} =$

$\vdots \quad \neq$

$m=3 \quad \neq \longrightarrow$ $\boxed{\text{ORDER : } p=4}$

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## STABILITY REGION OF LMM (ONLY 1 STEP)

1.) Aux. pol. $\boxed{\pi(z) = \rho(z) - \bar{h}\cdot\sigma(z)}$ ($\rho, \sigma$ = 1st, 2nd char. pol.)

stability region $\{ \bar{h} \ / \ |z(\bar{h})| < 1 \}$

$\pi(z) = 0 \longrightarrow z = z(\bar{h}) \longrightarrow |z(\bar{h})| < 1$

(ex.) a) stability region of Euler's M.

EULER: $y_{n+1} = y_n + f(t_n, y_n)\,h$

$y_{n+1} = y_n + h f_n$

a) $k = (n+1) - (n) = 1 \longrightarrow$ 1) $y_{n+1} = -\sum_{j=0}^{0} \alpha_j y_{n+j} + h \sum_{j=0}^{1} \beta_j f_{n+j}$

$y_{n+1} = -\alpha_0 y_n + h(\beta_0 f_n + \beta_1 f_{n+1})$

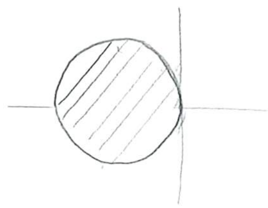
$y_{n+1} = y_n + h f_n$

$\begin{cases} \alpha_0 = -1 \ ; \ \alpha_1 = 1 \\ \beta_0 = 1 \ ; \ \beta_1 = 0 \end{cases}$

2) $\boxed{\rho(z) = -1 + z}$ $\boxed{\sigma(z) = 1}$

---

1) $\pi(z) = (-1+z) - \bar{h}\cdot 1 = -1 + z - \bar{h} = 0 \longrightarrow z(\bar{h}) = \bar{h} + 1$ $\begin{cases} |1 + \bar{h}| < 1 \\ |\bar{h} - (-1)| < 1 \end{cases}$



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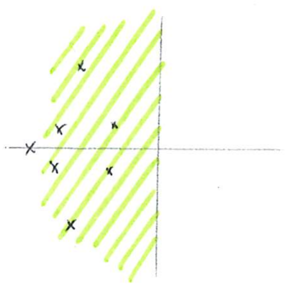
## UNCONDITIONAL STABILITY (A-STABLE)



Backward Euler (AM1) $\rightarrow$ order 1

Trapezoidal Method (AM) $\rightarrow$ order 2

* There's no a method with order > 2 that is unconditionally stable.

## CONVERGENCE OF PC METHODS

Adam's Method of order 4

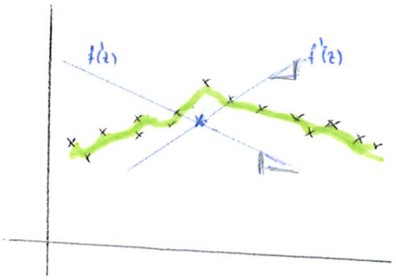
AB4 Expl.

AM4 Impl. → To improve precision (corrections)

* Each method has its corrections, not alway of the same order of conv.

# CH. 3. NUMERICAL DIFFERENTIATION

$$\left\{\begin{array}{l} k = \text{order of diff. of } f \\ z = \text{point we are estimating} \end{array}\right. \rightarrow f^{k}(z)$$



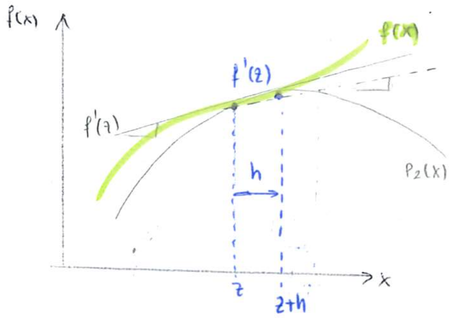

* we can have noise

* Then, we have more than one $f'(z)$ because it's not exact.

↳ we are going to be more accurate when working with numerical differenciation

(* we are going to calculate the approximate linear system.)

INTERPOLATION IDEA

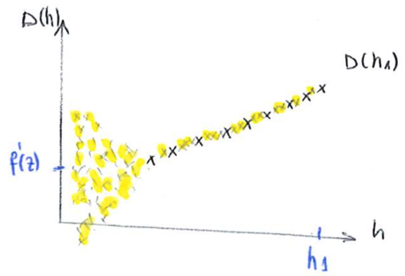



$k=1$

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$f'(z) \approx \underbrace{\frac{f(z+h)-f(z)}{h}}_{D} + O(h) \qquad OC = 1 \quad (\text{trunc})$$

↑ small $h$




- when $h$ is very small

* FORWARD DIFF. FORMULA:

$$\boxed{f'(z) = \frac{f(z+h)-f(z)}{h} + O(h)} \quad // \; OC = 1 \; (\text{trunc.})$$

$$\boxed{f'(z) = \frac{f(z+h)-f(z-h)}{2h} + O(h^2)} \quad // \; OC = 2$$

* BACKWARD DIFF. FORMULA:

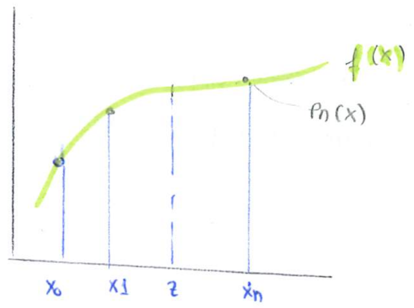
$$\boxed{f'(z) = \frac{f(z)-f(z-h)}{h} + O(h)} \quad // \; OC = 1$$

K=2

$$f''(z) \approx \frac{f'(z+h/2) - f'(z-h/2)}{h} \approx \frac{\dfrac{f(z+h)-f(z)}{h} - \dfrac{f(z)-f(z-h)}{h}}{h} = \frac{f(z+h)-2f(z)+f(z-h)}{h^2}$$

$$\boxed{f''(z) = \frac{f(z+h)-2f(z)+f(z-h)}{h^2} + O(h^2)} \quad // \quad OC=2$$

## INTERPOLATORY NUMERICAL DIFFERENCIATION FORMULA




$$f^{k)}(z) \simeq P_n^{k)}(z) = D$$

$$f^{k)}(z) = \underbrace{P_n^{k)}(z)}_{D} + E$$

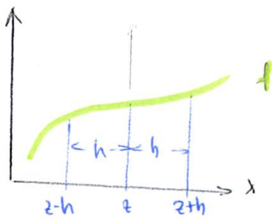
$$P_n(x) = L_0(x)\, f(x_0) + L_1(x)\, f(x_1) + \dots + L_n(x)\, f(x_n)$$

$$P_n^{k)}(x) = L_0^{k)}(x)\, f(x_0) + L_1^{k)}(x)\, f(x_1) + \dots + L_n^{k)}(x)\, f(x_n)$$

$$\boxed{D} = P_n^{k)}(z) = L_0^{k)}(z)\, f(x_0) + L_1^{k)}(z)\, f(x_1) + \dots + L_n^{k)}(z)\, f(x_n) =$$

$$= A_0 f(x_0) + A_1 f(x_1) + \dots + A_n f(x_n) = \boxed{\sum_{i=0}^{n} \underbrace{A_i}_{\text{COEFFS // WEIGHTS}} f(\overbrace{x_i}^{\text{NODES}})}$$

## EXAMPLE: D for $f'(z)$ using $z-h, z, z+h$ (CENTERED FORMULA)




$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot f(x_2) =$$

$$= \underbrace{\frac{(x-z)(x-z-h)}{-h(-2h)}}_{L_0(x)} f(x_0) + \underbrace{\frac{(x-z+h)(x-z-h)}{h(-h)}}_{L_1(x)} f(x_1) + \underbrace{\frac{(x-z+h)(x-z)}{2h \cdot h}}_{L_2(x)} \cdot f(x_2)$$

$A_0 = L_0'(z)$ ; $L_0'(z) = \dfrac{(x-z)+(x-z-h)}{2h^2} \longrightarrow A_0 = -1/2h$

$A_1 = L_1'(z)$ ; $L_1'(z) = \dfrac{(x-z+h)+(x-z-h)}{-h^2} \longrightarrow A_1 = 0$

$A_2 = L_2'(z)$ ; $L_2'(z) = \dfrac{(x-z+h)+(x-z)}{2h^2} \longrightarrow A_2 = 1/2h$

$$\boxed{D = \frac{f(z+h)-f(z-h)}{2h}}$$

*ME HE SALTADO UN EXAMPLE

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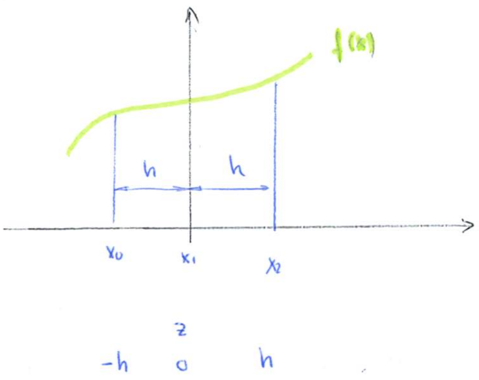
## EXAMPLE: D for $f''(z)$ CENTERED, 3 NODES




$f(x)\equiv 1 \to f''(0) = 0 = A_0\cdot 1 + A_1\cdot 1 + A_2\cdot 1 \longrightarrow A_0 + A_1 + A_2 = 0$

$f(x) = x \longrightarrow f''(0) = A_0(-h) + A_1\cdot 0 + A_2\cdot h = 0 \longrightarrow A_0 = A_2$

$f(x) = x^2 \to f''(0) = 2 = A_0(-h)^2 + A_1\cdot 0 + A_2 h^2 \longrightarrow A_0 + A_2 = 2/h^2$

$\hookrightarrow \boxed{A_0 = 1/h^2 = A_2} \longrightarrow \boxed{A_1 = -2/h^2}$

$$\boxed{D = \frac{1}{h^2} f(z-h) + \frac{(-2)}{h^2} f(z) + \frac{1}{h^2} f(z+h) = \boxed{\frac{f(z-h) - 2f(z) + f(z+h)}{h^2}}}$$

## TRUNCATION ERROR VIA INTERPOLATION

$$f^{k)}(z) = \underbrace{P_n^{k)}(z)}_{D} + E$$

$$\hookrightarrow \boxed{E = f^{k)}(z) - P_n^{k)}(z) = \left(f(z) - P_n(z)\right)^{k)} = e^{k)}(z) = \left(f[x_0, x_1, \dots, x_n, z]\,\Pi(z)\right)^{k)}}$$

*NEW THEOREM:

$$\boxed{f[x_0, x_1, \dots, x_n, x]^{k)}} = k!\, f[x_0, x_1, \dots, x_n, \overbrace{x, \underbrace{x, x, \dots, x}_{k}}^{k+1}] = \boxed{k!\,\frac{f^{(n+k+1)}(\xi)}{(n+k+1)!}}$$

f.s. $\xi$ between the nodes and $x$

CASUÍSTICA

- CASUISTRY : $f'(z)$ $(k=1)$

NEW THEOREM

$$\boxed{E = \left(f[x_0, x_1, x_2, \dots, x_n, z]\,\Pi(z)\right)' = f[x_0, x_1, \dots, x_n, z]'\,\Pi(z) + f[x_0, \dots, x_n, z]^{0}\,\Pi'(z) =}$$

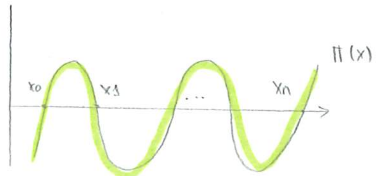
$$= 1!\,\frac{f^{(n+2)}(\xi_1)}{(n+2)!}\Pi(z) + 0!\,\frac{f^{(n+1)}(\xi_2)}{(n+1)!}\Pi'(z) = \boxed{\frac{f^{(n+2)}(\xi_1)}{(n+2)!}\Pi(z) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi'(z)}$$

$\xi_1, \xi_2$ between nodes and $z$

$f \in C^{n+2}$

$$\Pi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

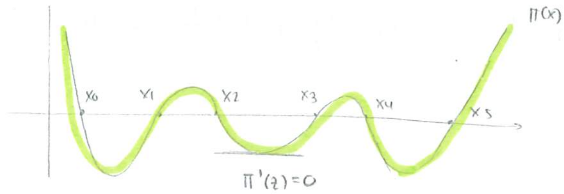
A) $\Pi(z)=0 \iff z$ is one of the nodes




$$\boxed{E = \frac{f^{n+1)}(\xi)}{(n+1)!}\,\Pi'(z)}$$

$\left\{\begin{array}{l} f(x)=x^n \longrightarrow f^{n+1)}(x)\equiv 0 \longrightarrow E=0 \\ f(x)=x^{n+1} \longrightarrow f^{n+1)}(x) = (n+1)! \neq 0 \longrightarrow E\neq 0 \end{array}\right\}$ $\boxed{N=n}$ if $z$ is a node (to estimate $f'(z)$)

B) $\Pi'(z)=0 \Longleftarrow$ Even number of nodes with central symm centered at $z$.




$\left\{\begin{array}{l} x^{n+1} \longrightarrow E=0 \\ x^{n+2} \longrightarrow E\neq 0 \end{array}\right\}$ $\boxed{N=n+1}$

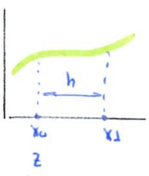
- CASUISTRY: $f''(z)$ $(k=2)$

$$\boxed{E} = \left(f[x_0,\dots,x_n,z]\,\Pi(z)\right)'' = \left(f[x_0,\dots,x_n,z]'\,\Pi(z) + f[x_0,\dots,x_n,z]\,\Pi'(z)\right)' = \dots =$$

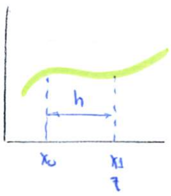
$$= \boxed{2\frac{f^{n+3)}(\xi_1)}{(n+3)!}\Pi(z) + 2\frac{f^{n+2)}(\xi_2)}{(n+2)!}\Pi'(z) + \frac{f^{n+1)}(\xi_3)}{(n+1)!}\Pi''(z)}$$

- CASES: $f'(z)$ 2 NODES

A)

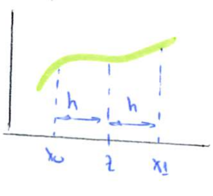



$\left\{\begin{array}{l} x_0=z \\ x_1=z+h \end{array}\right.$



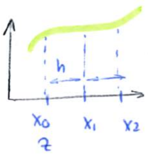

$\left\{\begin{array}{l} x_0=z-h \\ x_1=z \end{array}\right.$

B)




$\left\{\begin{array}{l} x_0=z-h \\ x_1=z+h \end{array}\right.$

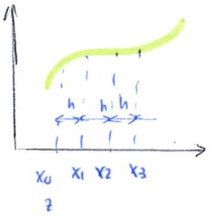
- CASES $f'(z)$ 3 NODES

A) 

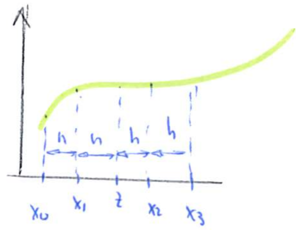
$$\begin{cases} x_0 = z \\ x_1 = z+h \\ x_2 = z+2h \end{cases}$$

~~B)~~

- CASES $f'(z)$ 4 NODES

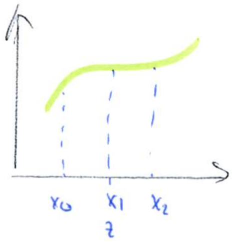
A) 

$$\begin{cases} x_0 = z \\ x_1 = z+h \\ x_2 = z+2h \\ x_3 = z+3h \end{cases}$$

B) 

$$\begin{cases} x_0 = z-2h \\ x_1 = z-h \\ x_2 = z+h \\ x_3 = z+2h \end{cases}$$

- CASES $f^{4}(z)$ $(k=2, n=2)$ $(nn=3)$



$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_2)(x_0-x_1)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) =$$

$$= \underbrace{\frac{(x-x_2)(x-h-z)}{-h(-2h)}}_{L_0(x)} f(x_0) + \underbrace{\frac{(x-z+h)(x-z-h)}{h(-h)}}_{L_1(x)} f(x_1) + \underbrace{\frac{(x-z+h)(x-z)}{2h \cdot h}} f(x_2)$$

$$\begin{cases} \pi(x) = (x-x_0)(x-x_1)(x-x_2) = (x-z+h)(x-z)(x-z-h) \\ \pi'(x) = (x-z)(x-z-h) + (x-z+h)(x-z-h) + (x-z+h)(x-z) \\ \pi(z) = 0 \ ; \ \pi'(z) = h^2 \ ; \ \pi''(z) = 0 \end{cases}$$

$$\boxed{E = 2 \cdot \frac{f^{4)}(\xi)}{4!}(-h^2) = -\frac{f^{4}(\xi)h^2}{12} = O(h^2)}$$

11/05/21

## THE TAYLOR WAY ("Ad-hoc")

- $f(z+h) = f(z)+f'(z)h + \frac{f''(\xi_1)}{2!}h^2$ f.s. $\xi_1 \in (z, z+h)$ if $f(\xi) \in C^2([z, z+h])$

  $\hookrightarrow f'(z) = \underbrace{\frac{f(z+h)-f(z)}{h}}_{D} - \underbrace{\frac{f''(\xi_1)}{2}h}_{E}$

- $f(z-h) = f(z) - f'(z)h + \frac{f''(z)}{2!}h^2$ f.s. $\xi_2$ ....

  $\hookrightarrow f'(z) = \underbrace{\frac{f(z)-f(z-h)}{h}}_{D} - \underbrace{\frac{f''(\xi_2)}{2}h}_{E}$

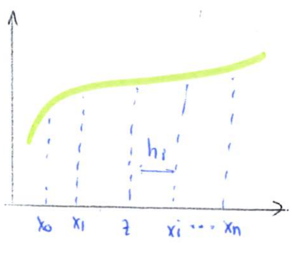
- $f(z+h) = f(z)+f'(z)h+\frac{f''(z)h^2}{2!} + \frac{f'''(\xi_1)h^3}{3!}$
- $f(z-h) = f(z) - f'(z)h + \frac{f''(z)h^2}{2!} - \frac{f'''(\xi_2)h^3}{3!}$

$$\boxed{f'(z) = \frac{f(z+h)-f(z-h)}{2h} - \frac{f'''(\xi_1)+f'''(\xi_2)}{2}\frac{h^2}{3!} = \underbrace{\frac{f(z+h)-f(z-h)}{2h}}_{D} - \underbrace{\frac{f'''(\xi)h^2}{6}}_{E}}$$

- $f(z+h) = f(z) + \dots + \frac{f'''(z)}{3!} + \frac{f^{4)}(\xi_1)}{4!}$
- $f(z-h) = f(z) - \dots - \frac{f'''(z)h^3}{3!} + \frac{f^{4}(\xi_1)}{4!}$

$$\boxed{f''(z) = \underbrace{\frac{f(z+h)-2f(z)+f(z-h)}{h^2}}_{D} - \underbrace{\frac{f^{4}(\xi)}{12}h^2}_{E}}$$

## TAYLOR GENERAL




- $f(x_i) = f(z+h_i) = f(z)+f'(z)h_i + \frac{f''(z)}{2!}h_i^2 + \dots + \frac{f^{n)}(z)}{n!}h_i^n + \dots + \frac{f^{m)}(z)}{m!}h_i^m + \frac{f^{m+1)}(\xi_i)h_i^{m+1}}{(m+1)!}$

$$\begin{cases} x_i = z+h_i \\ f^{k)}(z) = P_n^{k)}(z) + E = \sum_{i=0}^{n} A_i f(x_i) + E \end{cases}$$

$\rightarrow f^{k)}(z) = \sum_{i=0}^{n} A_i \cdot f(x_i) + E$

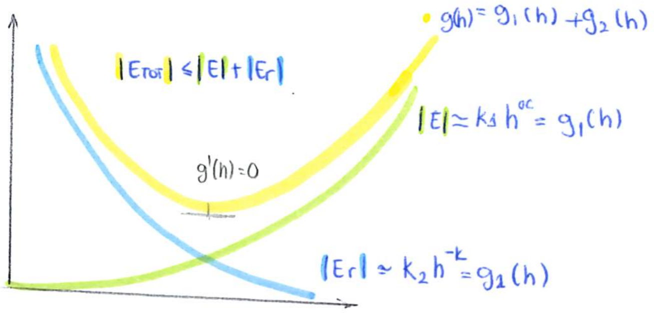
* Assuming $\begin{cases} n \geq k \\ m \geq n \end{cases}$ $f \in C^{m+1}$

$$f^{k)}(z) = f(z)\cancel{\sum_{i=0}^{n} A_i}^{\,0} + f'(z)\cancel{\sum_{i=0}^{n} A_i h_i}^{\,0} + f''(z)\frac{\cancel{\sum_{i=0}^{n} A_i h_i^2}^{\,0}}{2!} + \dots + f^{k)}(z)\frac{\overbrace{\sum_{i=0}^{n} A_i h_i^k}^{1}}{k!} + \dots + f^{n)}(z)\frac{\cancel{\sum_{i=0}^{n} A_i h_i^n}^{\,0}}{n!} + \underbrace{f^{n+1)}(z)\frac{\sum A_i h_i^{n+1}}{(n+1)!} + \dots + f^{m)}(z)\cdot\frac{\sum A_i h_i^m}{(m)!} + f^{m+1)}(z)\frac{\sum A_i h_i^{m+1}}{(m+1)!} + \epsilon}_{=0}$$

$$\left\{\begin{aligned} &\sum_{i=0}^{n} A_i = 0 \\ &\sum_{i=0}^{n} A_i h_i = 0 \\ &\sum_{i=0}^{n} A_i h_i^k = k! \\ &\sum_{i=0}^{n} A_i h^n = 0 \end{aligned}\right. \longrightarrow A_i$$

$$\boxed{E = -f^{n+1)}(z)\frac{\sum A_i h_i^{n+1}}{(n+1)!} - \dots - f^{m)}(z)\frac{\sum_{i=0}^{n} A_i h_i^m}{m!} - \frac{1}{(m+1)!}\sum_{i=0}^{n} A_i f^{m+1)}(\xi_i)\, h^{m+1}}$$

## OPTIMAL VALUE OF h ($h_{opt}$)




$$\bullet\ D = \sum_{i=0}^{n} \underbrace{A_i}_{A_i = \frac{C_i}{h^k}} f(x_i)$$

$$\left\{\begin{aligned} &E = \text{Truncation error} \\ &E_r = \text{Rounding error} \\ &E_{TOT} = \text{Total error} \ (E_{TOT} = E + E_r) \end{aligned}\right.$$

not exact value

$$\boxed{E_{TOT} = f^{k)}(z) - \bar{D} = f^{k)}(z) - \sum_{i=0}^{n} A_i \bar{f}(x_i)}$$

$$E_{TOT} = f^{k)}(z) - \bar{D} = \underbrace{f^{k)}(z) - D}_{E} + \underbrace{D - \bar{D}}_{E_r}$$

↳ Adding abs values:

- $|E_{TOT}| = |f^{k)}(z) - D + D - \bar{D}|$
- $|E_r| = |D - \bar{D}| = |\sum_{i=0}^{n} A_i(f_i - \bar{f}_i)| \leq \sum_{i=0}^{n} A_i \underbrace{|f_i - \bar{f}_i|}_{\varepsilon} = \varepsilon \cdot A_F \longrightarrow \boxed{|E_r| \leq \varepsilon A_F = g_2(h)}$

$\varepsilon$ ↳ UPPER BOUN OF ERROR IN NODAL ORDINATES

Example

CALCULATE $h_{opt}$ for : $f'(z) = \dfrac{f(z+h) - f(z-h)}{2h} + \underbrace{\dfrac{(-h^2)}{6} f'''(\xi)}$

$\hookrightarrow M \geq |f'''(\xi)|$

- $|E| = \left| \dfrac{-h^2}{6} f'''(\xi) \right| \leq \dfrac{h^2 M}{6} = g_1(h)$

- $AF = \sum_{i=0} |A_i| = \left|\dfrac{1}{2h}\right| + \left|\dfrac{-1}{2h}\right| = \dfrac{1}{h}$

- $|Er| = \mathcal{E} \cdot AF = \dfrac{\mathcal{E}}{h} = g_2(h)$

- $g(h) = g_1(h) + g_2(h) = \dfrac{h^2 M}{6} + \mathcal{E}h^{-1}$

$\hookrightarrow g'(h) = \dfrac{2hM}{6} - \mathcal{E}h^{-2} = 0 \rightarrow \dfrac{\cancel{2}hM}{\cancel{6}\,3} = \dfrac{\mathcal{E}}{h^2} \rightarrow h^3 = \dfrac{3\mathcal{E}}{M} \rightarrow \boxed{h = \sqrt[3]{\dfrac{3\mathcal{E}}{M}}}$

$\boxed{g(h_{opt}) = \dfrac{\sqrt[3]{9}}{2} \mathcal{E}^{2/3} \cdot M^{1/3}}$ (IT HAS TO BE A SINGLE TERM)

EULER: $y_{k+1} = y_k + f(t_k, y_k)\,h$

Ⓔ

ENHANCED EULER (HENN'S): $y_{k+1} = y_k + \dfrac{(k_1 + k_2)}{2}$ $\quad \begin{cases} k_1 = h \cdot f(t_k, y_k) \\ k_2 = h\, f(t_k + h_k, y_k + k_1) \end{cases}$

Ⓔ

(MOD. EULER) MIDPOINT: $y_{k+1} = y_k + k_2$ $\quad \begin{cases} k_1 = h\, f(t_k, y_k) \\ k_2 = h\, f(t_k + h/2, y_k + k_1/2) \end{cases}$

Ⓔ

RK4 $y_{k+1} = y_k + \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6}$ $\quad \begin{cases} k_1 = h \cdot f(t_k, y_k) \\ k_2 = h \cdot f(t_k + h/2, y_k + k_1/2) \\ k_3 = h\, f(t_k + h/2, y_k + k_2/2) \\ k_3 = h\, f(t_k + h, y_k + k_3) \end{cases}$

Ⓔ

TRAPEZOIDAL $y_{k+1} = y_k + \dfrac{(f(t_k, y_k) + f(t_{k+1}, y_{k+1}))}{2}\, h$

Ⓘ

BACKWARD: $y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\,h$ como EULER pero con $f_{k+1}$

Ⓘ

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$n$ par (EVEN) $\Longrightarrow \pi'(z) = 0$ SUFFICIENT

$z$ is a node $\Longleftrightarrow \pi(z) = 0$ NECESSARY AND SUFFICIENT