MEKANIKA eta UHINAK

1. INDAR ZENTRALAK

Magnitude Kolektiboak:

Magnitudea:	Partikularena:	Sistemarena:
Masa	m_i	$M \equiv \sum_{i=1}^{N} m_i$
Masa-momentua	$g_i = m_i r_i$	$G \equiv \sum_{i=1}^{N} g_i$
Momentu lineala	$p_i = m_i \dot{r}_i$	$P \equiv \sum_{i=1}^{N} p_i$
Indarra	$F_i = m_i \ddot{r}_i$	$F \equiv \sum_{i=1}^{N} F_i$
Momentu angeluarra	$L_i = m_i r_i \times \dot{r}_i$	$L \equiv \sum_{i=1}^{N} L_i$
Indar momentua	$N_i = r_i \times F_i$	$N \equiv \sum_{i=1}^{N} N_i$
Energia zinetikoa	$T_i = \frac{1}{2} m_i \dot{r}_i^2$	$T \equiv \sum_{i=1}^{N} T_i$
Potentzia	$P_i = F_i \cdot \dot{r}_i$	$P = \sum_{i=1}^{N} P_i$

Kontserbazio printzipioa:

Sistema bakartua, $F^{(K)} = N^{(K)} = 0$

Kanpo indarra

$P = Kte$ $\quad L = Kte$

Talkak:

$P_2 = P_1$

$L_2 = L_1$

$T_2 = T_1 + Q$

$$Q = \int_1^2 \sum_{i=1}^{N} \sum_{j \neq i} F_{ji} \cdot \dot{r}_i \, dt$$

- Talka endoenergetikoa : $T_2 < T_1$
- Talka exoenergetikoa: $T_2 > T_1$
- Talka elastikoa : $T_2 = T_1$

Bulkada:

$$J = \int_{t_1}^{t_2} F \, dt \Rightarrow P_2 - P_1$$

Bulkada angeluarra:

$$M = \int_{t_1}^{t_2} N \, dt \Rightarrow L_2 - L_1$$

Masa-zentroa:

$$\bar{R} = \frac{\sum_{i=1}^{N} m_i \cdot r_i}{M}$$

Masa-zentroaren posizioa

$T = T^*$

$m = m^*$

$r_i = r_i^* + R$

$\dot{r}_i = \dot{r}_i^* + \dot{R}$

$\ddot{r}_i = \ddot{r}_i^* + \ddot{R}$

$P^* = 0$

$*$: Masa-zentroaren sistemaN

König:

$P = M\dot{R}$

$L = L^* + R \times P$

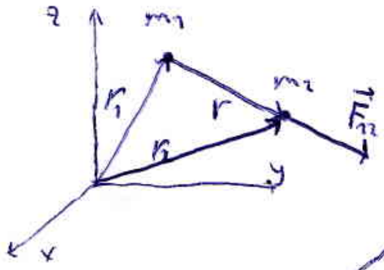
$$T = T^* + \frac{1}{2} M \dot{R}^2 = T^* + \frac{P}{2M}$$

- Masa zentroaren sistematik neurtua
- Masa zentroaren translazioa E.S.-rekiko
- Kanpoko E.S.-TIK

$N^* = \dot{L}^*$

Indar-zentralak:

Bi gorputzen problema:



$r = r_2 - r_1$

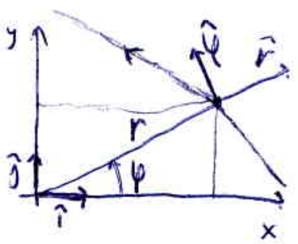
$F = F_1 = F(r)\hat{r}$

$m = \dfrac{m_1 m_2}{m_1 + m_2}$ ← masa laburbildua

$F = m\ddot{r} = F(r)\hat{r}$

$\dot{L} = 0 \Rightarrow L = \text{kte}$

OXY planoan:



$x = r\cos\varphi$

$y = r\sin\varphi$

$r = |r| = \sqrt{x^2 + y^2}$

$\varphi = \arctan\dfrac{x}{y}$

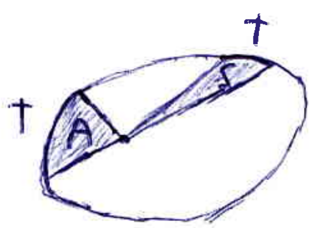
$\hat{r} = \dfrac{r}{|r|}$

$\hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$

$\vec{L} = m \cdot r^2 \dot{\varphi}\,\hat{k}$

Keplerren 2. legea:

Azalera-abiadura: $V_A = \dfrac{L}{2m} \overset{\frac{dS}{dt}}{=} \dfrac{1}{2} r^2 \dot{\varphi} = \text{kte}$



$A = S$

Indar Kontserbatzaileak eta energia potentziala:

- $F = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\imath}, \frac{\partial V}{\partial y}\hat{\jmath}, \frac{\partial V}{\partial z}\hat{k}\right)$

- $\nabla \times F = 0$

Energia mekanikoa

$$\underline{\underline{E}} = T + V = \underline{\underline{Ktea}}$$

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r)$$

Orbitaren ekuazioa:

$$\varphi - \varphi_0 = \pm \int_{r_0}^{r} \frac{L\, dr}{m r^2 \sqrt{\frac{2}{m}\left[E - V(r) - \frac{L^2}{2mr^2}\right]}}$$

$$V_e(r) = V(r) + \underbrace{\frac{L^2}{2mr^2}}_{\text{Energia potentzial zentrifugoa}}$$

Energia potentzial eraginkorra

$$F_z = -m\omega \times (\omega \times r)$$

Indar zentrifugoa

Binet:

$$u = \frac{1}{r}$$

$$u'' + u = -\frac{m}{L^2 u^2} F\left(\frac{1}{u}\right) \Rightarrow F\Big(\underbrace{\frac{1}{u}}_{r}\Big) = -\frac{L^2}{m}(u'' + u)u^2$$

Orbita zirkularrak:

Existitu: $V_e'(r_0) = 0$

Egonkorrak: $V_e''(r_0) > 0$

Oszilazio erradial txikiak:

$$\ddot{q} + \omega^2 q = 0, \quad \omega = \sqrt{\frac{V_e''(r_0)}{m}}, \quad q = r - r_0$$

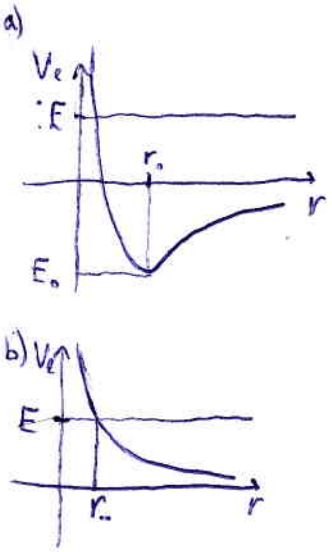
Harmonikoak

Indar newtondarrak:

$$V(r) = -\frac{K}{r} \Leftrightarrow F = -V'(r)\hat{r} = -\frac{K}{r^2}\hat{r}$$

a) Erakarlea: $K>0$ $\quad (K = G m_1 m_2 > 0)$

b) Aldaratzailea: $K<0$ $\quad \left(K = -\frac{q_1 q_2}{4\pi\varepsilon_0}\right)$



$$V_e = -\frac{K}{r} + \frac{L^2}{2mr^2} \Rightarrow \text{Minimoa: } V_e' = 0 \Rightarrow r = \frac{L^2}{mK}$$

Orbita newtondarren ekuazioa:

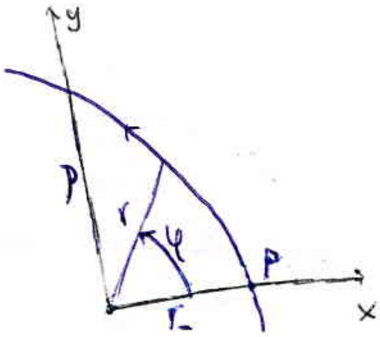
Eszentrikotasuna: $\varepsilon = \sqrt{1 + \frac{2L^2E}{mK^2}} = \sqrt{1 - \frac{E}{E_0}} \geq 0$

Foku-parametroa: $p = \frac{L^2}{m|K|}$ (semi-latus rectum)

Orbita:
$$r = \frac{p}{\varepsilon\cos(\varphi - \delta) + \frac{K}{|K|}}$$

Perizentroa: $r = r_- = \frac{p}{\varepsilon \pm 1}$

$$\varphi_- = \delta = 0$$



Polarretan:

$$r = \frac{p}{\varepsilon\cos\varphi + \frac{K}{|K|}}$$

Kartesiarretan:

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\} (1-\varepsilon^2)x^2 + 2\varepsilon p x + y^2 = p^2$$

Orbita motak:

- Periodikoak:

• Zirkularra:

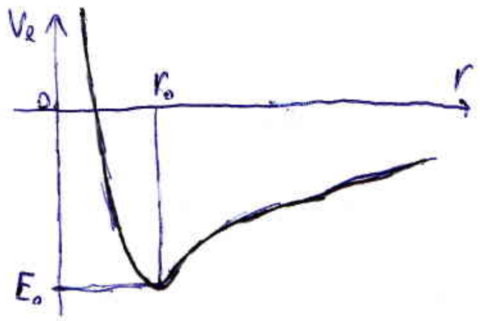
$E = E_0 = -\frac{mK^2}{2L^2}$

$\varepsilon = 0$

$r = r_0$

$E = -\frac{K}{2r_0}$

$x^2 + y^2 = p^2 = r_0^2$

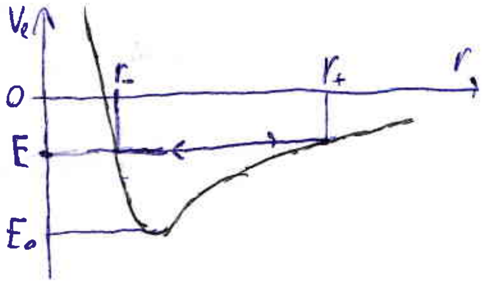


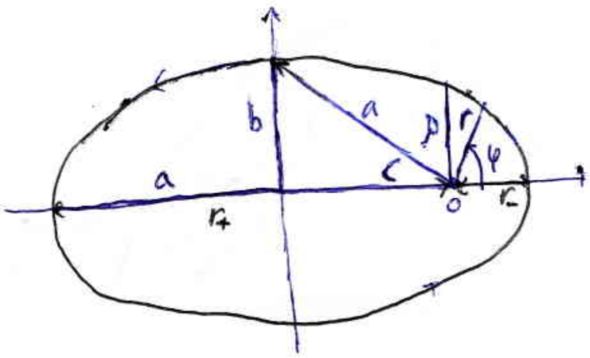
• Eliptikoa:

$E_0 \leq E < 0$

$0 \leq \varepsilon \leq 1$

$r_- < r = \frac{p}{1+\varepsilon\cos\varphi} \leq r_+$





$a = \frac{p}{1-\varepsilon^2} = \frac{r_- + r_+}{2} = -\frac{K}{2E}$

$b = \frac{p}{\sqrt{1-\varepsilon^2}} = \sqrt{pa}$

$c = \frac{\varepsilon p}{1-\varepsilon^2} = a - r_- = r_+ - a$

$\left(\frac{x+c}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Kepler 1. legea:

$\boxed{a^2 = b^2 + c^2, \quad \varepsilon = \frac{c}{a}}$

$$E = -\frac{K}{2a}$$

Elipsearen azalera: $S = \pi ab$

Azalera-abiadura:

$$V_a = \frac{\pi ab}{T} = \frac{L}{2m} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2 m}{K}$$

Planetetan:

$$M << M_{\odot} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{G M_{\odot}}$$ ← Keplerren 3. legea

Bertranden teorema:

Orbita bornatu guztiak itxiak badira, indar eremu zentrala newtondarra $\left(F \propto -\frac{1}{r^2}\right)$

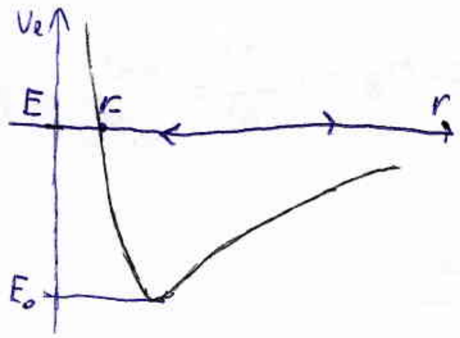
edo harmonikoa $(F \propto -r)$

- **Irekiak:**

a) $K>0$ (Erakarleak), orbita irekia $\Longleftrightarrow$ $v \geq v_i = \sqrt{\frac{2K}{mr}}$ (ihes abiadura)

b) $K<0$ (Aldarateraileak), Orbita beti irekia

• **Parabolikoa ($K>0$):**



$K>0$

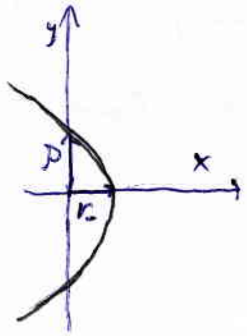
$\varepsilon=1$

$E=0$

$$r_- = \frac{p}{2} \leq r = \frac{p}{\cos\varphi+1} < +\infty$$

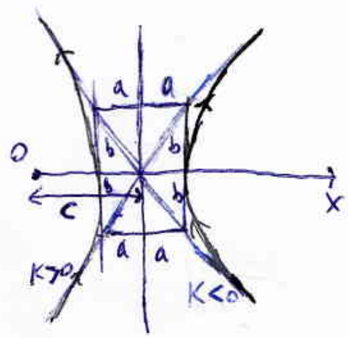
$-\pi < \varphi < \pi$

$y^2 = p^2 - 2px$



• **Hiperbolikoak ($K \gtrless 0$):**

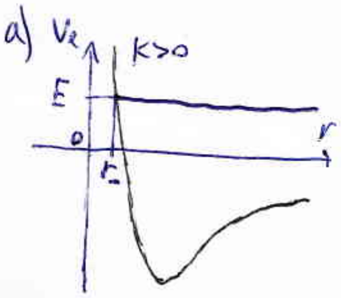
a) $K>0 \longrightarrow +$

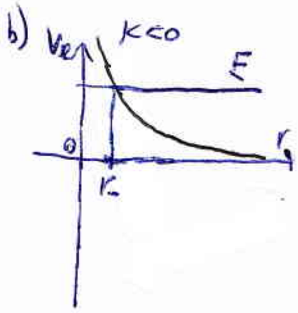


$\varepsilon>1$

$E>0$

$$r_- = \frac{p}{\varepsilon \pm 1} \leq r = \frac{p}{\varepsilon\cos\varphi \pm 1} < +\infty$$





$$a = \frac{p}{\varepsilon^2 - 1} = \frac{|K|}{2E}$$

$$b = \frac{p}{\sqrt{\varepsilon^2 - 1}} = \sqrt{pa}$$

$$c = \frac{\varepsilon p}{\varepsilon^2 - 1}$$

$$c^2 = a^2 + b^2, \quad \varepsilon = \frac{c}{a}$$

$$\left(\frac{x-c}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

$$E = \frac{|K|}{2a}$$

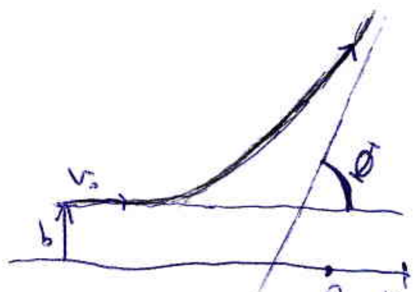
Asintotak:

$K>0 \quad -\pi < -\arccos\left(-\frac{1}{\varepsilon}\right) < \varphi < \arccos\left(-\frac{1}{\varepsilon}\right) < \pi$

$K<0 \quad -\frac{\pi}{2} < -\arccos\left(\frac{1}{\varepsilon}\right) < \varphi < \arccos\left(\frac{1}{\varepsilon}\right) < \pi$

Sekzio-eragilea:

-Sakabanatze newtondarra:



$K<0$

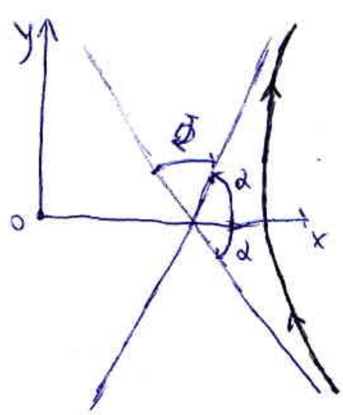
$E = \frac{1}{2} m v_0^2$

$L = m b v_0$ (b= jotze parametroa)

$$r = \frac{p}{\varepsilon \cos\varphi - 1}$$

$$\cos\alpha = \frac{1}{\varepsilon}$$

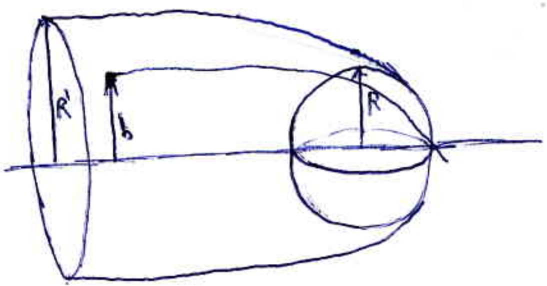
$$\cot\frac{\emptyset}{2} = \frac{m b v_0^2}{|K|} = \frac{2bE}{|K|}$$



$\emptyset$ = Sakabanatze angelua

Sekzio eragile osoa:

Harrapatze sekzioa:



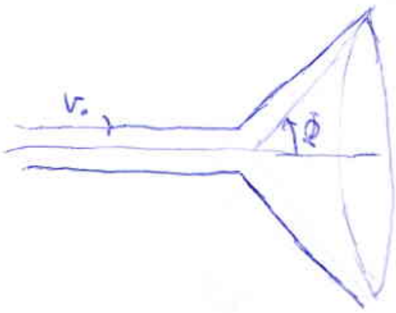
$$\sigma = \pi R'^2$$

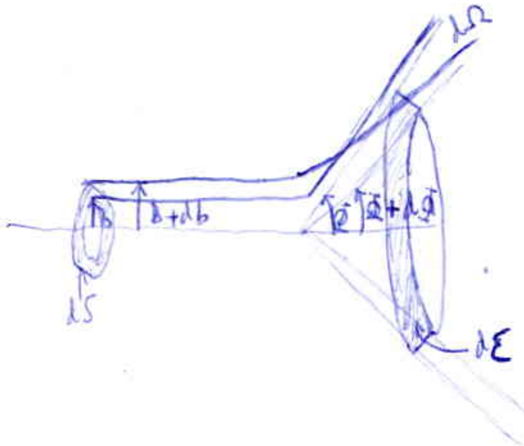
I = Intentsitatea

N = Denbora-unitatean harrapatutako partikula kopurua

$$\sigma = \frac{N}{I} \iff N = I\sigma$$

Sekzio eragile diferentziala:





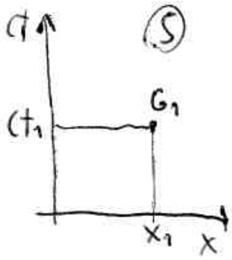
$$I = \frac{dN}{dS} = \frac{dN}{d\sigma}$$

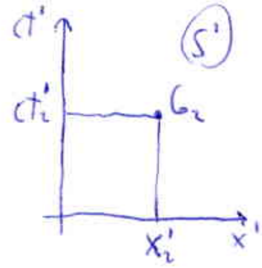
$$d\Omega = \frac{d\Sigma}{R^2} = -\frac{2\pi(R\sin\Phi)(R\,d\Phi)}{R^2} = -2\pi\sin\Phi\,d\Phi$$

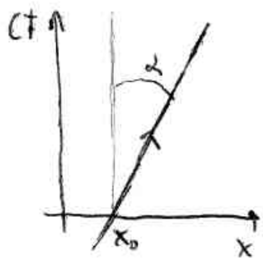
$$\frac{d\sigma}{d\Omega} = -\frac{b\,db}{\sin\Phi\,d\Phi}$$

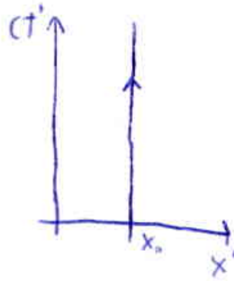
2. ERLATIBITATE BEREZIA

· Minkowskiren diagramak:



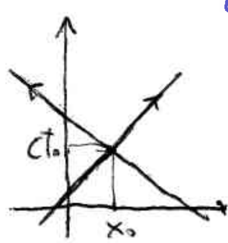




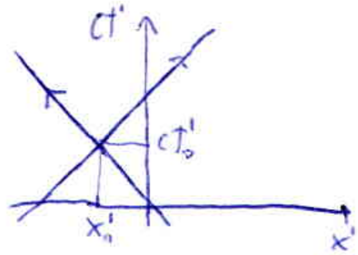


$\tan\alpha = \frac{v}{c}$

Fotoien unibertso lerroak:



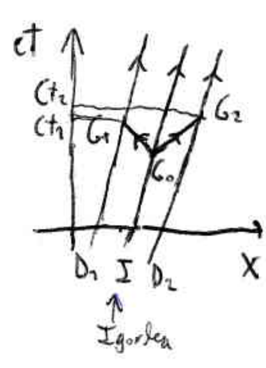
$c = v$
$\Downarrow$
$45°$

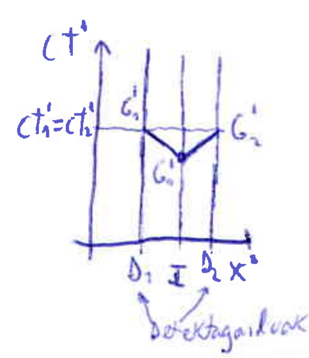


Aldiberekotasun erlatiboa:

Fotoien igorpena: $G_0 \sim (t_0, x_0) \sim (t'_0, x'_0)$

Fotoien detekzioa: $G_i \sim (t_i, x_i) \sim (t'_i, x'_i)$ $i = 1,2$



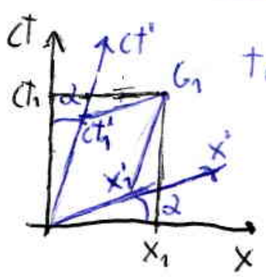


Lorentzen transformazioak:

$$\begin{cases} t = \gamma\left(t' + \frac{v}{c^2}x'\right) \\ x = \gamma\left(x' + vt'\right) \\ y = y' \\ z = z' \end{cases} \qquad \begin{cases} t' = \gamma\left(t - \frac{v}{c^2}x\right) \\ x' = \gamma\left(x - vt\right) \\ y' = y \\ z' = z \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

S'-ko ardatzak S-ko ardatzekiko:



$\tan\alpha = \frac{v}{c}$

Tarteak:

$\Delta t = t_2 - t_1$ $\qquad$ $\Delta t' = t'_2 - t'_1$

$\Delta x = x_2 - t_1$ $\qquad$ $\Delta x' = x'_2 - x'_1$

$$\begin{cases} \Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \\ \Delta x = \gamma\left(\Delta x' + v\Delta t'\right) \\ \Delta y = \Delta y' \\ \Delta z = \Delta z' \end{cases} \qquad \begin{cases} \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \\ \Delta x' = \gamma\left(\Delta x - v\Delta t\right) \\ \Delta y' = \Delta y \\ \Delta z' = \Delta z \end{cases}$$

- Espazio denborako tartea (Aldaezin erlatibista):

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = \Delta s'^2$$

- Lorentz eta FitzGerald-en uzkurdura:

S' sisteman, neurtutako objektuarekin batera mugitzen dena, luzera propioa

$$\underset{\substack{\uparrow \\ \text{Geldi dagoen sistema} \\ \text{objektua uzkurtu}}}{\Delta x} = \frac{\Delta x^*}{\gamma} < \Delta x^* \leftarrow \text{Luzera propioa}$$

- Denboraren zabalkuntza:

S' sisteman, denbora mantsoago pasatzen da S sisteman baino

(Mugitzen ari denarentzat urte bat, geldi dagoenarentzat 10)

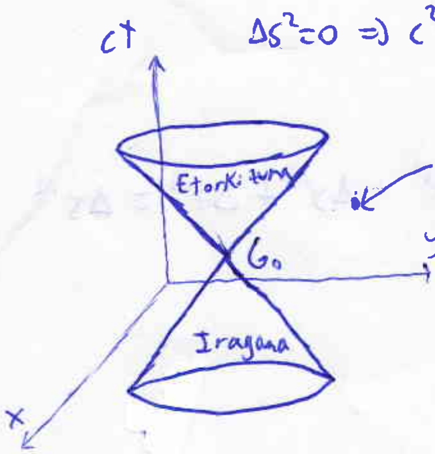
$$\underset{\substack{\uparrow \\ \text{Geldi dagoen sisteman} \\ \text{pasatako denbora}}}{\Delta t} = \gamma \Delta t^* > \Delta t^* \leftarrow \text{Denbora propioa}$$

- Espazio-denborako tarte motak:

- Espazio motako tartea: $\Delta s^2 > 0 \Rightarrow$ Denbora ordenarik eta erlazio kausalik ez
- Denbora motako tartea: $\Delta s^2 < 0 \Rightarrow$ Denbora ordena eta (agian) erlazio kausala
- Argi motako tartea: $\Delta s^2 = 0 \Rightarrow$ Denbora ordena eta (agian) erlazio kausala

Argi-konoa:

$$\Delta s^2 = 0 \Rightarrow c^2(t-t_0)^2 = (x-x_0)^2 + (y-y_0)^2$$



Espazio motako tartea, denbora ordena da erlazio y kausal gabea

- Abiadura-transformazioa:

$$\frac{df}{dt} = \frac{1}{\gamma\left(1+\frac{v}{c^2}\dot{x}'\right)}\frac{df}{dt'}$$

$$\begin{cases} \dot{x} = \dfrac{\dot{x}'+v}{1+\dfrac{v\dot{x}'}{c^2}} \\[2ex] \dot{y} = \dfrac{\dot{y}'}{\gamma\left(1+\dfrac{v\dot{x}'}{c^2}\right)} \\[2ex] \dot{z} = \dfrac{\dot{z}'}{\gamma\left(1+\dfrac{v\dot{x}'}{c^2}\right)} \end{cases} \qquad \begin{cases} \dot{x}' = \dfrac{\dot{x}-v}{1-\dfrac{v\dot{x}}{c^2}} \\[2ex] \dot{y}' = \dfrac{\dot{y}}{\gamma\left(1-\dfrac{v\dot{x}}{c^2}\right)} \\[2ex] \dot{z}' = \dfrac{\dot{z}}{\gamma\left(1-\dfrac{v\dot{x}}{c^2}\right)} \end{cases}$$

v S' sistemaren abiadura S-n

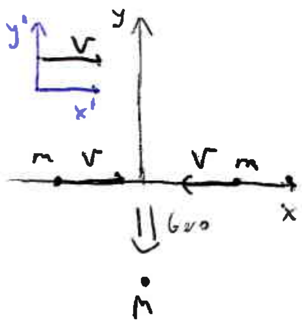
$\dot{x}$ partikularen abiadura S-n.

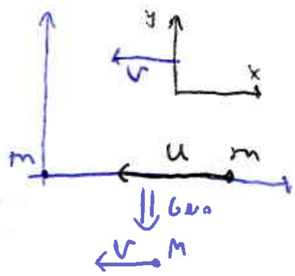
$\dot{x}'$ partikularen abiadura S'-n

- Momentu lineala:

$$p = m\gamma\dot{r} = \frac{m \cdot \dot{r}}{\sqrt{1-\frac{\dot{r}^2}{c^2}}}$$

- Talka simetriko inelastikoa:





$\dot{x} = -v$

$$\dot{x}' = -u = \frac{-v-v}{1-\frac{v(-v)}{c^2}} \Rightarrow u = \frac{2v}{1+\frac{v^2}{c^2}}$$

p. Kontserbatu:

$$\begin{cases} m\cdot\gamma_v + m\gamma_v = M\gamma_0 \\ m\gamma_0 + m\gamma_u = M\gamma_v \end{cases}$$

• Momentu lineala eta Energia:

$$\begin{cases} p = m\gamma\dot{r} \\ E = m\gamma c^2 \\ E_0 = mc^2 \\ p = \frac{E}{c^2}\dot{r} \\ E = \gamma E_0 \end{cases}$$

Aldaezin erlatibista, E_0^2:

$$m^2c^4 = E^2 - c^2p^2 = E'^2 - c^2p'^2$$

- Energia eta momentu lineal transformazioak:

$$\begin{cases} E = \gamma(E' + v p'_x) \\ p_x = \gamma\left(p'_x + \frac{v}{c^2} E'\right) \\ p_y = p'_y \\ p_z = p'_z \end{cases} \qquad \begin{cases} E' = \gamma(E - v p_x) \\ p'_x = \gamma\left(p_x - \frac{v}{c^2} E\right) \\ p'_y = p_y \\ p'_z = p_z \end{cases}$$

- (ct, x, y, z) eta (E, cp_x, cp_y, cp_z) tetrabektoreak:

$$\beta = \frac{v}{c}$$

Transformazioetan aplikatu

- Energia zinetikoa:

$$T = E - E_0 = m(\gamma - 1)c^2$$

- Indarra eta potentzia:

$$\dot{T} = F \cdot \dot{r}$$

$$F = \dot{p} = m\gamma\ddot{r} + m\gamma^3 \frac{\dot{r}\cdot\ddot{r}}{c^2}\dot{r}$$

$$E = \gamma(E' + v p'_x)$$

$$p_x = \gamma\left(p'_x + \frac{v}{c^2} E'\right)$$

$$p_y = p'_y$$

$$p_z = p'_z$$

- Fotoiak:

$$E = h\nu$$ maiztasuna

$$\lambda = \frac{c}{\nu} \Rightarrow p = \frac{E}{c} = \frac{h}{\lambda}$$

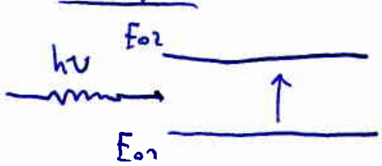
Igorpena:

$$E_{02} = h\nu + E$$

$$E_{01}^2 = E^2 - c^2p^2 = (E_{02} - h\nu)^2 - (h\nu)^2 = \ldots$$

$$h\nu = \Delta E\left(1 - \frac{\Delta E}{2E_{02}}\right) < \Delta E$$

Xurgapena:



$$E_{01} + h\nu = E$$

$$h\nu = \Delta E\left(1 + \frac{\Delta E}{2E_{01}}\right) > \Delta E$$

Doppler efektua:

$E_{iturria} = h\nu$, $p_{it.} = \frac{h\nu}{c}$

$E_{behatzailea} = h\nu'$, $p_{be} = \frac{h\nu'}{c}$

$\Rightarrow$ Energiaren transformazioak

$$h\nu' = \gamma\left(h\nu - v\frac{h\nu}{c}\right), \quad \frac{h\nu'}{c} = \gamma\left(\frac{h\nu}{c} - \frac{v}{c^2}h\nu\right)$$

$$\Downarrow$$

$$\nu' = \sqrt{\frac{c-v}{c+v}}\,\nu$$

4. MEKANIKA ANALITIKOA

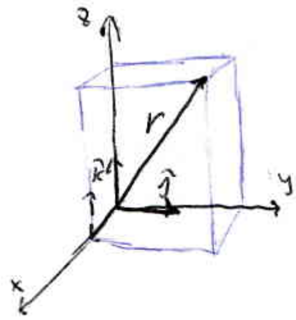
• **Koordenatuak:**

- Kartesiarrak:

$r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\dot{r} = \dot{x} + \dot{y} + \dot{z}$

$\dot{r}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$



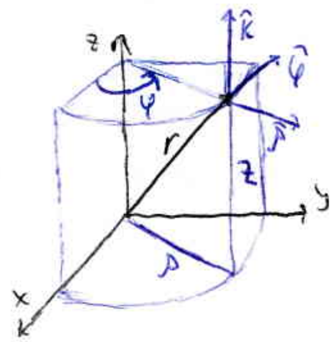
- Zilindrikoak:

$r = \rho\hat{\rho} + z\hat{k}$

$\dot{r} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{k}$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$\dot{r}^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2$



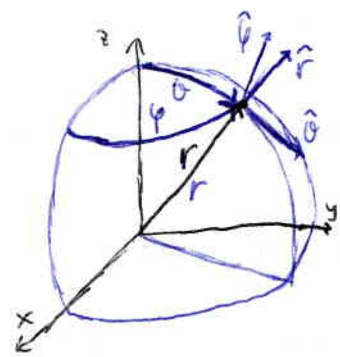
- Esferikoak:

$r = r\hat{r}$

$\dot{r} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$\dot{r}^2 = \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$



- **Loturak eta koordenatu orokortuak:**

$N =$ partikula kopurua

$L =$ lotura kopurua

$3N - L = n \rightarrow$ Askatasun gradua $\longrightarrow$ Koordenatu orokortuak aukeratu $(q_1, q_2, \dots, q_n) \Rightarrow \{(q_1, q_2, \dots, q_n)\}$ ← Konfigurazio espazioa

- **Lotura holonomoak:** Lai honetako gutziak

$f(t, r_1, r_2, \dots) = 0 \quad , g(t, x_1, y_1, z_1, \dots) = 0$

- Lotura egonkorra: $\frac{\partial f}{\partial t} = 0, \ \frac{\partial g}{\partial t} = 0$ • Lotura higikorra: Bestela

- **Estatika analitikoa:**

$$\frac{\partial V}{\partial q_i} = 0 \quad (i = 1, 2, \dots, n)$$

Oreka egonkor/ezegonkorra, energia potentzialaren minimo/maximo/inflexio puntuetan gertatu

Torricelli:

Indar eragilea pisua $\Longrightarrow$ Oreka egonkorra (ezegonkorra) masa zentroa ahalik eta baxuen (altuen) dagoenean, loturak apurtu gabe

$$\frac{\partial z}{\partial q_i} = 0 \quad (i = 1, 2, \dots, n), \qquad z = \frac{m_1 z_1 + m_2 z_2 + \dots + m_N z_N}{m_1 + m_2 + \dots + m_N}$$

- **Hamiltonen printzipioa:**

Ekintza: $I[q] = \int_{t_1}^{t_2} L[t, q(t), \dot{q}(t)]\, dt$

Lagrangearra: $\boxed{L(t, q, \dot{q}) = T - V}$

T: $\quad T(t, q, \dot{q}) = \frac{1}{2} \sum_{k=1}^{N} m_k \dot{r}_k \dot{r}_k$

V: $\quad V(t, q)$

- **Ekintza minimoaren printzipioa:**

  - Transformazio-ekuazioak erabiliz T, V, L eta I $(t, q_i, \dot{q}_i)$-ren mempe bakarrik

  - Hipotesia: Marruskadura lehor dinamikorik ez

  - Hipotesia: Indar eragileak kontserbakorrak

    $\Rightarrow$ Bi konfigurazio artean ditezken ibilbideen artean, ekintza minimoa duena da sistemaren higidura $(\delta I = 0)$

- **Lagrangeren ekuazioak:**

  Hamiltonen printzipioa + Aldakuntzen kalkulua

  $$\Downarrow$$

  $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \qquad i = (1, 2, \dots, n)$$

Problemak ebazteko:

$\longrightarrow$ Koordenatu orokortu egokiak aukeratu

$\longrightarrow$ Transformazio-ekuazioak eta abiadurak idatzi

$$r_k = r_k(t, q_1, q_2, \dots, q_n)$$

$$\dot{r}_k = \frac{dr_k}{dt} = \frac{\partial r_k}{\partial t} + \sum_{j=1}^{n} \frac{\partial r_k}{\partial q_j} \dot{q}_j \qquad k = (1, 2, \dots, N)$$

$\longrightarrow$ T, V eta $L = T - V$ koordenatu orokortuetan idatzi

$\longrightarrow$ Kalkulatu $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

$\longrightarrow$ Ebatzi (edo aztertu) higidura

- Momentu Kanoniko Konjugatuak:

$$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

Koordenatu ziklikoa:

$$\frac{\partial L}{\partial q_i} = 0 \Rightarrow \dot{p}_i = 0 \Rightarrow p_i \text{ Ktea}$$ → Koordenatu ziklikoa bati dagokion momentu Kanoniko Konjugatua higidura konstantea da.

- Hamiltondarra:

$$\boxed{H \equiv \sum_{i=1}^{n} \dot{q}_i p_i - L = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}$$

Hamiltondarra eta energia mekanikoa:

Sistema naturala:

1. Indar eragileak kontserbakorrak
2. Transformazio ekuazioetan denbora ez da explizituki agertzen.

$$\frac{\partial r_k}{\partial t} = 0$$ ← Lotura egonkorrak

$$\Rightarrow H = T + V = E$$

Jacobiren integrala:

Higiduran: $$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

H = Ktea: $$\frac{\partial L}{\partial t} = 0$$

S. Naturala	$\frac{\partial L}{\partial t}=0$	$H=E$	$\dot{H}=0$	Adibidea
Bai	Bai	Bai	Bai	Pendulu matematikoa
Ez	Bai	Ez	Bai	Hari biratorra
Bai	Ez	Bai	Ez	$L=\frac{1}{2}m\dot{x}^2-V(t,x)$
Ez	Ez	Ez	Ez	Hari biratorra eta $w=\alpha t$

• Legendren transformazioak:

Lagrange

Oinarrizko aldagaiak

$t, q_i, \dot{q}_i$

$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}(t, q_i, \dot{q}_i)$

Hamilton

t, q_i, p_i

$\dot{q}_i = \dot{q}_i(t, q_i, p_i)$

$$\boxed{H(t,q,p)=\sum_{i=1}^{n} p_i\,\dot{q}_i(t,q,p)-L[t,q,\dot{q}(t,q,p)]}$$

Oreka:

Oreka posizioak: $\frac{\partial V}{\partial q_i}=0$

Oszilazio txikiak:

$\theta \sim \theta_0 + \delta\theta$

Oreka posizioa

$\dot{\theta} \sim \delta\dot{\theta}$

$\ddot{\theta} \sim \delta\ddot{\theta}$

$$\underset{\text{Lortu}}{\delta\ddot{\theta}+w^2\sin\delta\theta} \overset{\text{Oszilazio txikiak}}{\Rightarrow} \delta\ddot{\theta}+w^2\delta\theta \Rightarrow w=2\pi \underset{\text{Maiztasuna}}{f}$$

- Hamiltonen ekuazio kanonikoak:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

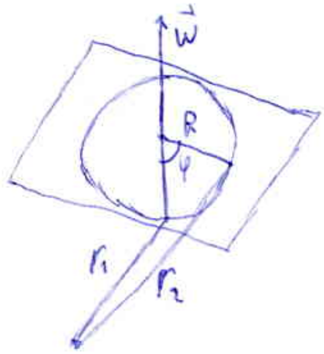
- Jacobiren integrala:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\frac{\partial H}{\partial t} = 0 \Leftrightarrow \frac{\partial L}{\partial t} = 0 \Leftrightarrow \frac{dH}{dt} = 0$$

3. SOLIDO ZURRUNA

Gehienez 6 askatasun gradu

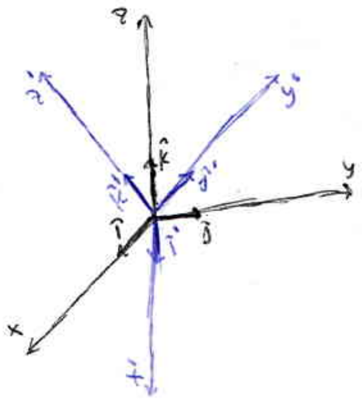


$$w = \frac{d\varphi}{dt}$$

$$|r_2 - r_1| = Kte \Rightarrow (r_2 - r_1)^{\cdot} = w \times (r_2 - r_1)$$

1. Solidoaren edozein bi punturen arteko higidura biraketa hutsa
2. Solido zurrunean barne indarrek lanik ez
3. Solidoaren bi puntu arteko abiadura angeluar erlatiboa: Solidoaren abiadura angeluarra
4. Solidoaren puntu baten abiadura ezagututa, beste batena: Abiadura erenua

Coriolis:



$$u = u_x \hat{i} + u_y \hat{j} + u_z \hat{k} = u'_x \hat{i}' + u'_y \hat{j}' + u'_z \hat{k}'$$

$$\Downarrow$$

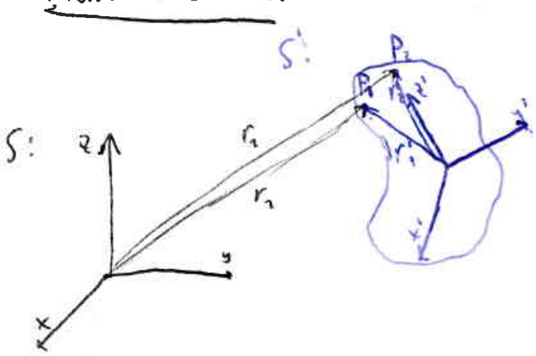
$$\left(\frac{du}{dt}\right)_S = \left(\frac{du}{dt}\right)_{S'} - w \times u$$ → Edozein bektorerekin, adibidez:

Sist. osoaren abiad. angeluarra

$$\vec{N} = \dot{\vec{L}} + \vec{w} \times \vec{L}$$ → Sist. biraketan

Laborat. Sist. birak.

Abiadura erenua:



$$\dot{r}_2 = \dot{r}_1 + w \times (r_2 - r_1)$$

Higidura ekuazioak:

E.S. mertsatuan:

$$\dot{p} = F = \sum_{i=1}^{N} F_i^{(k)} \qquad \dot{L} = N = \sum_{i=1}^{N} r_i \times F_i^{(k)}$$

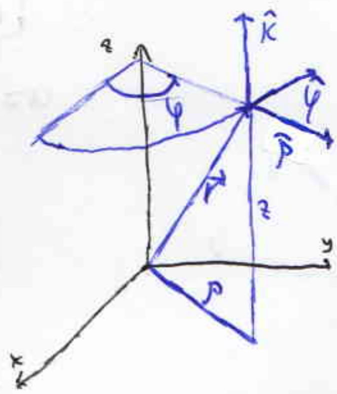
6 askatasun gradu eta 6 higidura ekuazio

Masa zentroaren sisteman:

$$p^* = 0$$

$$\dot{L}^* = N^* = \sum_{i=1}^{N} r_i^* \times F_i^{(k)}$$

Koordenatu zilindrikoak:



$$r = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \rho\hat{\rho} + z\hat{k}$$

$$\rho = \sqrt{x^2 + y^2}$$

$$x = \rho\cos\varphi$$

$$y = \rho\sin\varphi$$

$$\hat{\rho} = \cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}$$

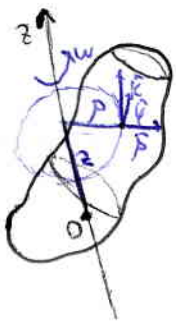
$$\varphi = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

$$\dot{\hat{\rho}} = \dot{\varphi}\,\hat{\varphi}$$

$$\dot{r} = \dot{\rho} + \rho\dot{\varphi} + \dot{z}$$

$$\dot{r}^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2$$

Ardatz finkoaren inguruko biraketa:



$r = \rho\hat{\rho} + z\hat{k}$

$\dot{\rho} = \dot{z} = \dot{\hat{k}} = 0$

$\dot{r} = \rho\dot{\varphi}\hat{\varphi}$

$w = \dot{\varphi}\hat{k}$

$\dot{r} = w \times r$

Momentu angeluar osoa:

$$L = \int dL = w\overbrace{\left[\int \rho^2 dm\right]}^{\text{Inertzia momentua}}\hat{k} - w\left[\int z(x\hat{i} + y\hat{j})dm\right]$$

Momentu angeluarra eta inertzia-momentua:

$I = \int \rho^2 dm$

$L = Iw + wI_{xz}\hat{i} + wI_{yz}\hat{j}$ $\qquad L = r \times p = r \times \dot{r}m$

$I_{xz} = -\int xz\,dm \qquad I_{yz} = -\int yz\,dm$

Ardatza finko:

$N = \dot{L} \neq I\dot{w}$

$w = kte \not\Rightarrow N = 0$

$L_z = Iw$

Energia zinetikoa:

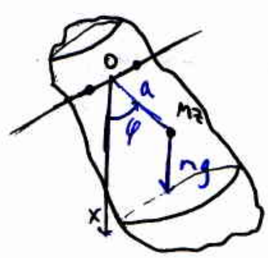
$T = \frac{1}{2} I w^2 \longrightarrow$ Ez beti

$T = \frac{1}{2} m k^2 \dot{\varphi}^2$

$k = \sqrt{\frac{I}{m}}$, $I = mk^2$, k = Biraketa erradioa

$T = \frac{1}{2} L \cdot w$

Pendulu fisikoa:



a = esekidura distantzia

$$\boxed{\ddot{\varphi} + \frac{ga}{k^2} \sin\varphi = 0}$$

l = luzera baliokidea $\Rightarrow$ $l \equiv \frac{k^2}{a}$

Steiner: $I = \underset{\text{Masa zentroa}}{I^*} + ma^2$

5. SOLIDO ZURRUNA

- **Inertzia-tentsorea:** $L = \int_V dL = \int_V [r^2 w - (r \cdot w) r] \, dm$

Inertzia-matrizea:

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \int_V \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dm \cdot \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

$I_{ij} = \int_V (r^2 \delta_{ij} - x_i x_j) \, dm \rightarrow$ Inertzia matrizea

$I_{ii} = \int_V (r^2 - x_i^2) \, dm \rightarrow$ Ardatzekiko inertzia-momentuak

$I_{ij} = -\int_V x_i x_j \, dm \rightarrow$ Inertzia-biderkadurak

Energia-zinetikoa:

$$T = \frac{1}{2} L \cdot w = \frac{1}{2} w^T \cdot I \cdot w$$

- **Inertzia-ardatz nagusiak:**

$L \parallel w$ direnko ardatzak.

Hauek lortzeko, balio eta bektore propioen problema:

$$I \cdot w = \lambda w$$

Inertzia-momentu nagusiak:

$$|I - \lambda \mathbb{1}| = \begin{vmatrix} I_{11}-\lambda & I_{12} & I_{13} \\ I_{21} & I_{22}-\lambda & I_{23} \\ I_{31} & I_{32} & I_{33}-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = I_1, I_2, I_3$$ inertzia-momentu nagusiak

Inertzia-norabide nagusiak:

$$\begin{cases} (I_{11}-I_i)u_{1i} + I_{12}u_{2i} + I_{13}u_{3i} = 0 \\ I_{21}u_{1i} + (I_{22}-I_i)u_{2i} + I_{23}u_{3i} = 0 \\ I_{31}u_{1i} + I_{32}u_{2i} + (I_{33}-I_i)u_{3i} = 0 \end{cases}$$

$$\hat{e}_i^* = \begin{pmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \end{pmatrix}$$ → Ardatz-nagusiekiko paraleloak diren bektore unitarioak

w Ardatz-nagusi batekiko paraleloa $\Rightarrow L = I \cdot w = I_i w$

Triedro-nagusia:

$$I_{ij}^* = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad L = I_1 w_1^* \hat{i}^* + I_2 w_2^* \hat{j}^* + I_3 w_3^* \hat{k}$$

↓
Erref. sistemaren ardatzak, ardatz-nagusiekiko paralelo.

- Ziba simetrikoa: $I_1 = I_2 \Rightarrow \alpha \hat{i}^* + \beta \hat{j}^*$ plano nagusi bat osatu
- Ziba esferikoa: $I_1 = I_2 = I_3 \Rightarrow$ Norabide guztiak nagusiak

Simetriak eta ardatz nagusiak:

z ardatz nagusia $\Rightarrow$ $I_x + I_y \geq I_z$

Objetu laua: $I_x + I_y = I_z$

- Simetria planoak ardatz nagusiekiko perpendikularrak

- Simetria-ardatza nagusia da:
    - Simetria ordena 3 edo handiagoa, plano perpendikularra nagusia eta ziba simetrikoa

Ardatz paraleloen teorema:

S sistematik M.Z.-tik

$$I_{ij} = I^{*}_{ij} + m\left(R^2 \delta_{ij} - X_i X_j\right)$$

R fortu nahi den posiziotik masa-zentrorako bektorea

$R = (X, Y, Z)$

$$I = I^{*} + mR^2 \mathbb{1} - m \begin{pmatrix} X^2 & XY & XZ \\ XY & Y^2 & YZ \\ XZ & YZ & Z^2 \end{pmatrix}$$

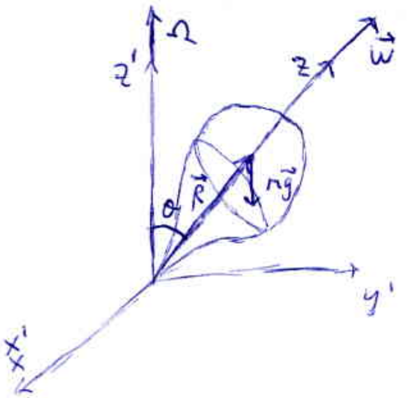
Ardatz orokor baten inguruko inertzia-momentua:

$$I = \hat{n} \cdot I \cdot \hat{n} = \sum_{i,j=1}^{3} I_{ij} n_i n_j$$

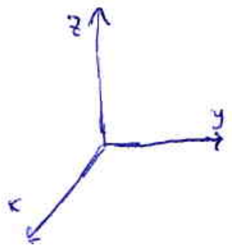
Ziba simetrikoaren prezesioa:

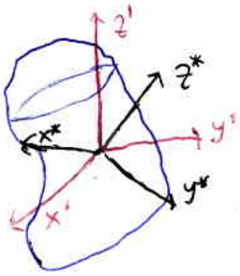
$$N = \dot{L} \Rightarrow \dot{L} = \Omega \times L \Rightarrow \dot{w} = \Omega \times w$$

$$\Omega = \frac{mgR}{I_z w} \hat{k}$$



Erreferentzia sistemak:





- Laborategiko sistema
- Espazioaren sistema (Translazioa baina ez biraketa)
- Solidoaren sistema (Solidoarekin trasladatu eta biratu)

Eulerren ekuazioak:

Solidoaren sisteman, OX, OY eta OZ ardatzak nagusiak eta inertzia-matrizea Ktea izanda.

$$N_x = I_x \dot{w}_x + (I_z - I_y) w_y w_z$$

$$N_y = I_y \dot{w}_y + (I_x - I_z) w_x w_z$$

$$N_z = I_z \dot{w}_z + (I_y - I_x) w_x w_y$$

Higidura askea:

$$I_x \dot{w}_x + (I_z - I_y) w_y w_z = 0$$

$$I_y \dot{w}_y + (I_x - I_z) w_x w_z = 0$$

$$I_z \dot{w}_z + (I_y - I_x) w_x w_y = 0$$

Higidura askearen egonkortasuna:

- Ardatz nagusi inguruan: $w_x = w_y = 0 \longrightarrow \dot{w}_z = 0$

- $|w_x|, |w_y| << |w_z|$

$$(1) \rightarrow \dot{w}_x = \frac{I_y - I_z}{I_x} w_y w_z$$

$$(2)^{\cdot} \rightarrow \ddot{w}_y + \alpha w_y = 0 \quad , \quad \alpha = \frac{(I_x - I_z)(I_y - I_z)}{I_x I_y} w_z^2$$

$$(2) \rightarrow w_x = \frac{I_y \dot{w}_y}{(I_z - I_x) w_z}$$

$$(3) \; \dot{w}_z = 0$$

$\rightarrow I_z > I_x, I_y$ edo $I_z < I_x, I_y$:

$$\alpha = \Omega^2$$

$$\ddot{w}_y + \Omega^2 w_y = 0 \Rightarrow w_y = A \sin(\Omega t + \varphi_0)$$

Egonkorra: w_x eta w_y txikiak mantendu

$\rightarrow I_x > I_z > I_y$ edo $I_x < I_z < I_y$:

$$\alpha = -\lambda^2 > 0$$

$$\ddot{w}_y - \lambda^2 w_y = 0 \Rightarrow w_y = A e^{\lambda t} + B e^{-\lambda t}$$

Ez-egonkorra: w_x eta w_y handitzen joan.

6. Oszilazio txikiak

Oreka egonkorra eta ostiladore harmonikoa:

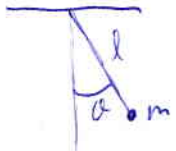
$V'(x_0)=0 \quad K \equiv V''(x_0)>0$

$V(x)=V(x_0)+\frac{1}{2}K(x-x_0)^2+\dots \Rightarrow V(x)\simeq\frac{1}{2}Kx^2,\ F(x)\simeq -Kx$

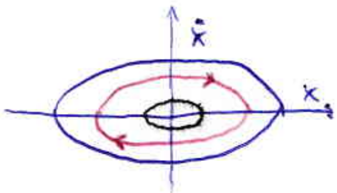
$$\ddot{x}+w^2x=0$$

Pultsazio propioa: $w=\sqrt{\frac{K}{m}}$

$V(\theta)=mgl(1-\cos\theta)\simeq\frac{1}{2}mgl\theta^2$



Fase ibilbidea: $\frac{x^2}{A^2}+\frac{\dot{x}^2}{w^2A^2}=1$



-, Fasoreak:

$x=Ce^{iwt}+De^{-iwt}$

- Posizioa: $z=Ce^{iwt}=Ae^{i(wt+\varphi_0)}$

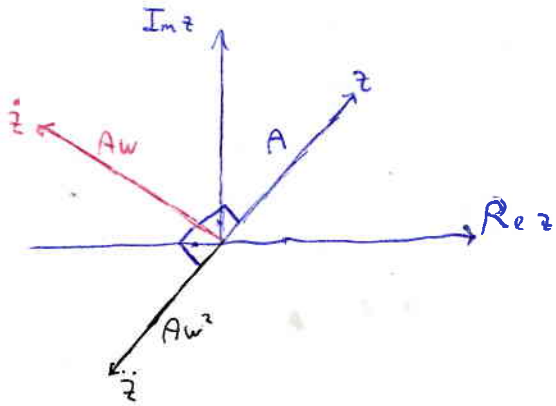
  Soluzioa: $A\cos(wt+\varphi_0)$
  ($\mathrm{Re}\,z$)

- Abiadura: $\dot{z}=iwCe^{iwt}=wAe^{i(wt+\varphi_0+\frac{\pi}{2})}$

  Soluzioa: $-Aw\sin(wt+\varphi_0)$

- Azelerazioa: $-w^2Ce^{iwt}=w^2Ae^{i(wt+\varphi_0+\pi)}$

  Soluzioa: $-Aw^2\cos(wt+\varphi_0)$



- Osziladore indargetua:

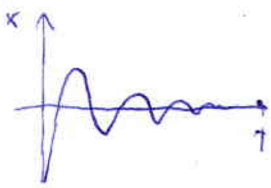
$$\boxed{\ddot{x}+2\gamma\dot{x}+w^2x=0}$$

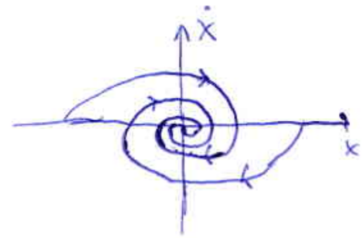
$$z=Ce^{\lambda t} \Rightarrow \lambda=-\gamma\pm\sqrt{\gamma^2-w^2}$$

- Indargetze ahula ($\gamma<w$):

$$\lambda=-\gamma\pm w_r \qquad w_r=\sqrt{w^2-\gamma^2}$$

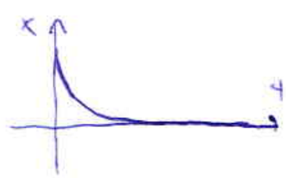
$$x=\mathrm{Re}\, z = A e^{-\gamma t}\cos(w_1 t+\varphi_0)$$

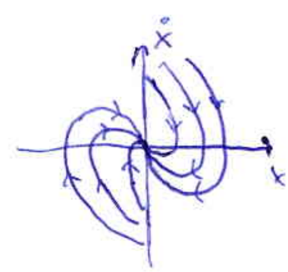




- Gainindargetzea ($\gamma>w$):

$$x=C_1 e^{-\lambda_1 t}+C_2 e^{-\lambda_2 t}$$

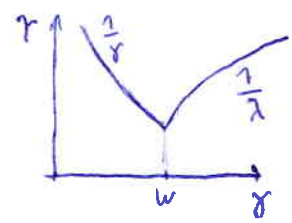




- Indargetze kritikoa ($\gamma=w$):

$$x=(C+Dt)e^{-wt}$$

$$\tau=\frac{1}{w}=\frac{1}{\gamma}$$

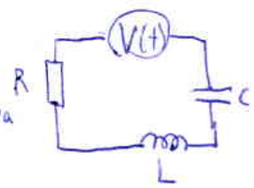


- Osziladore bortxatua:

$$\boxed{\ddot{x}+2\gamma\dot{x}+w^2x=f(t)}$$

Adibidez:

RLC zirkuitua



$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{dV}{dt}(t)$$

Indar Sinusoidala:

$$\ddot{x}+2\gamma\dot{x}+\omega^2x=f_0\cos(\Omega t+\alpha)$$

Soluzio orokorra: $x=x_g+x_v \xrightarrow{t\to+\infty} x_v$

$$z=Ae^{i(\Omega t+\varphi_0)}$$

$$\ddot{x}+2\gamma\dot{x}+\omega^2x=f_0e^{i(\Omega t+\alpha)} \Rightarrow Ae^{i\Omega t}e^{i\varphi_0}\left(-\Omega^2+2i\gamma\Omega+\omega^2\right)=f_0e^{i\Omega t}e^{i\alpha}$$

Iraunkorra:

$$x_v=A\cos(\Omega t+\varphi_0)$$

$$A=\frac{f_0}{\Delta}$$

$$\sin(\varphi_0-\alpha)=-\frac{2\gamma\Omega}{\Delta}$$

$$\cos(\varphi_0-\alpha)=-\frac{\Omega^2-\omega^2}{\Delta}$$

$$\Delta=\sqrt{(\Omega^2-\omega^2)^2+4\gamma^2\Omega^2}$$

Inpedantzia mekanikoa:

$$F=m(\ddot{x}+2\gamma\dot{x}+\omega^2x)$$

$$F=Z\dot{x}$$

$$Z=m\left[2\gamma+i\left(\Omega-\frac{\omega^2}{\Omega}\right)\right]$$

Inpedantzia

$$|Z|=\sqrt{R^2+\left(\Omega L-\frac{1}{\Omega C}\right)^2}$$

$$\delta=\operatorname{Arg} Z=\arctan\frac{1}{R}\left(\Omega L-\frac{1}{\Omega C}\right)$$

$$V=Z\,I=|Z|I_0e^{i(\Omega t+\delta)}$$

Anplitude-erresonantzia:

$$A=\frac{f_0}{\sqrt{(\Omega^2-\omega^2)^2+4\gamma^2\Omega^2}},\quad \gamma<\frac{\omega}{\sqrt{2}}$$

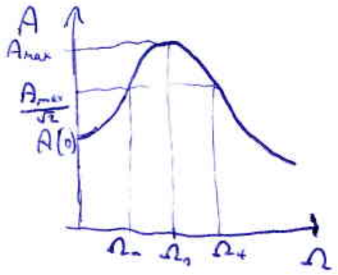
Hondarra

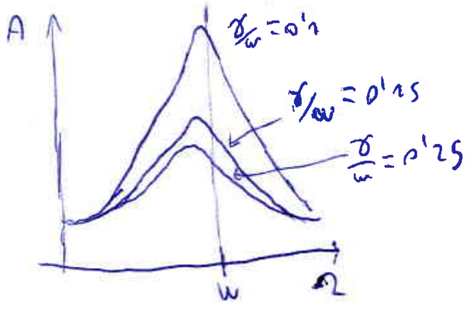
$$\Omega_0=\sqrt{\omega^2-2\gamma^2}=\omega-\frac{\gamma^2}{\omega}+O(\gamma^4)$$

$$A_{max}=A(\Omega_0)=\frac{f_0}{2\gamma\sqrt{\omega^2-\gamma^2}}=\frac{f_0}{2\gamma\omega}+O(\gamma)$$

$$\Delta\Omega\equiv\Omega_+-\Omega_-=2\gamma+O(\gamma^3)$$

$$\varphi_0-\alpha=-\arcsin\sqrt{\frac{\omega^2-2\gamma^2}{\omega^2-\gamma^2}}=-\frac{\pi}{2}+\frac{\gamma}{\omega}+O(\gamma^2)$$





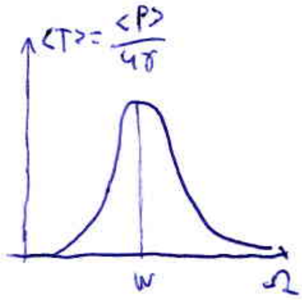
Erresonantzia-energia:

$$\langle V\rangle = \frac{1}{4}\omega^2 A^2$$

$$\langle T\rangle = \frac{1}{4}A^2\Omega^2$$

$$\langle P\rangle = \langle F\dot{x}\rangle = \frac{\gamma f_0^2}{\left(\Omega-\frac{\omega^2}{\Omega}\right)^2+4\gamma^2}$$

$$\langle T\rangle = \frac{\langle P\rangle}{4\gamma}$$



Gainezarmen printzipioa:

$$\left.\begin{aligned}\ddot{x}_1+2\gamma\dot{x}_1+\omega^2x_1=f_1\\ \ddot{x}_2+2\gamma\dot{x}_2+\omega^2x_2=f_2\end{aligned}\right\} \longrightarrow \begin{aligned}&x=\alpha_1x_1+\alpha_2x_2\\ &\ddot{x}+2\gamma\dot{x}+\omega^2x=\alpha_1f_1+\alpha_2f_2\end{aligned}$$

Fourier:

$$f(t)=\sum_{n=-\infty}^{\infty}C_n e^{in\Omega t} \qquad \Omega=\frac{2\pi}{T}$$

$$C_n=\frac{1}{T}\int_0^T f(t)e^{-in\Omega t}\,dt$$

$$f(t)=\int_{-\infty}^{\infty}F(\Omega)e^{i\Omega t}\,d\Omega$$

$$F(\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\,e^{-i\Omega t}\,dt$$

Green:

$$G(t-u)=\frac{1}{\omega_\gamma}e^{-\gamma(t-u)}\sin\omega_\gamma(t-u) \qquad \omega_\gamma=\sqrt{\omega^2-\gamma^2}$$

$$x_u=\int_{-\infty}^{t}G(t-u)\,f(u)\,du$$

$$f(t)=\int_{-\infty}^{\infty}f(u)\,\delta(t-u)\,du$$

- Taupadak:

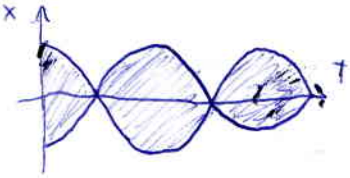
$$x = R_e(z_1 + z_2) = A_1 \cos(w_1 t + \varphi_{10}) + A_2 \cos(w_2 t + \varphi_2)$$

$$\delta = (w_2 - w_1)t + (\varphi_{20} - \varphi_{10})$$

$$\delta \bmod 2\pi = 0 \Rightarrow |z| = A_1 + A_2$$

$$\delta \bmod 2\pi = \pi \Rightarrow |z| = |A_1 - A_2|$$

$\delta = 0$



$$x = 2A_2 \cos(\varepsilon t + \mu)\cos(wt + \xi) + (A_1 - A_2)\cos(w_1 t + \varphi_{10})$$

$$\varepsilon = \frac{w_1 - w_2}{2} \qquad \mu = \frac{\varphi_{10} - \varphi_{20}}{2}$$

$$w = \frac{w_1 + w_2}{2} \qquad \xi = \frac{\varphi_{10} + \varphi_{20}}{2}$$

$\delta \neq 0$

$A_1 = A_2$ eta $w_1 \approx w_2$:

$$x = A(t)\cos(wt + \xi), \quad A(t) = 2A_1 \cos(\varepsilon t + \mu) \qquad w \approx w_1 \simeq w_2$$

$$\nu_t = |\nu_1 - \nu_2| = \frac{|w_1 - w_2|}{2\pi}$$

- Osziladore harmoniko anisotropoa:

$$V(x,y) = \frac{1}{2}(K_x x^2 + K_y y^2)$$

$$w_x = \sqrt{\frac{K_x}{m}} \qquad w_y = \sqrt{\frac{K_y}{m}}$$

Lagrangearra $\rightarrow L = \frac{1}{2}m(\dot{x} - w_x^2 x) + \frac{1}{2}m(\dot{y} - w_y^2 y)$

Higidura ekuazioak $\longrightarrow$

$$\ddot{x} + w_x^2 x = 0 \Rightarrow x = A_x \cos(w_x t + \varphi_{x0})$$

$$\ddot{y} + w_y^2 y = 0 \Rightarrow y = A_y \cos(w_y t + \varphi_{y0})$$

$\longrightarrow$ Lissajousen irudiak

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_x e^{iw_x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_y e^{iw_y} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Lehen eta bigarren modu normalak

•Oszilazio mihiztatuak:

→ Lagrangearra aurkitu

Kasu partikularrean → Higidura ekuazioak planteatu

→ $x_1 = C_1 e^{i\omega t}$

$x_2 = C_2 e^{i\omega t}$

$$\begin{pmatrix} K+K'-m\omega^2 & -K' \\ -K' & K+K'-m\omega^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$C_1 = 0$, $C_2 = 0$ beti egin daiteke

$= 0 \Rightarrow \omega_1 = \sqrt{\frac{K}{m}}$, $\omega_2 = \sqrt{\frac{K+2K'}{m}}$ → Pultsazio normalak

→ Higidura ekuazioetatik C_2 C_1-en mepe

→ $(\omega = \omega_1)$: $x_1 = C e^{i\omega_1 t}$, $x_2 = C e^{i\omega_1 t}$ $\qquad$ $(\omega = \omega_2)$: $x_1 = D e^{i\omega_2 t}$, $x_2 = -D e^{i\omega_2 t}$

Modu normalak

→ Soluzio orokorra:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + D \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t}$$

→ $Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2)$ $\quad Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2)$ → Koordenatu normalak

$$x_1 = \frac{Q_1 + Q_2}{\sqrt{2m}} \qquad x_2 = \frac{Q_1 - Q_2}{2m}$$

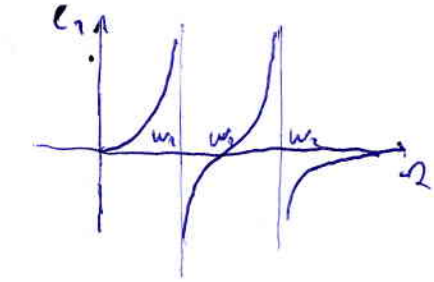
→ Oszilazio abstraktuak, koordenatu horietako, independenteak:

$$\ddot{Q}_1 + \omega_1^2 Q_1 = 0 \quad , \quad Q_1 = C_1 e^{i\omega_1 t}$$
$$\ddot{Q}_2 + \omega_2^2 Q_2 = 0 \quad , \quad Q_2 = C_2 e^{i\omega_2 t}$$

Osziloatio mihiztatu bortxatuak:

$$m\ddot{x}_1+(K+K')x_1-K'x_2=F_0\cos(\Omega t+\alpha)$$

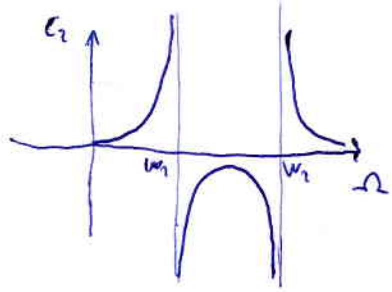
$$m\ddot{x}_2-K'x_1+(K+K')x_2=0$$



$\gamma=0\,,\quad \varphi_0-\alpha=0,\pi$

Marruskadurarik ez

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}C_1\\C_2\end{pmatrix}\cos(\Omega t+\alpha)$$



$$(-m\Omega^2+K+K')\,C_1-K'C_2=F_0$$

$$-K'C_1+(-m\Omega^2+K+K')C_2=0$$

$$C_1=\frac{\omega_1^2+\omega_2^2-2\Omega^2}{(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)}\frac{F_0}{2m}\qquad C_2=\frac{\omega_2^2-\omega_1^2}{(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)}\frac{F_0}{2m}$$

- Oszilazio txikiak eta Koordenatu normalak:

$$\left.\frac{\partial V}{\partial q_i}\right|_0=0\qquad V=\left(V_{ij}=\left.\frac{\partial^2 V}{\partial q_i\,\partial q_j}\right|_0\right)$$

Koadratikoa: $V(q)=V(0)+\frac{1}{2}\sum_{i,j=1}^{n}\left.\frac{\partial^2 V}{\partial q_i\,\partial q_j}\right|_0 q_i q_j+\cdots$

Zurruntasun matrizea

$$V\approx\frac{1}{2}\sum_{i,j=1}^{n}V_{ij}\,q_iq_j\qquad (V|_0=0)$$

T masa-matrizea eta V zurruntasun-matrizea Kteak, simetrikoak eta positiboak

(Koordenatu orokortu batzuetan ... balioa positiboa batzuetan)

$$T\approx\frac{1}{2}\sum_{i,j=1}^{n}T_{ij}\,\dot{q}_i\dot{q}_j\qquad (\text{Sist nat.})$$

Masa-matrizea

$$L\approx\frac{1}{2}\sum_{i,j=1}^{n}\left(T_{ij}\dot{q}_i\dot{q}_j-V_{ij}q_iq_j\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_j}=\sum_{j=1}^{n}\left(T_{ij}\ddot{q}_j+V_{ij}q_j\right)=0$$

$$T\cdot\ddot{q}+V\cdot q=0$$

$\sin\theta \simeq \theta$
$\cos\theta \simeq 1 + \frac{\theta^2}{2}$ $\Rightarrow$ - T eta V kalkulatzean 2. ordena baina handiagoko gaiak arbiatu
- Inertzia momentuak masa zentrotik hartu.

Modu normalak:

Maiztasun bakarreko soluzioak:

$$q_j = C_j e^{i\omega t}$$

$$\sum_{j=1}^{n} \left(T_{ij}\ddot{q}_j + V_{ij}q_j\right) = 0 \Rightarrow \sum_{j=1}^{n} \left(V_{ij} - \omega T_{ij}\right) = 0$$

$$U = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} \qquad V \cdot U = \lambda T \cdot U$$

$(V - \lambda T) \cdot U = 0 \longrightarrow$ Balio eta bektore orokortuak

$|V - \lambda T| = 0 \longrightarrow$ Pultsazio (maiztasun) normalak: $\omega_K = \sqrt{\lambda_K}$

$\lambda = \lambda_K$ bada izateko, $(V - \lambda T) \cdot U = 0$

$$\begin{cases} (V_{11} - T_{11}) u_1 + \dots + (V_{1n} - \lambda T_{1n}) u_n = 0 \\ (V_{21} - T_{21}) u_1 + \dots + (V_{2n} - \lambda T_{2n}) u_n = 0 \\ \vdots \end{cases}$$

$u_i = C_i$ Lortzen n bektore "ortonormal"

$S \equiv (U_1 U_2 U_3 \dots)$ $S^T \cdot T \cdot S = 1 \longrightarrow U_i$-ren normalizazioa egin

Koordenatu normalak:

$$q = S \cdot Q$$

$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ $(U_1 U_2 \dots)$ $\begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \end{pmatrix}$

Oszilatzaile independenteak:

$$L = \sum_{i=1}^{n} L_i \qquad L_i = \frac{1}{2}\dot{Q}_i^2 - \frac{1}{2}\omega_i^2 Q_i^2$$

$$\ddot{Q}_i + \omega^2 Q_i = 0$$

$$Q_j = C_j e^{i\omega_K t}$$

-nd-

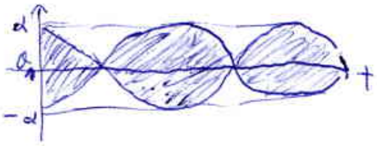
Taupadak:

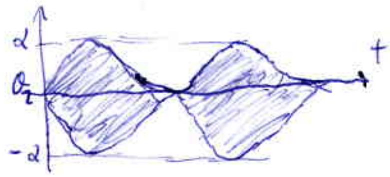
$\theta_1 = (\alpha \cos \varepsilon t) \cos wt$

Hasierako baldintzetatik $\longrightarrow$ $\theta_2 = (\alpha \sin \varepsilon t) \sin wt = \left[\alpha \cos\left(\varepsilon t - \frac{\pi}{2}\right)\right] \sin wt$

$$\varepsilon = \frac{w_2 - w_1}{2} \quad , \quad w = \frac{w_2 + w_1}{2}$$

Mihiztadura ahula: $\frac{K}{m} << \frac{g}{l} \Rightarrow w_1 \simeq w_2 \simeq w >> \varepsilon$





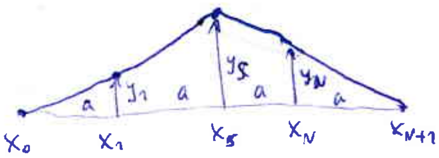
Modu normal nuluak:

$w_i = 0$

$\ddot{Q}_i = 0 \Rightarrow Q_i = C_i t + D_i$

- Traslazio simetria
- Orekaneutroa
- Verdidenbidu positiboa

- Tentsiopeko soka diskretua:



$y_0 = y_{N+1} = 0$

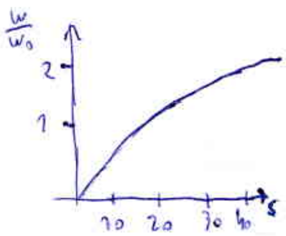
$|y_s| << a \Rightarrow T = Ktea$

$$a + dl_s = \sqrt{(x_{s+1} - x_s)^2 + (y_{s+1} - y_s)^2} = \sqrt{a^2 + (y_{s+1} - y_s)^2} \simeq a + \frac{(y_{s+1} - y_s)^2}{2a}$$

$$V_s = -\underset{\text{Tentsioa}}{T} \cdot dr_s = T dl_s$$

$$E_z \nearrow T = \frac{1}{2} m (\dot{y}_1^2 + \dot{y}_2^2 + \dots + \dot{y}_N^2)$$

$$V = \frac{1}{2} m w_0^2 \left[y_1^2 + (y_2 - y_1)^2 + \dots + (y_N - y_{N-1})^2 + y_N^2\right]$$



$\ddot{y}_1 = w_0^2 (y_2 - 2 y_1)$

$\ddot{y}_2 = w_0^2 (y_3 - 2y_2 + y_1)$

$\vdots$

$\ddot{y}_N = w_0^2 (-2 y_N + y_{N-1})$

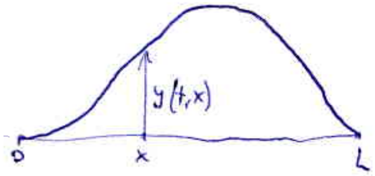
$y_s = C_s e^{iwt}$ saiatuz $\longrightarrow$ $N=1 \quad w_1 = \sqrt{2}\, w_0$

$N=3 \quad w_1^2 = (2 - \sqrt{2}) w_0^2$

$w_2^2 = 2 w_0^2$

$w_3^2 = (2 + \sqrt{2}) w_0^2$

Tentsiopeko soka jarraitua:



$$\mu = \lim_{a\to 0} \frac{m}{a} = \lim_{N\to\infty} \frac{(N+1)m}{L}$$

$\vdots$

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \qquad v = \sqrt{\frac{T}{\mu}}$$

$y_s = C_s e^{i\omega t}$ saiatuz $\longrightarrow$ $C''(x) + K^2 C(x) = 0 \qquad K = \frac{\omega}{v} \geq 0$

$$C(x) = A\cos Kx + B\sin Kx$$

$y(t,0) = 0 \Rightarrow A = 0$

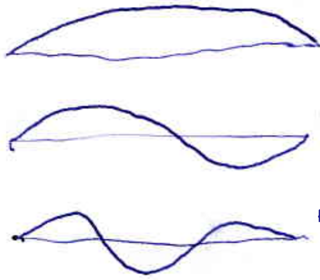
$y(t,L) = 0 \Rightarrow \sin KL = 0 \Rightarrow KL = n\pi$

$K_n = \frac{n\pi}{L} \qquad \omega_n = n\omega_1$

$\omega_1 = \frac{\pi v}{L}$

$$y = A_n \sin\frac{n\pi x}{L}\, e^{i\frac{n\pi v t}{L}}$$

Modu normalak:



n=1

n=2

n=3

Gainezarmena:

$$y = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\, e^{i\frac{n\pi v t}{L}}$$

$$\left(\nu = \frac{v}{2L}\right)$$

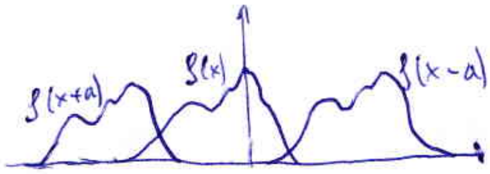
7. UHIN HIGIDURA

- Uhin ekuazioa:

$$u(t,x) = f(x \mp vt)$$

$$\frac{\partial u}{\partial t} = \mp v \frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$



Soluzio orokorra: $u(t,x) = f(x-vt) + g(x+vt)$

- Uhin harmonikoa:

$$f(x) = u(0,x) = C e^{i(Kx \mp wt)} \left[= A\cos(Kx \mp wt + \varphi_0) \right]$$

Uhin zenbakia: $K = \frac{2\pi}{\lambda}$

Pultsazioa: $w = |Kv|$

Periodoa: $T = \frac{2\pi}{w}$

Maiztasuna: $\nu = \frac{1}{T} = \frac{w}{2\pi} = \frac{|Kv|}{2\pi}$

Uhin luzera: $\lambda = \frac{2\pi}{|K|} = \frac{|v|}{\nu}$

v hedapen abiadura $\longrightarrow$ Fase abiadura

- Uhin periodikoa:

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in(Kx \mp wt)}$$

$$C_n = \frac{1}{T}\int_0^T u(t,0)\, e^{\pm inwt}\, dt$$

- Fourier:

$$u(t,x) = \int_{-\infty}^{\infty} F(k)\, e^{ik(x-vt)}\, dk$$

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(0,x)\, e^{-ikx}\, dx$$

- Talde-abiadura:

$K_1 \simeq K_2 \simeq K, \quad w_1 \simeq w_2 \simeq w$

$$u = u_0 \cos(K_1 x - w_1 t) + u_0 \cos(K_2 x - w_2 t) = 2 u_0 \cos\left[\frac{K_1 - K_2}{2}\left(x - \frac{w_1 - w_2}{K_1 - K_2} t\right)\right] \cos\left[\frac{K_1 + K_2}{2}\left(x + \frac{w_1 + w_2}{K_1 + K_2} t\right)\right]$$

Fase abiadura: $V_f = \frac{w}{K}$

Talde-abiadura: $V_T = \frac{dw}{dK}$

Ingurune ~~xx~~ sakabanatzailea: K ezberdineko uhin monokromatikoek abiadura ezberdina

$$V_T = V_f + K \frac{dV_f}{dK}$$

- Luzetarako uhinak:

- Barra elastikoa:

- Estatika: Deformazio eremua: $u(x)$, $u(0) = 0$ $u(l) = \Delta l$

Esfortzua: $\tau = \frac{F}{A}$ — A Azalera

Hooke: $F \propto A \frac{u(x)}{x} = A \frac{\Delta l}{l}$

Youngen modulua: $\tau = E \frac{u(x)}{x}$

Young (Presioa)

Energia potentziala: (lana) $V = \frac{1}{2} A E l \left(\frac{\Delta l}{l}\right)^2$

- Dinamika: $F(t,x) = A\,\tau(t,x) = EA \frac{\partial u}{\partial x}(t,x) \Rightarrow dF = EA \frac{\partial^2 u}{\partial x^2}(t,x)\,dx$

Masa dentsitatea

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2}$$

$$V = \sqrt{\frac{E}{\rho}}$$

Tentsio-uhina:

$$\tau = E \frac{\partial u}{\partial x} \longrightarrow \frac{\partial^2 \tau}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \tau}{\partial t^2} = 0$$

Momentu lineala:

$$dp = \rho A \frac{\partial u}{\partial x} dx \longrightarrow \frac{\partial^2 \left(\frac{dp}{dx}\right)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \left(\frac{dp}{dx}\right)}{\partial t^2} = 0$$

Energia:

$$dT = \frac{1}{2}\rho A u_0^2 \sin^2(kx-wt)\,dx$$

$$dV = \frac{1}{2}\rho A u_0^2 \sin^2(kx-wt)\,dx$$

$$\langle dT + dV \rangle = \frac{1}{2}\rho A w^2 u_0^2\,dx$$

Energia dentsitatea $\rightarrow U = \frac{\langle dT+dV\rangle}{A\,dx} = \frac{1}{2}\rho w^2 u_0^2$

Potentzia:

$$P = -EA\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} = v\frac{dT+dV}{dx}$$

$$\langle P \rangle = vAU$$

Intentsitatea $\rightarrow I = \frac{\langle P\rangle}{A} = vU$

- Gas-zutabea:

$p(x,t)$ presio erenua

$\rho(x,t)$ dentsitate uenua

$$\Delta(dV) = \frac{\partial u}{\partial x}dV \Rightarrow \frac{\Delta(dV)}{dV} = \frac{\partial u}{\partial x}$$

Hooke: $\Delta p = -B\frac{\Delta V}{V}$, B Konprimiagarritasun-modulua

$$p = \underset{\text{Orekan}}{p_0} - B\frac{\partial u}{\partial x}$$

$$dF = \rho_0 A \frac{\partial^2 u}{\partial t^2}dx$$

Presio-uhina: $\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} = 0 \qquad v = \sqrt{\frac{B}{\rho_0}}$$

- Soinua:

Gas idealetan:

$B = \gamma p_0$, $\gamma = \frac{C_p}{C_v}$ berretzaile adiabatikoa

$$v = \sqrt{\frac{\gamma p}{\rho}} \qquad pV = nRT \Leftrightarrow p\overset{\text{masa molarra}}{m} = \rho RT \quad \left(m = \frac{\rho V}{n}\right)$$

$$\overset{\shortparallel}{v} = \sqrt{\frac{\gamma RT}{m}} \overset{\text{Airean}}{\Rightarrow} 20'055\sqrt{T}$$

Soinua: $u = u_0 e^{i(kx-wt)}$

Presio uhina: $p - p_0 = -B\frac{\partial u}{\partial x} = wv\rho u_0 e^{i(kx-wt-\frac{\pi}{2})}$

Anplitudea: $\mathcal{I} = vw\rho u_0$

Intentsitatea: $I = vU = \frac{\mathcal{I}^2}{2\rho v}$

Energia-dentsitatea: $U = \frac{1}{2}\rho w^2 u_0^2 = \frac{\mathcal{I}}{2\rho v^2}$

$10\log_{10}\frac{I}{I_0}(dB)$, $I_0 = 10^{-12}\frac{W}{m^2}$

• Zeharkako uhinak:

-Tentsiopeko soka:

μ luzera unitateko masa

$T_x = T\cos\alpha$ $\qquad \tan\alpha = \frac{\partial u}{\partial x}$ $\qquad dT_x = 0$

$T_y = T\sin\alpha$ $\qquad\qquad\qquad\qquad dT_y = \mu\, dx \frac{\partial^2 u}{\partial t^2}$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 u}{\partial t^2} = 0 \qquad v = \sqrt{\frac{T}{\mu}}$$

Inpedantzia:

$x=0$ puntuan $F = F_0 e^{i\omega t}$ indarra

$u = u_0 e^{i(kx-\omega t+\varphi_0)}$

Anplitude txikiak

$$Z = \frac{F}{\dot{y}} \qquad Z = \mu v$$

Inpedantzia Karakteristikoa

Polarizazioa:

$y(t,x) = u_{0y} e^{i(kx-\omega t+\varphi_{0y})}$

$z(t,x) = u_{0z} e^{i(kx-\omega t+\varphi_{0z})}$

$u(t,x) = y(t,x)\hat{j} + z(t,x)\hat{k}$

Puntu baten higidura:

Eliptikoa:

Lineala:

$\varphi_{0x} = \varphi_{0y}$

Zirkularra:

$u_{0y} = u_{0z}$

$\varphi_{0x} - \varphi_{0y} = \pm\frac{\pi}{2}$

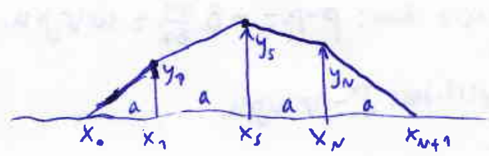
-Egitura periodikoan:

$\ddot{y}_s = \omega_0^2(y_{s+1} - 2y_s + y_{s-1}) \qquad \omega_0 = \sqrt{\frac{T}{ma}}$

Soka jarraitua: $a \to 0$: $v \approx \omega_0 a = \sqrt{\frac{T}{\mu}}$

$y_s = C e^{i(ksa-\omega t)}$

$-\omega^2 = -4\omega_0^2 \sin^2\frac{ka}{2}$



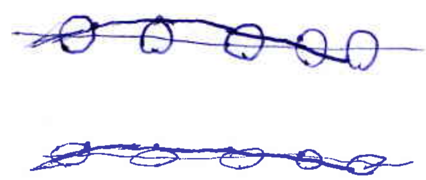
$x_s = sa \qquad N \to \infty$

Sakabanatze erlazioa: $\omega(k) < 2\omega_0$

Ingurune sakabanatzailea: $v = \frac{\omega}{k} = \frac{2\omega_0}{k}\sin\frac{ka}{2} = v(k)$

-Kanal batean:

$$v=\sqrt{\left(\frac{g\lambda}{2\pi}+\frac{2\pi\gamma}{\rho\lambda}\right)\tanh\frac{2\pi h}{\lambda}}$$



ρ masa-dentsitatea
γ Gainazal-tentsioa
h Altuera

• Uhinak bi eta hiru dimentsiotan:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}=0 \iff \nabla^2 u-\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}=0$$

Uhin harmoniko laua: $u(t,r)=u_0 e^{iK(\hat{n}\cdot r-vt)}=u_0 e^{i(K\cdot r-wt)}$

Uhin bektorea: $\mathbf{K}\equiv K\hat{n}$

-Uhin zirkular eta esferikoak:

$u(t,r)$

$$\frac{\partial u}{\partial x}=\frac{x}{r}\frac{\partial u}{\partial r} \qquad \left(\frac{\partial r}{\partial x}=\frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x_i^2}=\frac{x_i^2}{r^2}\frac{\partial^2 u}{\partial r^2}+\left[\frac{1}{r}-\frac{x_i^2}{r^3}\right]\frac{\partial u}{\partial r}\quad , \quad x_i=x,y,z$$

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}-\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}=0 \qquad u(t,r)=\frac{1}{r}f(r\mp vt) \text{ soluzio orokorra}$$

Esferiko harmonikoa: $u=\frac{u_0}{r}e^{iK(r-vt)}$

$$|u|=\frac{u_0}{r}$$

Presio-uhin esferikoak:

(Fluido isotropoan)

Uhin esferiko harmonikoa: $p - p_0 = \frac{\mathcal{Z}}{r} e^{i(kr - \omega t - \frac{\pi}{2})}$

$kr \propto \frac{r}{\lambda} >> 1 \Rightarrow u = \frac{u_0}{r} e^{i(kr - \omega t)}$ $\qquad u_0 = \frac{\mathcal{Z}}{v \omega \rho_0}$

Energia dentsitatea: $U = \frac{1}{2} \frac{\rho_0 \omega^2 u_0^2}{r^2} = \frac{\mathcal{Z}^2}{2 \rho_0 v^2 r^2}$

Intentsitatea: $I = vU = \frac{I_0}{r^2}$

Energia kontserbazioa: $\oint I\, dS = IS = 4\pi r^2 I = \frac{2\pi \mathcal{Z}^2}{\rho_0 v^2} = Kte$

-Uhin elektromagnetiko lauak:

$\vec{E} = E(x,y,z,t)\hat{\jmath}, \ \vec{B} = B(x,y,z,t)\hat{k}$

Maxwellen ekuazioak:

Gauss: $\nabla \cdot E = 0 \Rightarrow \frac{\partial E}{\partial y} = 0$

$\nabla \cdot B = 0 \Rightarrow \frac{\partial B}{\partial z} = 0$

Faraday-Henry: $\nabla \times E = -\frac{\partial B}{\partial t} \Rightarrow \begin{cases} \frac{\partial E}{\partial z} = 0 \\ \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \end{cases}$

Ampere-Maxwell: $\nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \Rightarrow \begin{cases} \frac{\partial B}{\partial y} = 0 \\ \frac{\partial B}{\partial x} = -\varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \end{cases}$

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

$$\frac{\partial^2 B}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = 0 \qquad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\frac{\partial E}{\partial x} = \pm c \frac{\partial B}{\partial x}$$

$$\frac{\partial E}{\partial t} \pm c \frac{\partial B}{\partial t}$$

Polarizazioa:

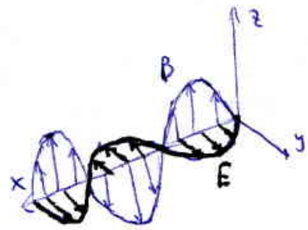
Bi soluzio: $E = f(x \mp ct)\,\hat{j}$ $\quad B = f(x \mp ct)\,\hat{k}$

$E = g(x \mp ct)\,\hat{k}$ $\quad B = g(x \mp ct)\,\hat{j}$

$\longrightarrow$ Harmonikoa: $\vec{E} = A_1 e^{i(kx-\omega t)}\,\hat{j}\left[+A_2 e^{i(kx-\omega t+\frac{\pi}{2})}\,\hat{k}\right]$

$$\vec{B} = \frac{A_1}{c} e^{i(kx-\omega T)}\,\hat{k}\left[-\frac{A_2}{c} e^{i(kx-\omega t+\frac{\pi}{2})}\,\hat{j}\right]$$

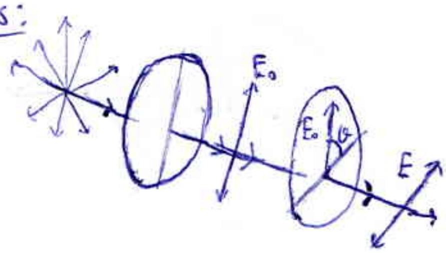
Lineala: $\vec{E}$ $\hat{j}$-n eta $\vec{B}$ $\hat{k}$-n bakarrik



Zirkularra:

$A_1 = A_2$

Malus:



$E = E_0 \cos\theta$

$I = I_0 \cos^2\theta$

Polarizazio maila: $P = \dfrac{I_{\parallel} - I_{\perp}}{I_{\parallel} + I_{\perp}}$

Poynting eta energia fluxua:

$$u = \frac{1}{2}\varepsilon_0 \left(E^2 + c^2B^2\right)$$

Uhinetan: $E^2 = c^2B^2$, $u = \varepsilon_0 E^2 = \dfrac{B^2}{\mu_0}$

Aldiuneko intentsitatea: $I = \varepsilon_0 c E^2$

Poynting: $G = \dfrac{1}{\mu_0} E \times B = \varepsilon_0 c^2 E \times B$

Fotoien momentu lineala: $P = \dfrac{G}{c^2}$

Argiaren abiadura ingurune materialean: $v = \dfrac{1}{\sqrt{\varepsilon\mu}}$, $\varepsilon = \varepsilon_r\varepsilon_0$, $\mu = \mu_r\mu_0$

Errefrakzio indizea: $n = \sqrt{\varepsilon_r\mu_r}$ $\quad v = \dfrac{c}{n}$

-Doppler:

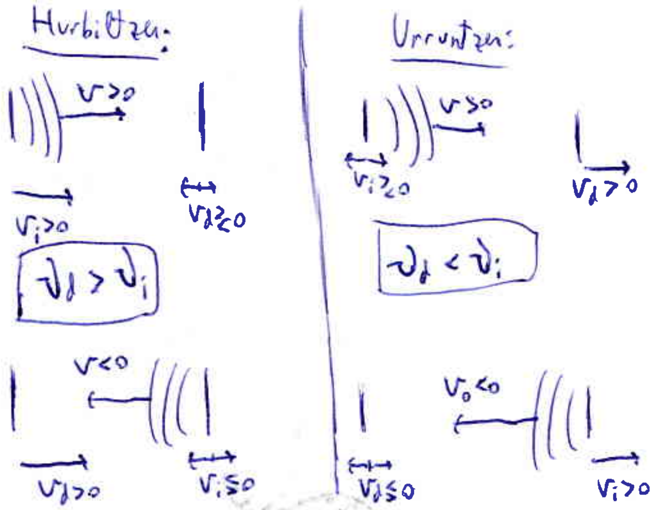
$$d = vt_0 - v_i T_i + v_d T_d = v(t_0 + T_d - T_i)$$

$$T_d = \frac{v - v_i}{v - v_d} T_i$$

$$\nu_d = \frac{v - v_d}{v - v_i} \nu_i = \left(1 - \frac{v_d - v_i}{v - v_i}\right) \nu_i$$

Taupadak: $\Delta\nu = |\nu_1 - \nu_0|$

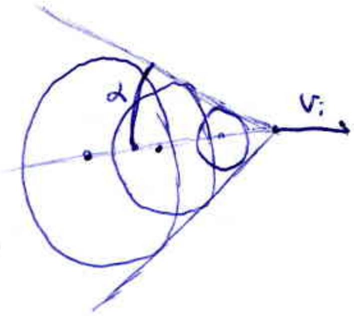
Bi maiztasunen gainezarmena



Talka uhinak:

Iturri supersonikoa: $v_i > v$

Uhin fronte konikoa: $\alpha = \arcsin \frac{v}{v_i}$



Doppler elektromagnetikoa:

$$T' = \sqrt{\frac{1+\beta}{1-\beta}}\, T \quad , \; \beta = \frac{v}{c}$$

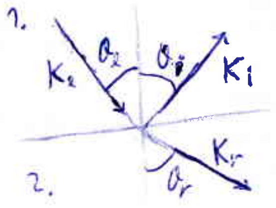
$$\nu' = \sqrt{\frac{1-\beta}{1+\beta}}\, \nu$$

$\beta > 0 \rightarrow$ Gorrirantz lerrotu

$\beta < 0 \rightarrow$ Urdinerantz lerrotu

8. UHIN FENOMENOAK

• Islapena eta errefrakzioa:



• Hiru izpiak plano berean

• $\theta_e = \theta_i$

• Snell: $\dfrac{\sin\theta_e}{\sin\theta_r} = \dfrac{V_1}{V_2} = n_{21}$

Uhin elektromagnetikoen errefrakzioa:

$n_1 = \dfrac{c}{V_1}$ $\quad n_2 = \dfrac{c}{V_2}$ $\quad n_{21} = \dfrac{V_1}{V_2} = \dfrac{n_2}{n_1}$ $\quad n_1 \sin\theta_e = n_2 \sin\theta_r$

Talde abiadura

$V_t = \dfrac{c}{n + w\dfrac{dn}{dw}}$

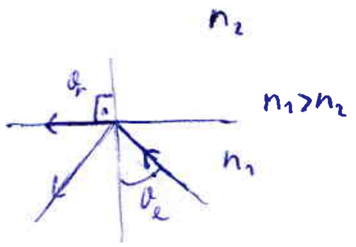
Sakabanatze: Normala: $\dfrac{dn}{dw} > 0$, $V_t < \dfrac{c}{n}$

Anomaloa: $\dfrac{dn}{dw} < 0$, $V_t > \dfrac{c}{n}$

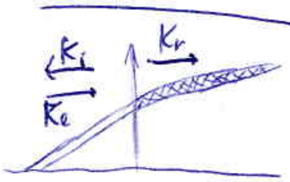
Islapen osoa: $V_2 > V_1$

Angelu kritikoa:

$\theta_r = \dfrac{\pi}{2} \Rightarrow \theta_e = \eta = \arcsin n_{21} = \arcsin \dfrac{V_1}{V_2}$



Islapen eta errefrakzioa bi soka tenkatutan:



Bi soka: $K_e = K_e\hat{\imath}$ $\quad K_i = -K_i\hat{\imath}$ $\quad K_r = K_r\hat{\imath}$

$U_e = U_{e0}\, e^{i(K_1x - wt)}$ $\quad U_i = U_{i0}\, e^{i(-K_1x - wt)}$ $\quad U_r = U_{r0}\, e^{i(K_2x - wt)}$

U jarraitua $x=0 \Rightarrow U_{e0} + U_{i0} = U_{r0}$

T_y jarraitua $x=0 \Rightarrow U_{i0} = \dfrac{K_1 - K_2}{K_1 + K_2} U_{e0}$ $\quad U_{r0} = \dfrac{2K_1}{K_1 + K_2} U_{e0}$

Islapen eta transmisio-koefizienteak:

$$v_i=\sqrt{\frac{T}{\mu_i}}$$

$$v_i=\frac{w}{k_i}$$

$$u_{i0}=\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}\,u_{e0} \qquad u_{r0}=\frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}\,u_{e0}$$

Islapen koefizientea: $R=\frac{u_{i0}}{u_{e0}}=\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}$

Transmisio koefizientea: $T=\frac{u_{r0}}{u_{e0}}=\frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}$

$$1+R=T$$

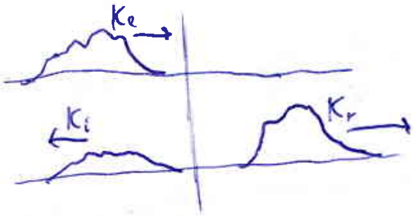
Errefraktatua

$u_{0i}=R\,u_{e0}$ $\qquad u_{r0}=T\,u_{e0}$

Islatua $\quad$ Erasotzailea

Uhin islatua:

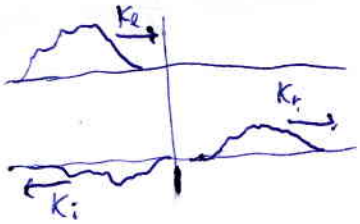
- $\mu_1>\mu_2 \iff v_1<v_2 \iff R>0$



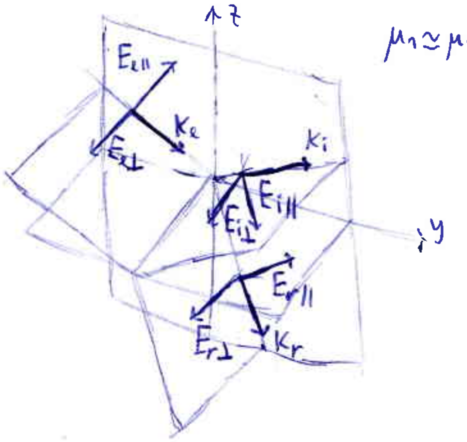
- $\mu_2\to\infty \quad v_2\to 0$

$T=0 \quad R=-1$

- $\mu_1<\mu_2 \iff v_1>v_2 \iff R<0$



Uhin elektromagnetikoen R eta T:



$\mu_1\simeq\mu_2\simeq\mu_0$

$$R_\parallel=\frac{E_{i\parallel}}{E_{e\parallel}}=\frac{n_1\cos\theta_r-n_2\cos\theta_e}{n_1\cos\theta_r+n_2\cos\theta_e}$$

$$R_\perp=\frac{E_{i\perp}}{E_{e\perp}}=\frac{n_1\cos\theta_e-n_2\cos\theta_r}{n_1\cos\theta_e+n_2\cos\theta_r}$$

$$T_\parallel=\frac{E_{r\parallel}}{E_{e\parallel}}=\frac{2n_1\cos\theta_e}{n_1\cos\theta_r+n_2\cos\theta_e}$$

$$T_\perp=\frac{E_{r\perp}}{E_{e\perp}}=\frac{2n_1\cos\theta_e}{n_1\cos\theta_e+n_2\cos\theta_r}$$

- Brewsteren legea; polarizazio angelua:

$$R_{\parallel}=0 \Leftrightarrow \theta_e+\theta_r=\frac{\pi}{2} \Leftrightarrow \theta_e=\arctan\frac{n_2}{n_1}$$

- Eraso normalean $(\theta_e=\theta_i=\theta_r=0)$

$$R_{\parallel}=R_{\perp}=\frac{n_1-n_2}{n_1+n_2} \qquad T_{\parallel}=T_{\perp}=\frac{2n_1}{n_1+n_2}$$

$V_2<V_1 \Leftrightarrow n_1<n_2 \Leftrightarrow R<0 \longrightarrow$ Uhin islatuan π balioko desfasea

$V_2>V_1 \Leftrightarrow n_1>n_2 \Leftrightarrow R>0$

- Eraso tangentean $\left(\theta_e=\theta_i \simeq \frac{\pi}{2}\right)$ uhin islatua kontrafasean: $R_{\perp}=-R_{\parallel}=-1$

- **Interferentzia:**

$$\left.\begin{array}{l} u_1=A_1 e^{i(\varphi_1-wt)} \\ u_2=A_2 e^{i(\varphi_2-wt)} \end{array}\right\} \Rightarrow u=u_1+u_2=A e^{i(\varphi-wt)}$$

$$A=\sqrt{A_1^2+A_2^2+2A_1A_2\cos(\varphi_2-\varphi_1)} \qquad \sin\varphi=\frac{A_1\sin\varphi_1+A_2\sin\varphi_2}{A} \qquad \cos\varphi=\frac{A_1\cos\varphi_1+A_2\cos\varphi_2}{A}$$

$$I=I_1+I_2+2\sqrt{I_1I_2}\cos(\varphi_2-\varphi_1)$$

- Interferentzia: - Eraikitzailea: $\varphi_2-\varphi_1=2n\pi \Leftrightarrow \overset{\text{Uhin erdietan}}{r_2-r_1}=n\lambda$

$$A=A_1+A_2$$

$$I=I_1+I_2+2\sqrt{I_1I_2}$$

- Suntsitzailea: $\varphi_2-\varphi_1=(2n+1)\pi \Leftrightarrow r_2-r_1=(2n+1)\frac{\lambda}{2}$ Erdietan

$$A=|A_1-A_2|$$

$$I=I_1+I_2-2\sqrt{I_1I_2}$$

Koherentzia:

• Iturri koherenteak (sinkronoak): $\dfrac{\partial(\varphi_2-\varphi_1)}{\partial t}=0$

• $\varphi_2-\varphi_1$ oso arin aldatu (zorian): $\cos(\varphi_2-\varphi_1)=0$

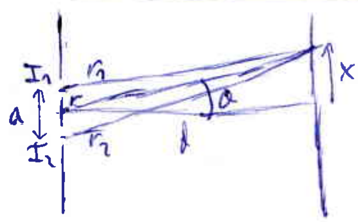
$$\langle I\rangle = I_1+I_1$$

• Koherentzia luzera
- Argi naturala $\sim 1\mu m$
- Deskarga lanpara $\sim 1mm$
- Laserra $\sim 100m$

Bi zirrikituen esperimentua / Youngen esperimentua:

zirrikituen zabalera

$b \ll \lambda \rightarrow$ zulo bakoitza iturri puntual.



Zirrikituak hurbil: $u_{10}=u_{20}$, $a,x \ll d$

$r_1 \simeq r_2 \simeq r$ $\qquad \varphi_2-\varphi_1 = k(r_2-r_1) \approx \dfrac{2\pi}{\lambda} a\sin\theta \approx \dfrac{2\pi}{\lambda} a\tan\theta = \dfrac{2\pi a x}{\lambda d}$

$A_1 \simeq A_2$

$$A \simeq A_1\sqrt{2[1+\cos k(r_2-r_1)]} = 2A_1\left|\cos\frac{k(r_2-r_1)}{2}\right| = 2A_1\left|\cos\frac{\pi a x}{\lambda d}\right|$$

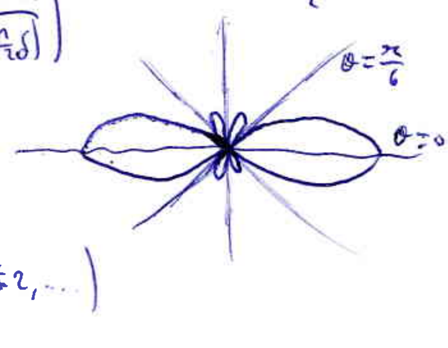
$$I \propto A^2 \Rightarrow I = 4I_1\cos^2\left(\frac{\pi a x}{\lambda d}\right) = 4I_1\cos^2\left(\frac{\pi a\sin\theta}{\lambda}\right)$$

• Maximoak: $a\sin\theta = n\lambda \Leftrightarrow x = n\dfrac{\lambda d}{a}$

• Minimoak: $a\sin\theta = (2n+1)\dfrac{\lambda}{2} \Leftrightarrow x = (2n+1)\dfrac{\lambda d}{2a}$

N iturri sinkrono:

$\delta = \dfrac{2\pi}{\lambda}a\sin\theta \qquad A = |u| = \left|\dfrac{\sin\left(\frac{1}{2}N\delta\right)}{\sin\left(\frac{1}{2}\delta\right)}\right| A_1 \Rightarrow I = I_1\left(\dfrac{\sin\left(\frac{1}{2}N\delta\right)}{\sin\left(\frac{1}{2}\delta\right)}\right)^2$



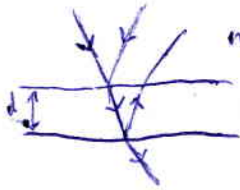
• Maximo nagusiak: $I = N^2 I_1$

$a\sin\theta = n\lambda$

• Minimoak: $a\sin\theta = \dfrac{m}{N}\lambda$, $(m \neq nN;\ m,n=\pm1,\pm2,\dots)$

Interferentzia film mehetan:

Eraso normala



$2d <$ Koherentzia-luzera

$$\Delta\varphi_0 = k_2 2d = \frac{4\pi d}{\lambda_2} = \frac{4\pi n_{21} d}{\lambda_1}$$

	$n_2 > n_1, n_3$ $n_2 < n_1, n_3$	$n_1 > n_2 > n_3$ $n_1 < n_2 < n_3$
$\Delta\varphi$	$2\pi\left(\frac{2n_{21}d}{\lambda_1} \pm \frac{1}{2}\right)$	$2\pi \frac{2n_{21}d}{\lambda_1} \ (+2\pi)$
Maximoak	$\frac{2n_{21}d}{\lambda_1} = m + \frac{1}{2} = \frac{2d}{\lambda_2}$	$\frac{2n_{21}d}{\lambda_1} = m = \frac{2d}{\lambda_2}$
Minimoak	$\frac{2n_{21}d}{\lambda_1} = m$	$\frac{2n_{21}d}{\lambda_1} = m + \frac{1}{2}$

$m = 0, 1, 2, 3, \dots$

• Uhin geldikorrak:

$$u(t,x) = u_0 e^{i(kx-\omega t)} + u_0' e^{i(-kx-\omega t)}$$

M.B. $\begin{cases} u(t,0) = 0 \Rightarrow u_0' = -u_0 \\ u(t,L) \Rightarrow k = k_n = n\frac{\pi}{L} \end{cases}$

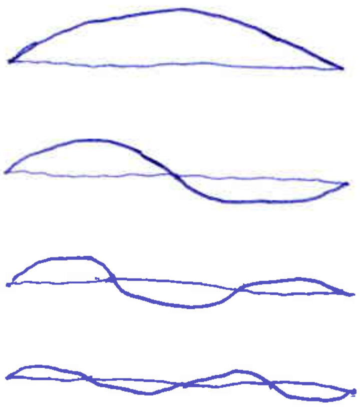
$$u = A_n \sin(k_n x)\, e^{-i\omega_n t}$$

$k_n = \frac{n\pi}{L}$ $\lambda_n = \frac{2L}{n}$ $\omega_n = \frac{n\pi v}{L}$ $\nu_n = n\frac{v}{2L}$, $n = 1, 2, 3, 4, \dots$

$$u = A_n \sin\frac{n\pi x}{L}\, e^{-i\omega_n t}$$

Nodoak: $x = \frac{m\pi}{k} = m\frac{\lambda}{2} = m\frac{L}{n}$

Antinodoak: $x = \frac{(2m+1)\pi}{2k} = (2m+1)\frac{\lambda}{4} = (2m+1)\frac{L}{2n}$



$n=1$

$n=2$

$n=3$

$n=4$

Modu normalak:

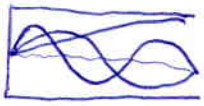
Uhin ekuazioa + Soluzio periodikoa $\longrightarrow$ Helmontz

Helmontz: $f'' + k^2 f = 0$ $\longrightarrow$ Bi ertzak lotuta; $f(x) = A_n \sin(k_n x)$

Gas zutabea:

Mutur itxiak $u=0$, mutur askeak $p=p_0$

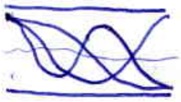
$$p - p_0 = -B \frac{\partial u}{\partial x} \rightarrow \left.\frac{\partial u}{\partial x}\right|_{mut} = 0$$



$u(t,x) = A\sin(Kx)\, e^{-i\omega t}$

$K = (2n-1)\frac{\pi}{2L}$ $\quad \lambda = \frac{4L}{2n-1}$

$\nu = (2n-1)\frac{v}{4L}$



$u(t,x) = A\cos(Kx)\, e^{-i\omega t}$

$K = n\frac{\pi}{L}$ $\quad \lambda = \frac{2L}{n}$

$\nu = n\frac{v}{2L}$

Erresonantzia:

Soka tenkatua, $u(t,L)=0$ eta $u(t,0) = u_0 \cos(\Omega t)$ (bortxatua)

$$\underset{\text{Saiatu}}{u(t,x) = f(x)\cos(\Omega t)} \Rightarrow f(x) = A\cos(Kx) + B\sin(Kx) \qquad K = \frac{\Omega}{v}$$

M.B. $f(0) = u_0 \Rightarrow A = u_0$

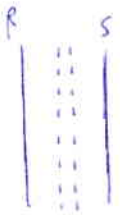
$f(L) = 0 \Rightarrow B = -u_0 \cot(KL)$ $\quad \Rightarrow u(t,x) = u_0 \frac{\sin(K(L-x))}{\sin(KL)} \cos\Omega t$

$u = \infty$ (Erresonantzia):

$$KL = n\pi \Rightarrow \Omega = \omega_n = n\frac{\pi v}{L} \quad (n = 1, 2, \dots)$$

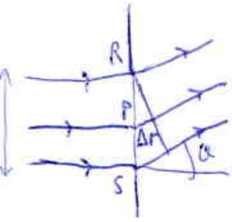
• **Difrakzioa:** $\lambda \gtrsim b$

Fraunhofer difrakzioa:



$$\theta = 0 \rightarrow (x, x+dx) \rightarrow du = \frac{A_0\, dx}{b} e^{i\varphi_R}$$

$$\theta \neq 0 \rightarrow (x, x+dx) \rightarrow du = \frac{A_0\, dx}{b} e^{i(\varphi_R + \delta)}$$

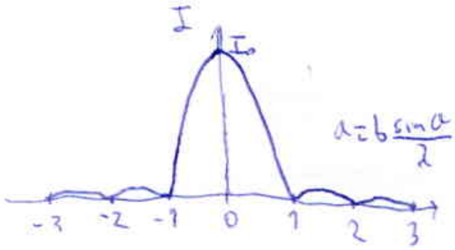


$$\delta = k \Delta r = \frac{2\pi}{\lambda} x \sin\theta$$

$$u = \int_R^S du = \frac{A_0}{b} e^{i\varphi_R} \int_0^b e^{i\left(\frac{2\pi x \sin\theta}{\lambda}\right)} dx = A_0 \frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} e^{i\left[\varphi_R + \left(\pi b \sin\frac{\theta}{\lambda}\right)\right]}$$

Amplitudea:
$$A = A_0 \left| \frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \right|$$

Intentsitatea:
$$I = I_0 \left[\frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \right]^2$$



- Minimoak: $\sin \pi\alpha = 0 \rightarrow b \sin\theta = n\lambda \quad (n = \pm 1, \pm 2, \ldots)$

  $\frac{b \sin\theta}{\lambda}$

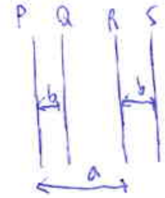
- I_a intentsitate osoa: $-\frac{\lambda}{b} < \theta < \frac{\lambda}{b}$

- $b \lesssim \lambda \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (Iturri puntuala)

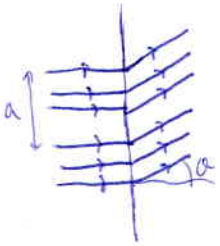
- Maximoak: $b \sin\theta \simeq \pm (2m+1) \frac{\lambda}{2}$

$$I \approx \frac{4 I_0}{(2m+1)^2 \pi^2} \quad m = 1, 2, \ldots$$

Bi zirrikituak:



$$A_1 = A_2 = A_0 \left| \frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \right|$$

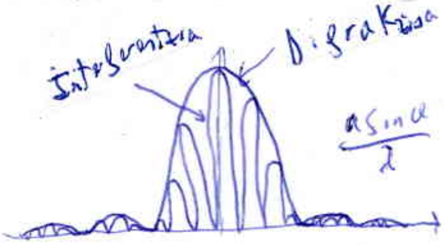


$$u_2 = u_1 e^{i\delta} \qquad \delta = K\Delta r = \frac{2\pi a \sin\theta}{\lambda}$$

$$u = u_1 + u_2 = u_1 e^{i\delta/2}\left(e^{i\delta/2} + e^{-i\delta/2}\right) = 2u_1 e^{i\delta/2}\cos\left(\frac{\delta}{2}\right)$$

$$A = 2A_0 \left| \frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \cos\left(\frac{\pi a \sin\theta}{\lambda}\right) \right|$$

$$I = 4I_1 \underbrace{\left[\frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \right]^2}_{\text{Difrakzioa}} \underbrace{\cos^2\left(\frac{\pi a \sin\theta}{\lambda}\right)}_{\text{Interferentzia}}$$



Difrakzio sareak:

N zirrikitu

$$I = I_1 \underbrace{\left[\frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \right]^2}_{\text{Difrakzioa}} \underbrace{\left[\frac{\sin\left(\frac{N\pi a \sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi a \sin\theta}{\lambda}\right)} \right]^2}_{\text{Interferentzia}} = I_1 \left[\frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \right]^2 \underbrace{\left[1 + 2\cos\left(\frac{2\pi}{\lambda} a \sin\theta\right) \right]^2}_{\text{Interferentzia}}$$