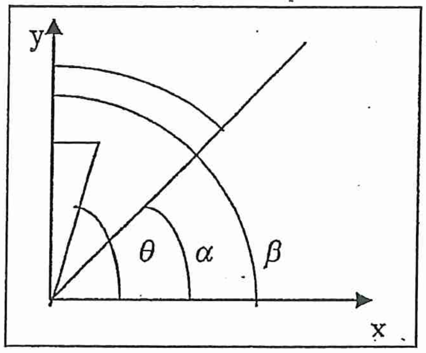
SECTOR CIRCULAR



$$Iy = \int_M x^2 dm$$

$$dm = \sigma r dr d\theta$$

$$Iy = \sigma \int r^3 Cos^2\theta dr d\theta$$

$$Iy = \sigma \int_0^R r^3 dr \int_\alpha^\beta Cos^2\theta d\theta$$

$$\int Cos^2\theta d\theta = \frac{1}{2}\theta + \frac{1}{4}Sen2\theta$$

$$Iy = \sigma \frac{R^4}{4}\left[\frac{1}{2}(\beta-\alpha) + \frac{1}{4}(Sen2\beta - Sen2\alpha)\right]$$

$$\sigma = \frac{M}{A} > A = \int_0^R r dr \int_\alpha^\beta d\theta = \frac{R^2}{2}(\beta-\alpha)$$

$$Ix = \int_M y^2 dm = \sigma \int_0^R r^3 dr \int_\alpha^\beta Sen^2\theta d\theta = \sigma \frac{R^4}{4}\left[\frac{1}{2}(\beta-\alpha) - \frac{1}{4}(Sen^2\beta - Sen^2\alpha)\right]$$

$$\int Sen^2\theta = \frac{1}{2}\theta - \frac{1}{4}Sen2\theta$$

$$Iz = \sigma \frac{R^4}{4}(\beta-\alpha)$$

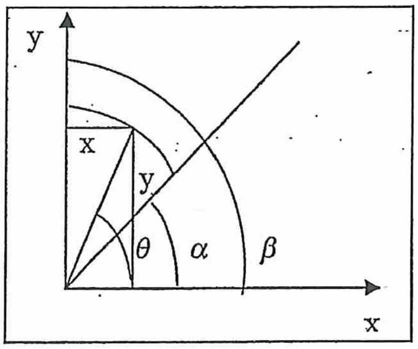
$$Cz = \sigma \int yx r d\theta dr$$

$$Cz = \sigma \int_0^R r^3 dr \int_\alpha^\beta Sen\theta Cos\theta d\theta = \sigma \frac{R^4}{4}(Sen^2\beta - Sen^2\alpha)\frac{1}{2}$$

$$Cz = \sigma \frac{R^4}{8}(Sen^2\beta - Sen^2\alpha)$$

$$\sigma = \frac{M}{A}$$

PORCIÓN DE ARO



$$Iy = \lambda R^3 \int_\alpha^\beta Cos^2\theta d\theta$$

$$\int Cos^2\theta d\theta = \frac{1}{2} + \frac{1}{4} Sen 2\theta$$

$$Iy = \lambda R^3 \left[\frac{1}{2}[\beta - \alpha] + \frac{1}{4}(Sen2\beta - Sen2\alpha) \right]$$

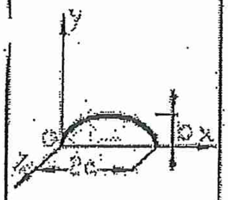
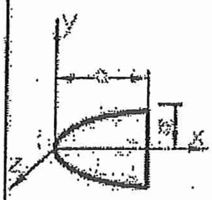
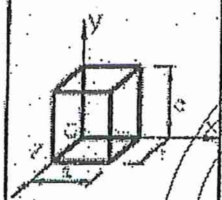
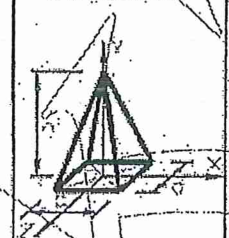
$$\lambda = \frac{M}{L}; L = R \int_\alpha^\beta d\theta = R(\beta - \alpha)$$

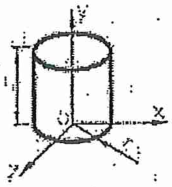
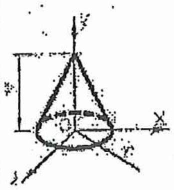
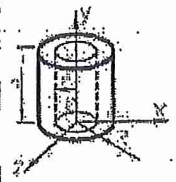
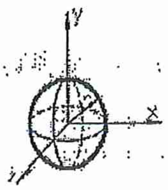
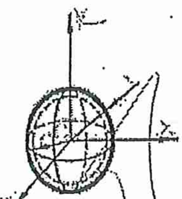
$$Ix = \lambda R^3 \left[\frac{1}{2}(\beta - \alpha) - \frac{1}{4}(Sen2\beta - Sen2\alpha) \right]$$

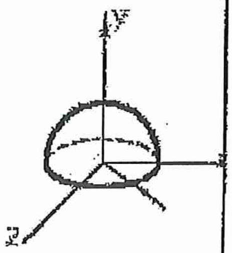
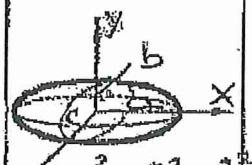
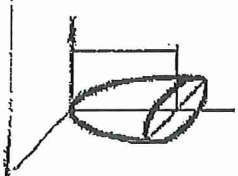
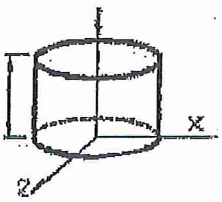
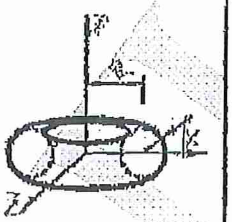
$$Iz = \lambda R^3 (\beta - \alpha)$$

$$Cz = \lambda R^3 \int_\alpha^\beta Cos\theta Sen\theta d\theta$$

$$Cz = \frac{\lambda R^3}{2}(Sen^2\beta - Sen^2\alpha)$$

Sólido	Longitud-Área-Volumen	Masa y baricentro	Momentos de inercia	Productos de inercia
SEMIELIPSE 	$S=\frac{\pi ab}{2}$	$m=\frac{\rho\pi ab}{2}$ X=a $Y=\frac{4b}{3\pi}$ Z=0	$I_x=\frac{mb^2}{4}$ $I_y=\frac{5}{4}ma^2$ $I_z=\frac{m}{4}\left[5a^2+b^2\right]$	$P_{xy}=\frac{4}{3}m\frac{ab}{\pi}$ $P_{xz}=0$ $P_{yz}=0$
PARABOLA 	$S=\frac{4}{3}ab$	m=p4/3 ab X=3/5 a Y=0 Z=0	$I_x=\frac{mb^2}{5}$ $I_y=\frac{3}{7}ma^2$ $I_z=m\left[\frac{3}{7}a^2+\frac{b^2}{5}\right]$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
SEMIPARABOLA 	$S=\frac{2}{3}ab$	m=p2/3 ab X=3/5 a Y=3/8 b Z=0	$I_x=\frac{mb^2}{5}$ $I_y=\frac{3}{7}ma^2$ $I_z=m\left[\frac{3}{7}a^2+\frac{b^2}{5}\right]$	$P_{xy}=\frac{m}{4}ab$ $P_{xz}=0$ $P_{yz}=0$
PRISMA 	S=2(ab+ac+bc) V=abc	m=ρ abc X=a/2 Y=b/2 Z=c/2	$I_x=\frac{m}{3}\left(b^2+c^2\right)$ $I_y=\frac{m}{3}\left(a^2+c^2\right)$ $I_z=\frac{m}{3}\left(a^2+b^2\right)$	$P_{xy}=\frac{m}{4}ab$ $P_{xz}=\frac{m}{4}ac$ $P_{yz}=\frac{m}{4}bc$
PIRAMIDE 	V=1/3 abh	m=ρ1/3 abh X=0 Y=0 Z=h/4	$I_x=\frac{m}{20}\left(b^2+2h^2\right)$ $I_y=\frac{m}{20}\left(a^2+b^2\right)$ $I_z=\frac{m}{20}\left(a^2+2h^2\right)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$

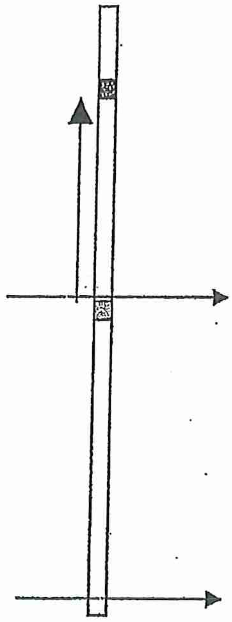
Sólido	Longitud-Área-Volumen	Masa y baricentro	Momentos de inercia	Productos de inercia
CILINDRO 	$S=2\pi rh+2\pi r^2$ $V=\pi r^2 h$	$m=\rho\,\pi r^2 h$ $X=0$ $Y=h/2$ $Z=0$	$I_x=\frac{m}{12}\left(3r^2+4h^2\right)$ $I_y=\frac{1}{2}mr^2$ $I_z=\frac{m}{12}\left(3r^2+4h^2\right)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
CONO 	$S=\pi r\sqrt{r^2+h^2}+\pi r^2$ $V=\frac{1}{3}\pi r^2 h$	$m=\frac{1}{3}\rho\pi r^2 h$ $X=0$ $Y=h/4$ $Z=0$	$I_x=\frac{m}{20}\left(3r^2+2h^2\right)$ $I_y=\frac{3}{10}mr^2$ $I_z=\frac{m}{20}\left(3r^2+2h^2\right)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
TUBO CILINDRICO 	$S=2\pi h(R+r)+2\pi(R^2-r^2)$ $V=\pi h(R^2-r^2)$	$m=\rho\,\pi h(R^2-r^2)$ $X=0$ $Y=h/2$ $Z=0$	$I_x=\frac{m}{12}\left(3R^2+3r^2+4h^2\right)$ $I_y=\frac{1}{2}m\left(r^2+R^2\right)$ $I_z=\frac{m}{12}\left(3R^2+3r^2+4h^2\right)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
ESFERA 	$S=4\pi r^2$ $V=\frac{4}{3}\pi r^3$	$V=\frac{4}{3}\rho\pi r^3$ $X=0$ $Y=0$ $Z=0$	$I_x=\frac{2}{5}mr^2$ $I_y=\frac{2}{5}mr^2$ $I_z=\frac{2}{5}mr^2$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
ESFERA HUECA 	$S=4\pi(R^2+r^2)$ $V=\frac{4}{3}\pi\left(R^3-r^3\right)$	$m=\rho 4/3\pi(R^3-r^3)$ $X=0$ $Y=0$ $Z=0$	$I_x=\frac{2}{5}m\frac{R^5-r^5}{R^3-r^3}$ $I_y=\frac{2}{5}m\frac{R^5-r^5}{R^3-r^3}$ $I_z=\frac{2}{5}m\frac{R^5-r^5}{R^3-r^3}$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$

Sólido	Longitud-Area-Volumen	Masa y baricentro	Momentos de inercia	Productos de inercia.
SEMIESFERA 	$S=2\pi r^2$ $V=\frac{2}{3}\pi r^3$	$m=\rho\ 2/3\ \pi r^3$ X=0 Y=3/8 r Z=0	$I_x=\frac{2}{5}mr^2$ $I_y=\frac{2}{5}mr^2$ $I_z=\frac{2}{5}mr^2$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
ELIPSOIDE  $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$V=4/3\ \pi\ abc$	$m=\rho\ 4/3\ \pi abc$ X=0 Y=0 Z=0	$I_x=\frac{m}{5}(b^2+c^2)$ $I_y=\frac{m}{5}(a^2+c^2)$ $I_z=\frac{m}{5}(a^2+b^2)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
PARABOLOIDE 	$V=\pi\ abc$	$m=\rho\ \pi abc$ X=2/3 a Y=0 Z=0	$I_x=\frac{m}{6}(b^2+c^2)$ $I_y=\frac{m}{6}(3a^2+c^2)$ $I_z=\frac{m}{6}(3a^2+b^2)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
CILINDRO ELIPTICO 	$V=\pi\ abh$	$m=\rho\ \pi\ abh$ X=0 Y=h/2 Z=0	$I_x=\frac{m}{12}(3b^2+4h^2)$ $I_y=\frac{m}{4}(a^2+b^2)$ $I_z=\frac{m}{12}(3a^2+4h^2)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$
TORO 	$S=4\pi^2 Rr$ $V=2\pi^2 Rr^2$	$m=\rho\ 2\pi^2 Rr^2$ X=0 Y=0 Z=0	$I_x=\frac{m}{8}(4R^2+5r^2)$ $I_y=m\left(R^2+\frac{3}{4}r^2\right)$ $I_z=\frac{m}{8}(4R^2+5r^2)$	$P_{xy}=0$ $P_{xz}=0$ $P_{yz}=0$

11.5 KASU BEREZIAK

Adibide moduan, jarraian ohizko sistema materialen inertzia momentuak azalduko dira.

11.5.1 M masako eta L luze den barra.

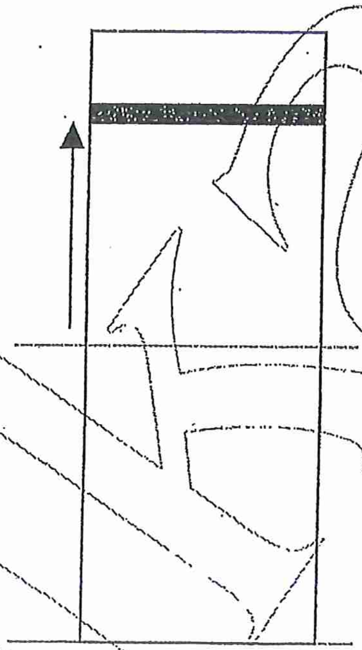


$$I_G = I_y = I_{xy_G} = \int z^2 dm = \rho \int_{-L/2}^{L/2} z^2 dz = \frac{ML^2}{12}$$

$$\rho = \frac{M}{L}$$

$$I_O = I_{y'} = I_{x'y'} = M \cdot \left(\frac{L}{2}\right)^2 + I_G = \frac{ML^2}{3}$$

11.5.2 M masako eta b luzerako laukizuzena.

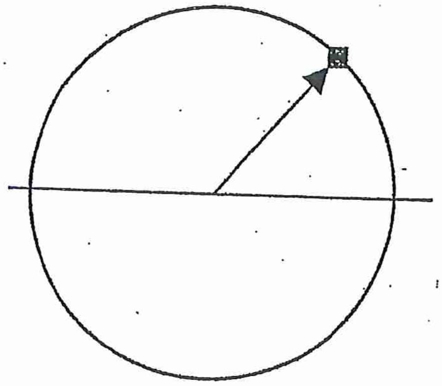


$$I_y = I_{xy} = \int z^2 dm = \rho \int_{-b/2}^{b/2} z^2 \cdot a \cdot dz = \frac{Mb^2}{12}$$

$$\rho = \frac{M}{a \cdot b}$$

$$I_{y'} = I_{x'y'} = M \cdot \left(\frac{b}{2}\right)^2 + I_y = \frac{Mb^2}{3}$$

11.5.3 M masako eta R erradioko aroa.

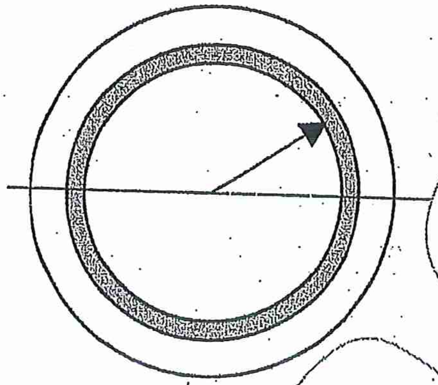


$$I_O = \int R^2 dm = MR^2$$

$$\rho = \frac{M}{2\pi R}$$

$$I_y = I_{xy} = \frac{MR^2}{2}$$

11.5.4 M masako eta R erradioko zirkulua.

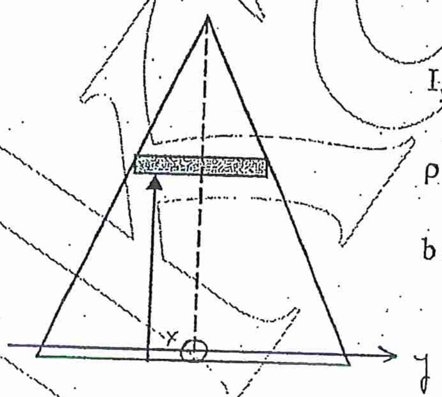


$$I_O = \int r^2 dm = \int_0^R r^2 \rho 2\pi r dr = \frac{MR^2}{2}$$

$$\rho = \frac{M}{\pi R^2}$$

$$I_y = I_{xy} = \frac{MR^2}{4}$$

11.5.5 M masako eta H altuerako triangelua.

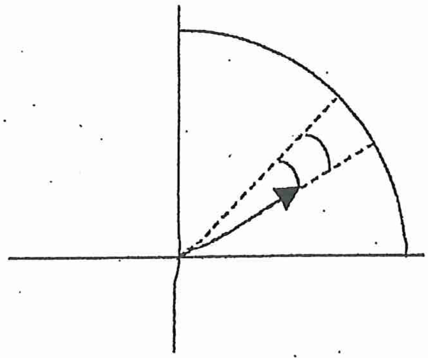


$$I_y = I_{xy} = \int z^2 dm = \rho \int_0^H z^2 \cdot b \cdot dz = \frac{MH^2}{6}$$

$$\rho = \frac{2M}{B \cdot H}$$

$$b = \frac{B \cdot (H - z)}{H}$$

11.5.6 M masako eta R erradioko zirkulu laurdena.



$$I_O = \int r^2 dm = \int_0^R \int_0^{\frac{\pi}{2}} r^2 \rho r d\theta dr = \frac{MR^2}{2}$$

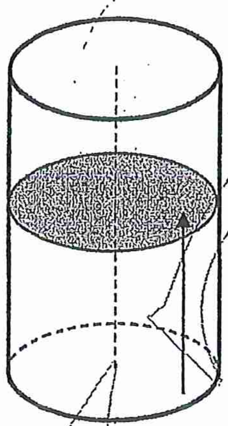
$$\rho = \frac{4M}{\pi R^2}$$

$$I_y = I_{xy} = \frac{MR^2}{4}$$

$$C_x = \int r^2 dm = \int_0^R \int_0^{\frac{\pi}{2}} r^2 \rho r d\theta dr = \frac{MR^2}{2\pi}$$

$$\rho = \frac{4M}{\pi R^2}$$

11.5.7 M masako, R erradioko eta H altuerako zilindroa.



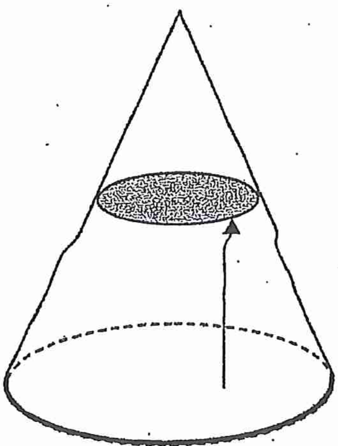
$$I_{xy} = \int z^2 dm = \rho \int_0^H z^2 A dz = \frac{MH^2}{3}$$

$$\rho = \frac{M}{A \cdot H}$$

$$I_z = \frac{MR^2}{2}$$

Oinarriko planoarekiko inertzi momentua, bere formaren menpeko ez da eta ondorioz edozein prisma zuzenen inertzi momentua zilindroarenaren berdina izango da.

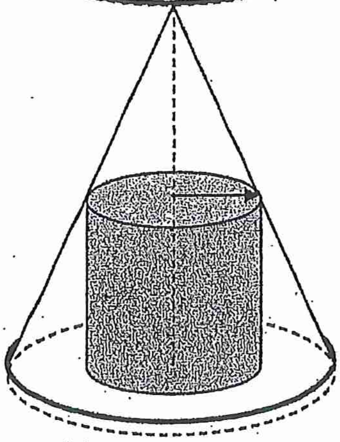
11.5.8 M masako, R erradioko eta H altuerako konoa.



$$I_{xy} = \int z^2 dm = \rho \int_0^H z^2 \cdot \pi \left(\frac{R \cdot (H-z)}{H} \right)^2 \cdot dz = \frac{MH^2}{10}$$

$$\rho = \frac{3M}{\pi R^2 \cdot H}$$

$$r = \frac{R \cdot (H-z)}{H}$$

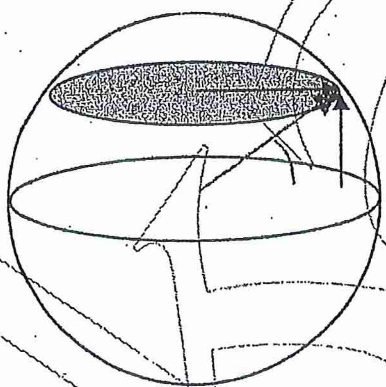


$$I_e = \int r^2 dm = \rho \int_0^R r^2 \cdot 2\pi r \cdot z \cdot dr = \frac{3MR^2}{10}$$

$$\rho = \frac{3M}{\pi R^2 \cdot H}$$

$$z = \frac{H \cdot (R-r)}{R}$$

11.5.9 R erradioko eta M masako esfera.



$$I_\pi = \int z^2 dm = \rho \int_0^H (R\, sen\, \theta)^2 \cdot \pi (R \cos \theta)^2 \cdot dz = \frac{MR^2}{5}$$

$$\rho = \frac{3M}{4\pi R^3}$$

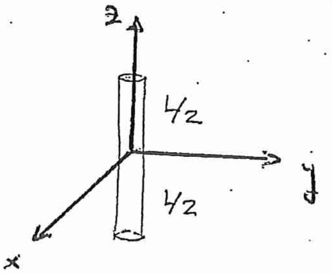
$$r = R \cos \theta$$

$$z = R\, sen\, \theta$$

$$dz = R \cos \theta d\theta$$

Momentos de inercia

+ VARILLA

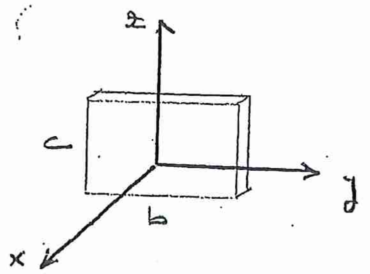


$$I_{xy} = \frac{1}{12} ML^2$$

$$I_{yz} = 0$$

$$I_{zx} = 0$$

+ PLACA DELGADA

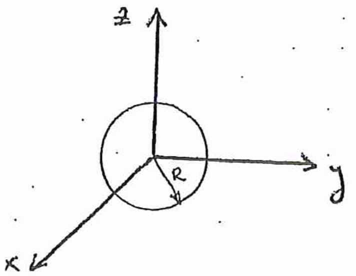


$$I_{xy} = \frac{1}{12} Mc^2$$

$$I_{yz} = 0$$

$$I_{zx} = \frac{1}{12} Mb^2$$

+ DISCO DELGADO

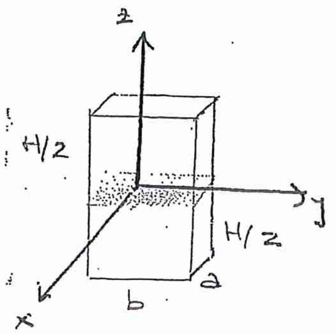


$$I_{xy} = \frac{1}{4} MR^2$$

$$I_{yz} = 0$$

$$I_{zx} = \frac{1}{4} MR^2$$

+ PRISMA



$$I_{xy} = \frac{1}{12} MH^2$$

$$I_{xy} = \delta \int_{prisma} z^2 \, dv = \frac{M}{abc} \int_0^c z^2 ab \, dz = \frac{1}{3} Mc^2$$

$$I_{yz} = \frac{1}{12} Ma^2$$

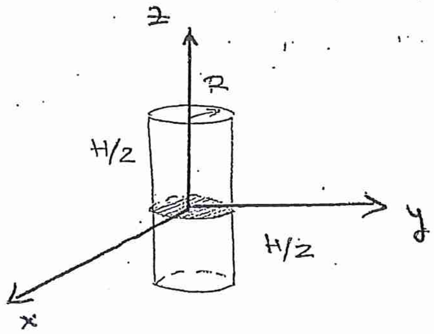
$$I_{zx} = \frac{1}{12} Mb^2$$

$$I_x = \frac{1}{3} M(b^2+c^2) \qquad I_o = \frac{1}{3} M(a^2+b^2+c^2)$$

$$I_y = \frac{1}{3} M(a^2+c^2) \qquad C_z = \delta \int_{prisma} xy \, dv = \frac{Mc}{abc} \int_0^b y\,dy \int_0^a x\,dx = \frac{1}{4} Mab$$

$$I_z = \frac{1}{3} M(a^2+b^2) \qquad C_x = \frac{Mbc}{4} \qquad C_y = \frac{Mca}{4} \rightarrow C_{xG} = C_{yG} = C_{zG} = 0$$

+ CILINDRO



$$I_{xy} = \frac{1}{12} MH^2$$

$$I_{yz} = \frac{1}{4} MR^2$$

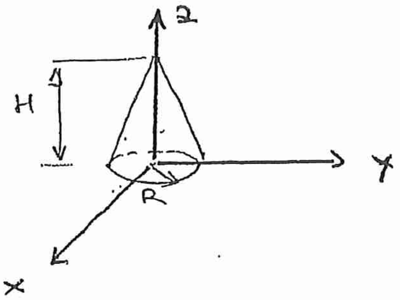
$$I_{zx} = \frac{1}{4} MR^2$$

SIMETRÍA $I_{yz} = I_{zx}$

$$I_z = \delta \int_{cil} r^2 dv = \frac{M}{\pi R^2 H} \int_0^R r^2 2\pi r H dr = \frac{1}{2} MR^2$$

$C_x = C_y = C_z = 0$ SIMETRÍA

+ CONO



$$I_{xy} = \frac{1}{10} MH^2$$

$$I_{xy} = \delta \int_c z^2 dv = \frac{3M}{\pi R^2 H} \int_0^H z^2 \pi \left[\frac{R}{H}(H-z)\right]^2 dz = \frac{1}{10} MH^2$$

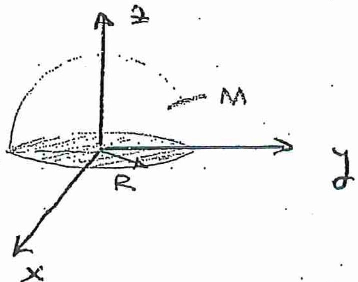
$$I_{yz} = \frac{3}{20} MR^2$$

$$I_{zx} = \frac{3}{20} MR^2$$

$$I_z = \delta \int_c r^2 dv = \frac{3M}{\pi R^2 H} \int_0^R r^2 2\pi r H \left(1 - \frac{r}{H}\right) dr = \frac{3}{10} MR^2$$

$C_x = C_y = C_z = 0$ SIMETRÍA

+ SEMIESFERA



$$I_{xy} = \frac{1}{5} MR^2$$

$$I_{yz} = \frac{1}{5} MR^2$$

$$I_{zx} = \frac{1}{5} MR^2$$

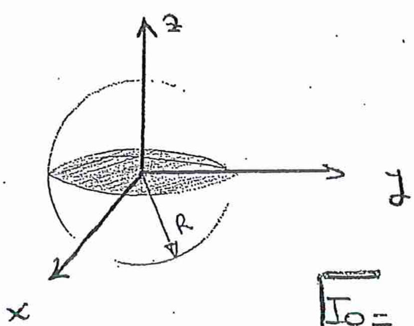
$I_{xy} = I_{yz} = I_{zx}$ SIMETRÍA RESPECTO OX, OY

$C_x = C_y = C_z = 0$

$$I_{xy\,esfera} = I_{xy\,semiesf} + I_{xy\,semiesf} \Rightarrow I_{xy\,semiesf} = \frac{1}{2} I_{xy\,esfera} = \frac{1}{2} \cdot \frac{1}{5} \cdot 2MR^2 = \frac{1}{5} MR^2$$

(2M: masa esfera)

+ ESFERA



$$I_{xy} = \frac{1}{5} MR^2$$

$$I_\pi = \int z^2 dm = \rho \int_0^H (R\sin\theta)^2 \cdot \pi (R\cos\theta)^2 dz = \frac{MR^2}{5}$$

$$I_{yz} = \frac{1}{5} MR^2$$

$$I_{zx} = \frac{1}{5} MR^2$$

$$I_O = \delta \int_{esfera} r^2 dv = \frac{3M}{4\pi R^3} \int_0^R r^2 (4\pi r^2) dr = \frac{3}{5} MR^2$$

+ SIMÉTRICA RESPECTO SU CENTRO

$$I_{xy} = I_{yz} = I_{zx} = \frac{1}{5} MR^2$$

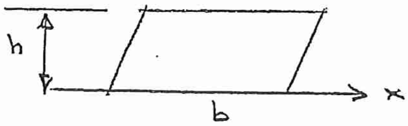
$$I_x = I_y = I_z = I_{xy} + I_{zx} = \frac{2}{5} MR^2$$

+ SIMÉTRICA RESPECTO PLANOS DIAMETRALES

$C_x = C_y = C_z = 0$

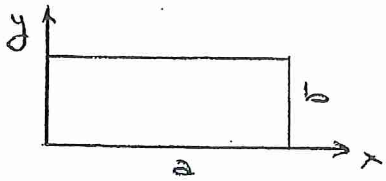
Momentos Segundos

+ PARALELOGRAMO



$$I_x = \frac{1}{3} b h^3$$

+ RECTÁNGULO

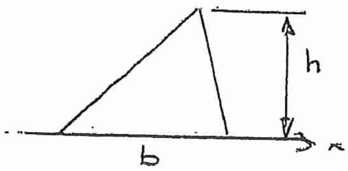


$$I_x = \frac{1}{3} a b^3$$

$$I_y = \frac{1}{3} a^3 b$$

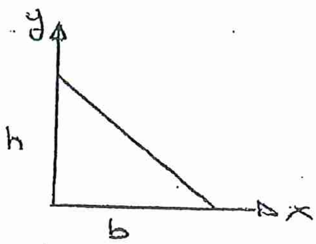
$$C_z = \frac{1}{4} a^2 b^2$$

+ TRIÁNGULO ESCALENO



$$I_x = \frac{1}{12} b h^3$$

+ TRIÁNGULO RECTÁNGULO

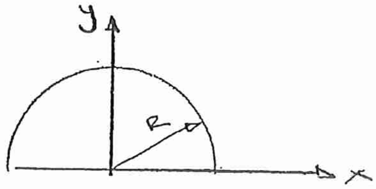


$$I_x = \frac{1}{12} b h^3$$

$$I_y = \frac{1}{12} b^3 h$$

$$C_z = \frac{1}{24} b^2 h^2$$

+ SEMICÍRCULO



$$I_x = \frac{1}{8} \pi R^4$$

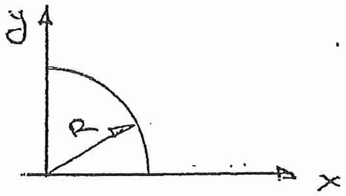
$$I_y = \frac{1}{8} \pi R^4$$

$$C_z = 0$$

OCTANTE ESFERA (volumen dx dy dz)

$$C_z = 8\int_{\text{octante}} xy\,dv = \frac{6M}{\pi R^3}\int_0^R dz \int_0^{\sqrt{R^2-z^2}} y\,dy \int_0^{\sqrt{R^2-y^2-z^2}} x\,dx = \frac{2}{5\pi} M R^2$$

4 CUADRANTE CIRCULO

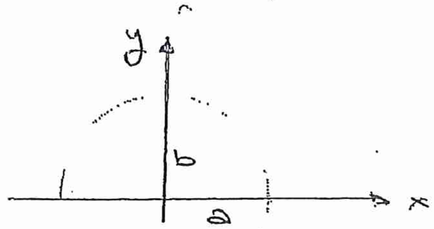


$$I_x = \frac{1}{16}\pi R^4$$

$$I_y = \frac{1}{16}\pi R^4$$

$$C_z = \frac{1}{8}R^4$$

+ SEMIELIPSE

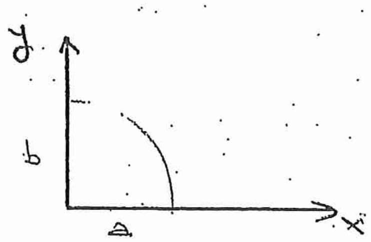


$$I_x = \frac{1}{8}\pi a b^3$$

$$I_y = \frac{1}{8}\pi a^3 b$$

$$C_z = 0$$

+ CUADRANTE ELIPSE



$$I_x = \frac{1}{16}\pi a b^3$$

$$I_y = \frac{1}{16}\pi a^3 b$$

$$C_z = \frac{1}{8}a^2 b^2$$

+ ELIPSOIDE

$$I_{xy} = \frac{1}{5}Mc^2 \qquad I_x = \frac{1}{5}M(b^2+c^2)$$

$$I_{yz} = \frac{1}{5}Ma^2 \qquad I_y = \frac{1}{5}M(a^2+c^2)$$

$$I_{zx} = \frac{1}{5}Mb^2 \qquad I_z = \frac{1}{5}M(a^2+b^2)$$

$$\boxed{I_o = \frac{1}{5}M(a^2+b^2+c^2)}$$

- Octante

$$C_z = ab\left(\frac{2}{5\pi}M\right) = \frac{2}{5\pi}Mab$$