CHAPTER 8. DIMENSIONAL ANALYSIS AND SIMILITUDE

1. Necessity of experimental measurements with scaled models
2. Homogeneity: Buckingham Pi theorem
3. Mechanical similitude of fluid flows

1. Necessity of experimental measurements with scaled models

✓ Different forms to solve a problem of Fluid Mechanics or Hydraulic Engineering:

- Analytical: Sometimes the solutions are unknown
- Computational: computation capacity is not infinite, necessity of empirical correlation
- Experimental: necessity of laboratory experiments with models or mock-ups.

2. Buckingham Pi theorem

✓ Formulation

- Physical phenomenon: $f\,(A,B,C,\ldots,N) = 0$
- "n" Physical variables: $(A,B,C,\ldots,N)$
- "m" fundamental dimensions: usually 3 in fluid mechanics: M, L y T
- It can be defined: $F\,(\Pi_1, \Pi_2,\ldots, \Pi_{n-m}) = 0$

With (n-m) dimensionless groups: $\Pi_i = A^{\alpha i}\, B^{\beta i}\, C^{\gamma i} \ldots N^{\nu i}$

2. Buckingham Pi theorem

✓ Method of repeating variables

1. Physical variables A, B, C,..., N
   (m) fundamental dimensions
   (n-m) number of dimensionless groups

2. Selection of repeating variables (m)

3. Generate the dimensionless groups Π

4. Calculation of the exponents

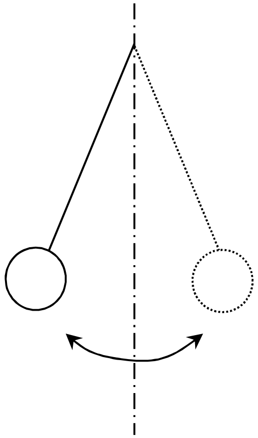
2. Buckingham Pi theorem

✓ Useful guidelines

- Dimensionless physical variable, dimensionless Π group
- Variables with the same dimensions, quotient dimensionless group
- Any Π number can be substituted by:
  - a power of itself
  - the product by a constant
  - the quotient of two Π groups
- Any Π number can be expressed as a function of other Π numbers

2. Buckingham Pi theorem

EXAMPLE 1: SIMPLE PENDULUM



STEP 1:
$n = 4$ variables
t, [T]
l, [L]
m, [M]
W, $[MLT^{-2}]$

m=3 fundamental dimensions: M, L, T

Nº of dimensionless groups: n-m = 4-3 = 1

STEP 2:
m = 3 repeating variables are selected: t, l y m

STEP 3:
Dimensionless group: $\Pi = t^{\alpha} l^{\beta} m^{\gamma} W$

STEP 4:
$[\Pi]=[T]^{\alpha} [L]^{\beta} [M]^{\gamma} [MLT^{-2}]^{1}$ ⇒ $\Pi = t^2 l^{-1} m^{-1} W$ ⇒ $t = \left(\frac{\Pi l m}{W}\right)^{1/2} = \Pi^{1/2}\left(\frac{l}{g}\right)^{1/2}$

$$t = 2\pi\sqrt{\frac{l}{g}}$$

2. Buckingham Pi theorem

EXAMPLE 2: DRAG RESISTANCE THAT A FLUID EXERTS ON THE MOTION OF A PARTIALLY SUBMERGED BODY

STEP 1: Acting variables: n = 7 variables

$F\ [MLT^{-2}]$ $U\ [LT^{-1}]$ $\varepsilon\ [L]$ $g\ [LT^{-2}]$ $l\ [L]$ $\rho\ [ML^{-3}]$ $\mu\ [ML^{-1}T^{-1}]$

m=3 fundamental dimensions: T, L y M No of dimensionless groups: n-m = 7-3 = 4

STEP 2: Repeating variables: ρ, l, U

STEP 3: (n-m) = 4 Π groups.

$\Pi_1 = \rho^{\alpha 1}\, l^{\beta 1}\, U^{\gamma 1}\, F$
$\Pi_2 = \rho^{\alpha 2}\, l^{\beta 2}\, U^{\gamma 2}\, \varepsilon$
$\Pi_3 = \rho^{\alpha 3}\, l^{\beta 3}\, U^{\gamma 3}\, \mu$
$\Pi_4 = \rho^{\alpha 4}\, l^{\beta 4}\, U^{\gamma 4}\, g$

STEP 4: $[\Pi_1] = [M\,L^{-3}]^{\alpha 1}\,[L]^{\beta 1}\,[LT^{-1}]^{\gamma 1}\,[MLT^{-2}]$

$[\Pi_2] = [M\,L^{-3}]^{\alpha 2}\,[L]^{\beta 2}\,[LT^{-1}]^{\gamma 2}\,[L]$

$[\Pi_3] = [M\,L^{-3}]^{\alpha 3}\,[L]^{\beta 3}\,[LT^{-1}]^{\gamma 3}\,[ML^{-1}T^{-1}]$

$[\Pi_4] = [M\,L^{-3}]^{\alpha 4}\,[L]^{\beta 4}\,[LT^{-1}]^{\gamma 4}\,[LT^{-2}]$

$$\Pi_1 = \frac{F}{\rho l^2 U^2}$$

$$\Pi_2 = \frac{\varepsilon}{l}$$

$$\Pi_3' = \Pi_3^{-1} = \frac{\rho l U}{\mu} = Re$$

$$\Pi_4' = \Pi_4^{-1/2} = \frac{U}{\sqrt{lg}} = Fr$$

2. Buckingham Pi theorem

EXAMPLE 2: DRAG RESISTANCE THAT A FLUID EXERTS ON THE MOTION OF A PARTIALLY SUBMERGED BODY

- Any Π number can be expressed as a function of other Π numbers

$\Pi_1=\varphi\,(\Pi_2, \Pi_3, \Pi_4)$

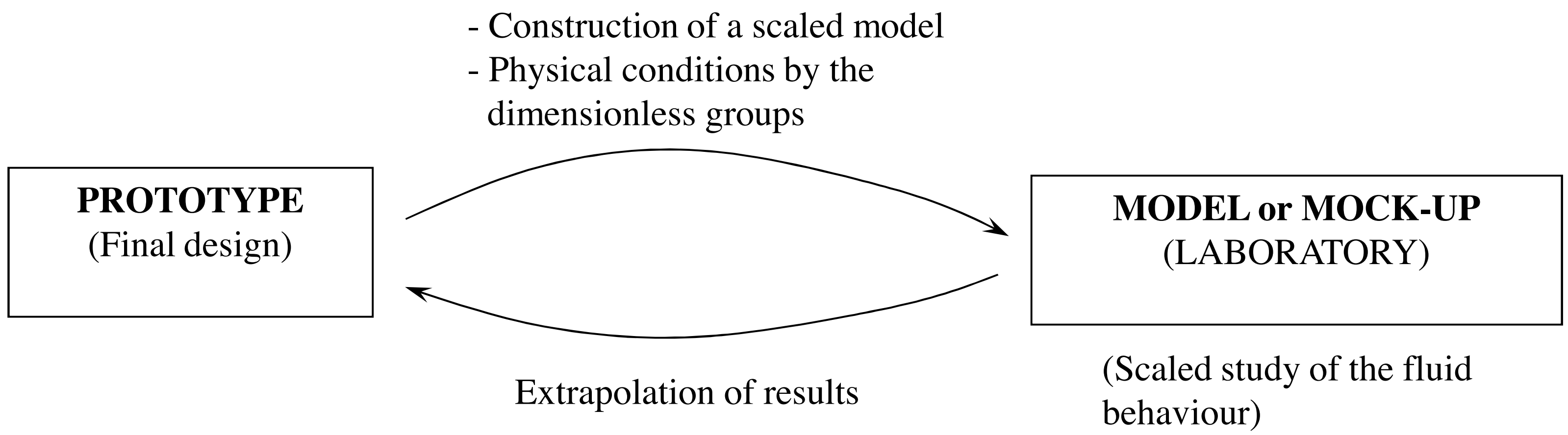
$$\frac{F}{\rho l^2 U^2} = \varphi\left(\frac{\varepsilon}{l}, \mathrm{Re}, \mathrm{Fr}\right) \Rightarrow F = \rho l^2 U^2 \cdot \varphi\left(\frac{\varepsilon}{l}, \mathrm{Re}, \mathrm{Fr}\right)$$

$$\Rightarrow F = \frac{1}{2} C_d \rho A U^2$$

$$\Rightarrow F = F_f + F_w = \frac{1}{2} C_{d,f} \rho A U^2 + \frac{1}{2} C_{d,w} \rho A U^2$$

3. Mechanical similitude of fluid flows

✓ Study of the fluid related phenomena in the laboratory



✓ Mechanical or absolute similitude

- Geometric similitude
- Kinematic similitude
- Dynamic similitude

✓ Implications for the laboratory experiments:

$$\mathbf{\Pi_{1m}} = \varphi\,(\Pi_{2m}, \Pi_{3m}, \ldots, \Pi_{km}) = \varphi\,(\Pi_{2p}, \Pi_{3p}, \ldots, \Pi_{kp}) = \mathbf{\Pi_{1p}}$$

3.1. Dimensionless parameters in Fluid Mechanics

✓ Calculation of the main dimensionless groups

- Equations

$$\rho\vec{F} - \vec{\nabla}p + \frac{1}{3}\mu\vec{\nabla}(\vec{\nabla}\vec{U}) + \mu\nabla^2\vec{U} = \rho\vec{a}$$ NAVIER – STOKES EQUATION

$p = C\,\rho^k$ PROCESS EQUATION

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{U}) = 0$$ CONTINUITY EQUATION

The same scale relationship between terms of the same equation in PROTOTYPE and MODEL

3.1. Dimensionless parameters in Fluid Mechanics

✓ Calculation of the main dimensionless groups

NAVIER – STOKES EQUATION

$$\frac{\rho_p g_p}{\rho_m g_m} = \frac{\rho_p a_p}{\rho_m a_m} = \frac{\nabla p_p}{\nabla p_m} = \frac{\mu_p \nabla(\nabla U_p)}{\mu_m \nabla(\nabla U_m)} = \frac{\mu_p \nabla^2 U_p}{\mu_m \nabla^2 U_m}$$

1. Gravity forces	2. Inertial forces	3. Pressure forces	4. Viscous forces	5. Viscous forces

1. Gravity forces	2. Inertial forces	$\frac{U_m}{\sqrt{g_m l_m}} = \frac{U_p}{\sqrt{g_p l_p}} = Fr$	**FROUDE**
2. Inertial forces	3. Pressure forces	$\frac{p_m}{\rho_m U_m^2} = \frac{p_p}{\rho_p U_p^2} = Eu$	**EULER**
2. Inertial forces	4. Viscous forces	$\frac{\rho_p U_p l_p}{\mu_p} = \frac{\rho_m U_m l_m}{\mu_m} = Re$	**REYNOLDS**

3.1. Dimensionless parameters in Fluid Mechanics

✓ Calculation of the main dimensionless groups

PROCESS EQUATION

$$\frac{p_{1p}}{p_{2p}} = \left(\frac{\rho_{1p}}{\rho_{2p}}\right)^{k_p}$$

$$\frac{p_{1m}}{p_{2m}} = \left(\frac{\rho_{1m}}{\rho_{2m}}\right)^{k_m}$$

$$\frac{p_{1m}}{p_{1p}} = \frac{p_{2m}}{p_{2p}} \qquad \frac{\rho_{1m}}{\rho_{1p}} = \frac{\rho_{2m}}{\rho_{2p}} \quad \Rightarrow \quad \frac{p_{1m}}{p_{2m}} = \frac{p_{1p}}{p_{2p}} \qquad \frac{\rho_{1m}}{\rho_{2m}} = \frac{\rho_{1p}}{\rho_{2p}}$$

$$\Rightarrow \boxed{\frac{k_m}{k_p} = 1}$$

"STATE CONDITION"

3.1. Dimensionless parameters in Fluid Mechanics

✓ Calculation of the main dimensionless groups

PROCESS EQUATION

NEWTON'S VELOCITY LAW

$$c^2 = \frac{dp}{d\rho} = \frac{kp}{\rho}$$

$$\frac{c_m^2}{c_p^2} = \frac{k_m p_m / \rho_m}{k_p p_p / \rho_p} \Rightarrow \frac{c_m^2}{c_p^2}\frac{U_p^2}{U_m^2} = \frac{k_m}{k_p}\frac{p_m}{\rho_m U_m^2}\frac{\rho_p U_p^2}{p_p} = \frac{Eu_m}{Eu_p} = 1$$

$$\Rightarrow \frac{c_m^2}{c_p^2}\frac{U_p^2}{U_m^2} = 1 \Rightarrow \frac{U_p}{c_p} = \frac{U_m}{c_m} = M$$

MACH

3.2. Physical meaning of dimensionless groups

NAME	EQUATION	MEANING	RELEVANCE
Reynolds	$Re = \frac{Ul}{\nu} = \frac{\rho Ul}{\mu}$	Relationship between inertial forces and viscous forces	- Differentiating the flow regime (turbulent / laminar). - Viscous effects
Froude	$Fr = \frac{U}{\sqrt{gl}}$	Relationship between inertial forces and gravity forces	- Open-channel flows
Mach	$M = \frac{U}{c}$	Relationship between inertial forces and elastic forces	- Compressible flow - Sound propagation (pressure waves)
Euler (Cavitation No if p-p_v instead of p)	$Eu = \frac{p}{\rho U^2}$	Relationship between pressure forces and inertial forces	- Cavitation phenomenon
Thermodynamic process exponent	k	Type of thermodynamic process a gas develops	- Compressible flow (gases)
Weber	$We = \frac{U^2 \rho L}{\sigma}$	Relationship between inertial forces and surface tension forces	- Flow with an interface between two liquids
Strouhal	$S = \frac{\omega l}{U}$	Importance of undulatory motion (angular velocity ω) in comparison to the main motion (average vel. U)	- Oscillating flows
Rossby	$Ro = \frac{U}{\Omega_{tierra} l}$	Relative importance of the Coriolis effect in the fluid flow	- geophysical flows
Relative roughness	$\frac{\varepsilon}{l}$	Roughness with respect to the main length of the body	- Turbulent flow - Friction
Friction coefficient or Fanning friction factor	$C_f = \frac{\tau_p}{\frac{1}{2}\rho U^2}$	Relationship between shear friction stress (per unita rea) and dynamic pressure	- Friction
Friction factor	$f = \frac{h_f}{(U^2/2g)(L/D)}$	Relationship between head loss and kinetic head	- Internal viscous flow (pipes)
Drag coefficient	$C_d = \frac{F_d}{(1/2)\rho A U^2}$	Relationship between drag force and dynamic force	- External flow (aerodynamics, hydrodynamics)
Lift coefficient	$C_L = \frac{F_L}{(1/2)\rho A U^2}$	Relationship between lift force and dynamic force	- External flow (aerodynamics, hydrodynamics)

3.3. Absolute and limited similitude. Froude and Reynolds incompatibility

- ✓ Mechanical or absolute similitude
  - Geometric similitude
  - Kinematic similitude
  - Dynamic similitude
- ✓ Implications for laboratory experiments:

$$\mathbf{\Pi_{1m}} = \varphi\,(\Pi_{2m}, \Pi_{3m},\ldots, \Pi_{km}) = \varphi\,(\Pi_{2p}, \Pi_{3p},\ldots, \Pi_{kp}) = \mathbf{\Pi_{1p}}$$

- ✓ Limited or incomplete similitude. Necessity

3.3. Absolute and limited similitude. Froude and Reynolds incompatibility

✓ Incompatibility between Froude and Reynolds

- Model construction according to Froude

$$Fr_m = Fr_p$$

$$\frac{U_m}{U_p} = \sqrt{\frac{l_m}{l_p}} = \lambda^{1/2}$$

- Model construction according to Reynolds

$$Re_m = Re_p$$

$$\frac{\nu_m}{\nu_p} = \frac{U_m}{U_p}\frac{l_m}{l_p}$$

$$\frac{\nu_m}{\nu_p} = \sqrt{\frac{l_m}{l_p}}\frac{l_m}{l_p} = \left(\frac{l_m}{l_p}\right)^{3/2} = \lambda^{3/2}$$

3.3. Absolute and limited similitude. Froude and Reynolds incompatibility

✓ Types of limited similitude

- **Mach**: in any case where the compressibility of the fluid is significant (gases with high velocity)
- **Froude**: in any case where the gravity forces are important (dams, channels, buoyancy, open flows)
- **Reynolds**: in any case where the viscous effect is preponderant (fluid flow in closed ducts, hydraulic machines, aerodynamic tunnels with low Mach No)
- **Euler**: in any case where the fluid pressure state is important (cavitation phenomenon)

3.4. Scale distortion

- ✓ Necessity of scale distortion
- ✓ Example: simulation of the flow of a river

$$\lambda_1 = \frac{\text{length}_m}{\text{length}_p} = \frac{\text{width}_m}{\text{width}_p}$$

$$\lambda_2 = \frac{h_m}{h_p}$$

- **Model according to Froude:**

$$\frac{U_m}{\sqrt{g_m h_m}} = \frac{U_p}{\sqrt{g_p h_p}}$$

$$\frac{U_m}{U_p} = \sqrt{\frac{h_m}{h_p}}$$

Velocity ratio

$$\frac{Q_m}{Q_p} = \frac{A_m U_m}{A_p U_p} = \frac{\text{width}_m \times h_m}{\text{width}_p \times h_p}\sqrt{\frac{h_m}{h_p}} = \lambda_1 \lambda_2 \lambda_2^{1/2} = \lambda_1 \lambda_2^{3/2}$$

Flow rate ratio

3.5. Applications: external flow

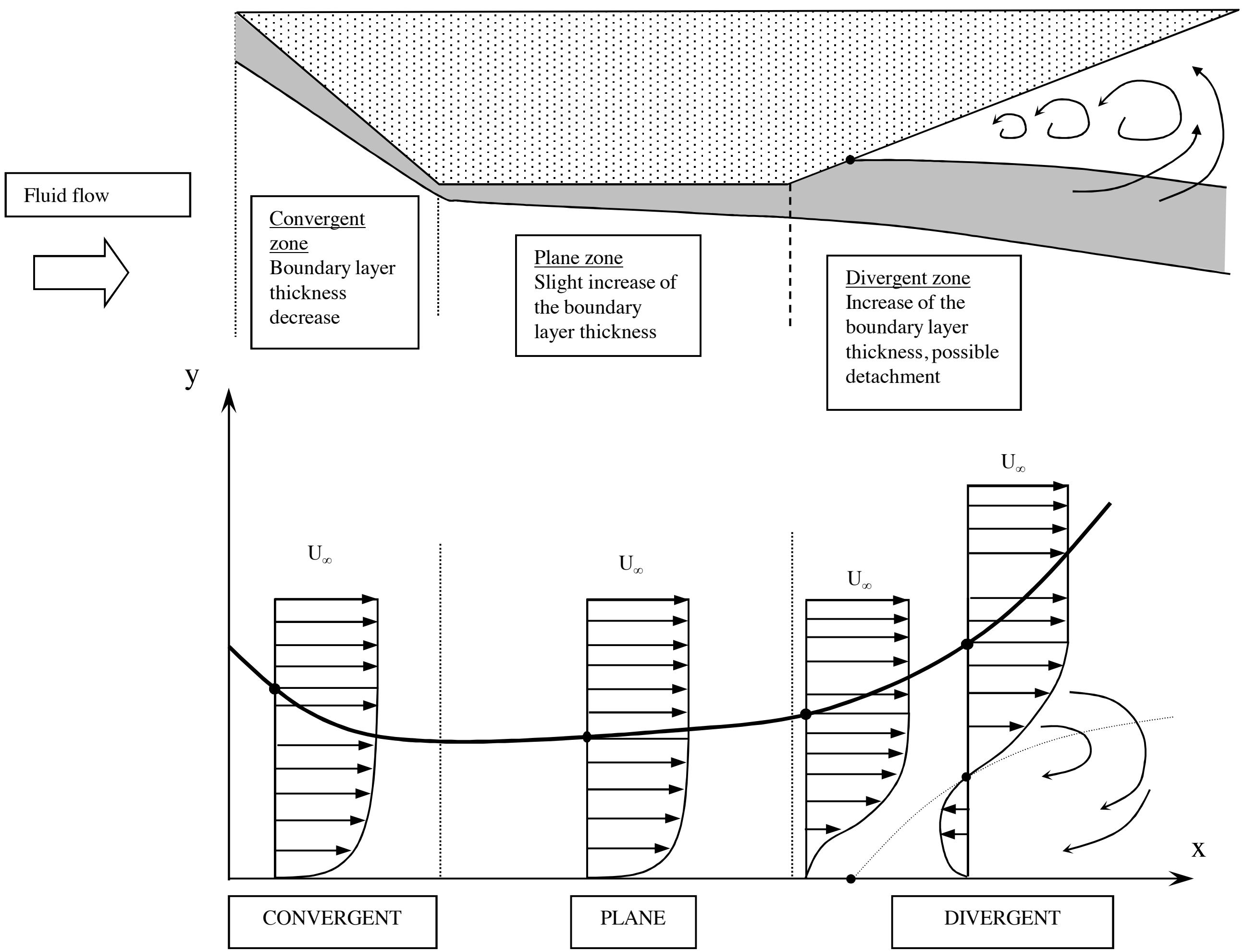
3.5.1. Drag force

3.5.2. Lift force

3.5.3. Flow induced vibration. Von Karman vortex street

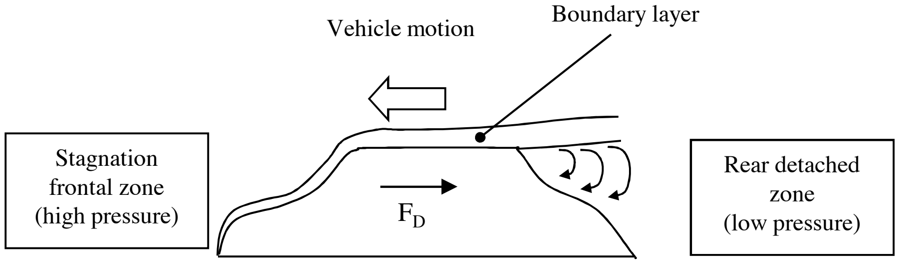
3.5.1. Drag force: friction and pressure

✓ Pressure drag or form drag: separation/detachment of the boundary layer



3.5.1. Drag force: friction and pressure

✓ Pressure drag or form drag: separation/detachment of the boundary layer



✓ General expression of the drag force

$$F_d = \frac{1}{2} C_d \rho A U^2$$

- Area:

1. Frontal area (projection on to the plane that is perpendicular to the motion direction) is used: in most cases, with totally immersed bodies.

2. Planform area (Proj. on to the horizontal plane) is used: with streamlined bodies (such as airfoils)

3. Wetted area (skin): with buoyant bodies (ships, boats)

3.5.1. Drag force: friction and pressure

- ✓ Friction and pressure:

$$F_d = F_{d,f} + F_{d,p}$$

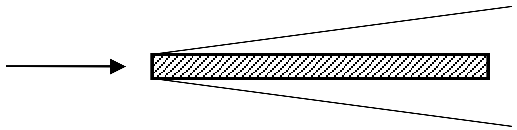
$$C_{d,f} = \frac{F_{d,f}}{\frac{1}{2}\rho U^2 A} \qquad C_{d,p} = \frac{F_{d,p}}{\frac{1}{2}\rho U^2 A} \quad \Rightarrow \quad C_d = C_{d,f} + C_{d,p}$$

- ✓ Partially submerged bodies: " wave drag"

$$F_w = \frac{1}{2} C_{d,w} \rho A U^2$$

3.5.1. Drag force: friction and pressure

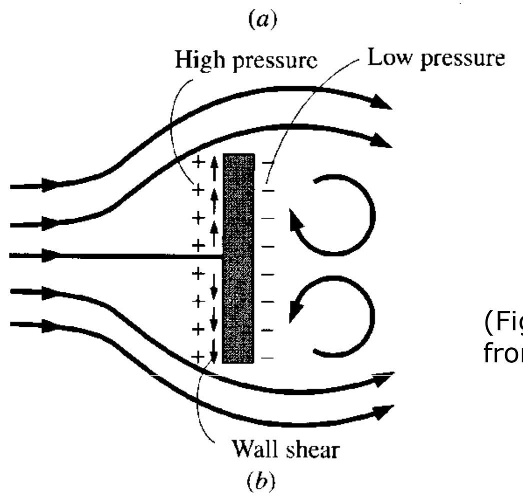
✓ Relative importance of pressure and friction drag:



Plane plate parallel to the flow: only friction

$$F_d = \iint_A \tau_p \, dA$$

"Friction drag"



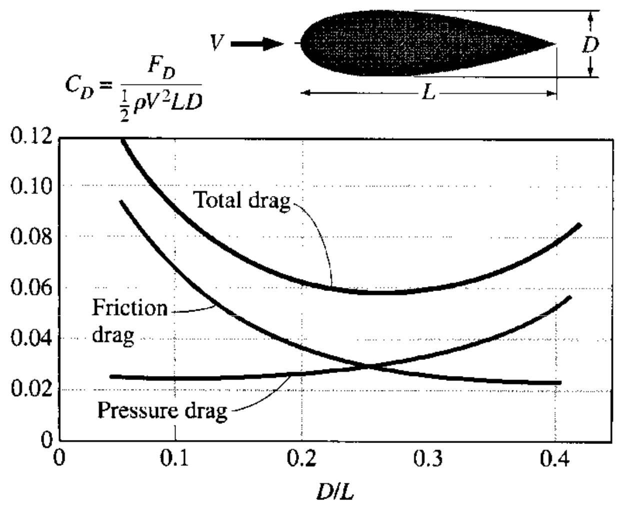
(Figure taken from Cengel)

Flat plate normal to the flow: only pressure

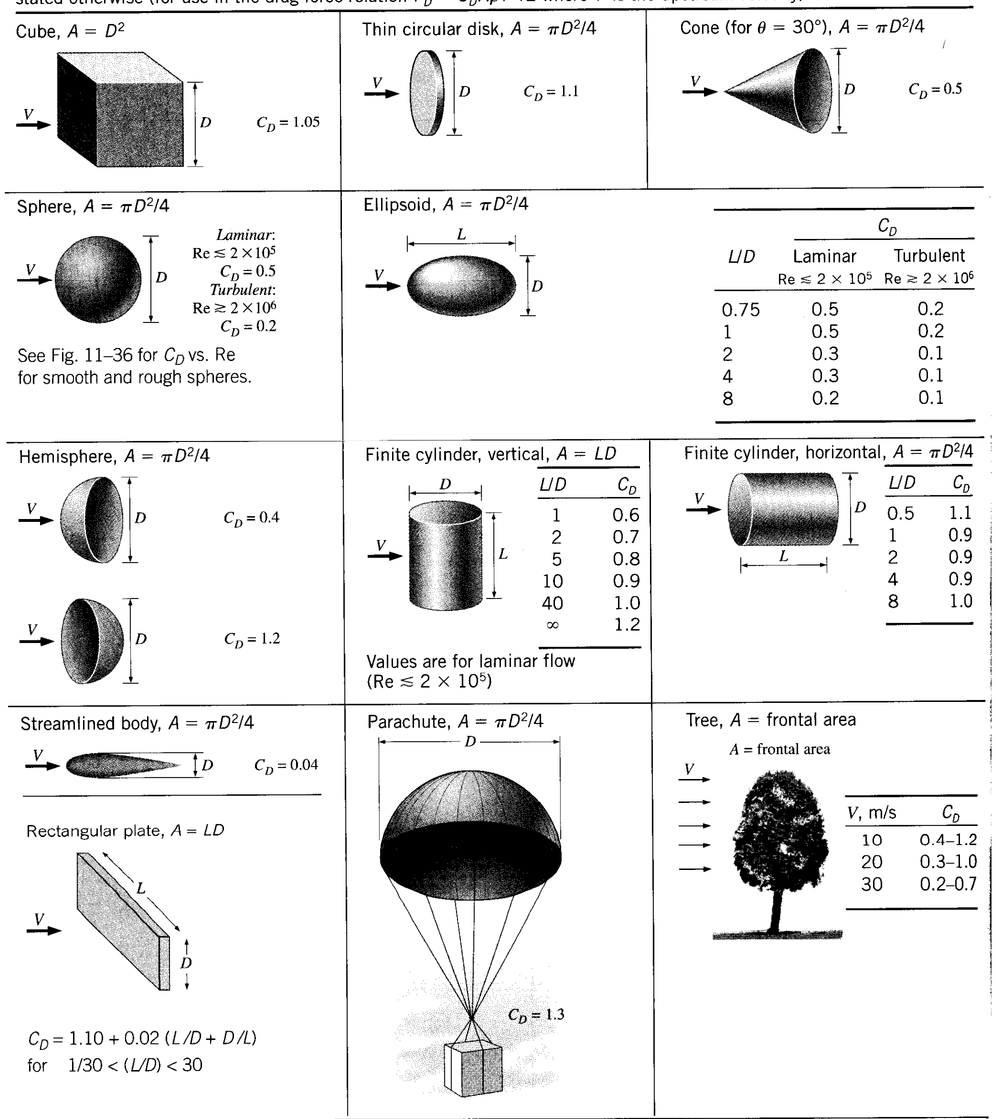
$$F_d = \iint_{A_1} p\,dA - \iint_{A_2} p\,dA$$

"Pressure drag"

✓ Drag coefficient: form influence



(Figures taken from Cengel)



Representative drag coefficients C_D for various three-dimensional bodies based on the frontal area for Re > 10^4 unless stated otherwise (for use in the drag force relation $F_D = C_D A \rho V^2/2$ where V is the upstream velocity)

Cube, $A = D^2$: $C_D = 1.05$

Thin circular disk, $A = \pi D^2/4$: $C_D = 1.1$

Cone (for $\theta = 30°$), $A = \pi D^2/4$: $C_D = 0.5$

Sphere, $A = \pi D^2/4$: Laminar: Re ≲ 2×10^5, $C_D = 0.5$; Turbulent: Re ≳ 2×10^6, $C_D = 0.2$

See Fig. 11–36 for C_D vs. Re for smooth and rough spheres.

Ellipsoid, $A = \pi D^2/4$

	C_D	
L/D	Laminar Re ≲ 2×10^5	Turbulent Re ≳ 2×10^6
0.75	0.5	0.2
1	0.5	0.2
2	0.3	0.1
4	0.3	0.1
8	0.2	0.1

Hemisphere, $A = \pi D^2/4$: $C_D = 0.4$; $C_D = 1.2$

Finite cylinder, vertical, $A = LD$

L/D	C_D
1	0.6
2	0.7
5	0.8
10	0.9
40	1.0
∞	1.2

Values are for laminar flow (Re ≲ 2×10^5)

Finite cylinder, horizontal, $A = \pi D^2/4$

L/D	C_D
0.5	1.1
1	0.9
2	0.9
4	0.9
8	1.0

Streamlined body, $A = \pi D^2/4$: $C_D = 0.04$

Rectangular plate, $A = LD$

$C_D = 1.10 + 0.02\,(L/D + D/L)$ for $1/30 < (L/D) < 30$

Parachute, $A = \pi D^2/4$: $C_D = 1.3$

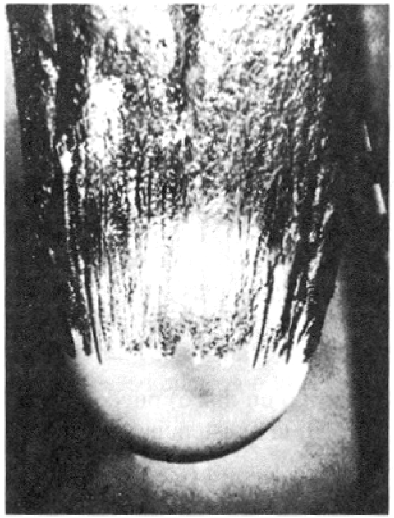
Tree, A = frontal area

V, m/s	C_D
10	0.4–1.2
20	0.3–1.0
30	0.2–0.7

(*continues*)

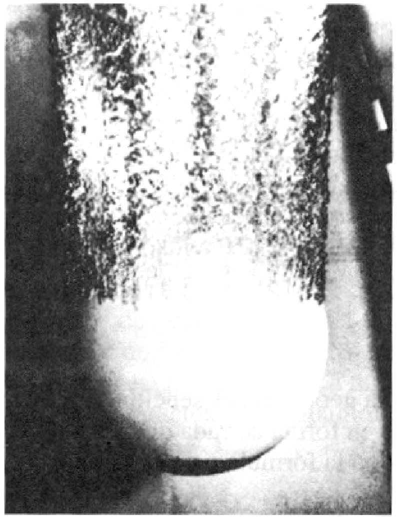
3.5.1. Flow over cylinders and spheres

✓ Type of boundary layer, detachment:



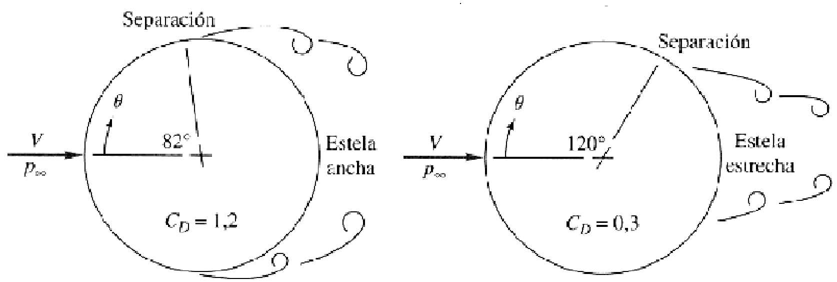
(a)

LAMINAR

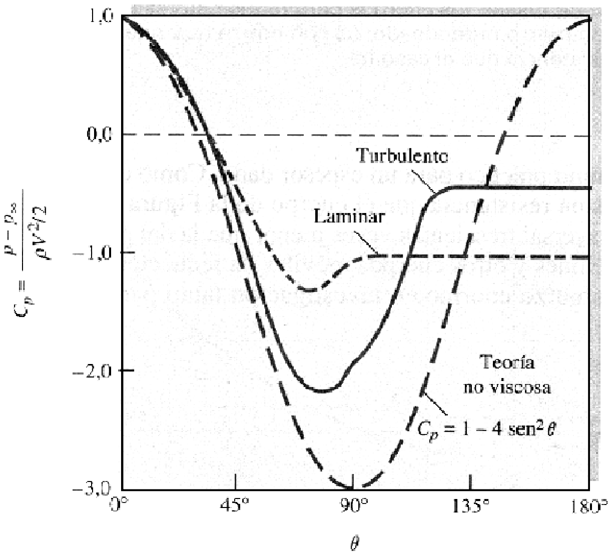


(b)

TURBULENT



(a) (b)

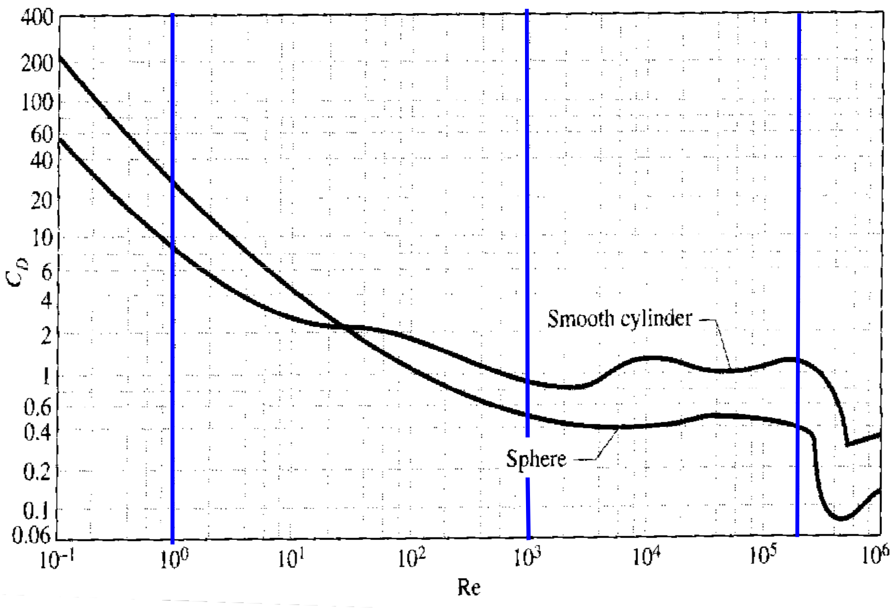


(c)

(Figures taken from White)

3.5.1. Flow over cylinders and spheres

✓ Reynolds number, type of regime



(Figure taken from Cengel)

✓ Regions:

Re<1: Dominant force: friction drag F_f.
It can be defined as: $C_d = k/Re$, in case of a sphere $C_d = 24/Re$

$F = 6\pi R\mu U$ ("Stokes formula")

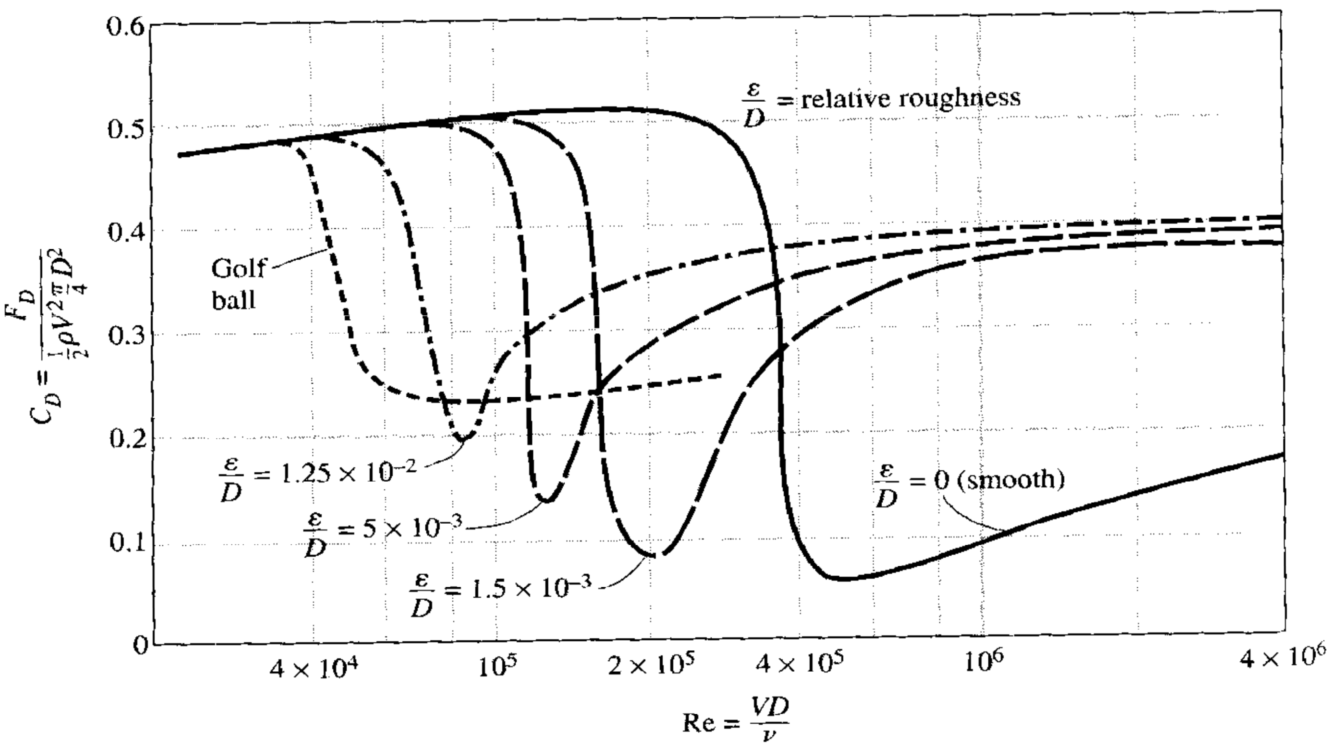
1<Re<1000: Transition zone, friction drag F_f and pressure drag F_p simultaneously intervening.

1000<Re<2x10⁵: Dominant force: pressure drag F_p

Re>2 10⁵: The boundary layer becomes turbulent

3.5.1. Flow over cylinders and spheres

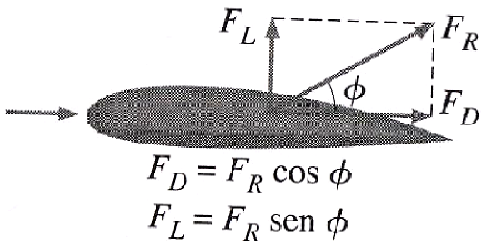
✓ Effect of roughness



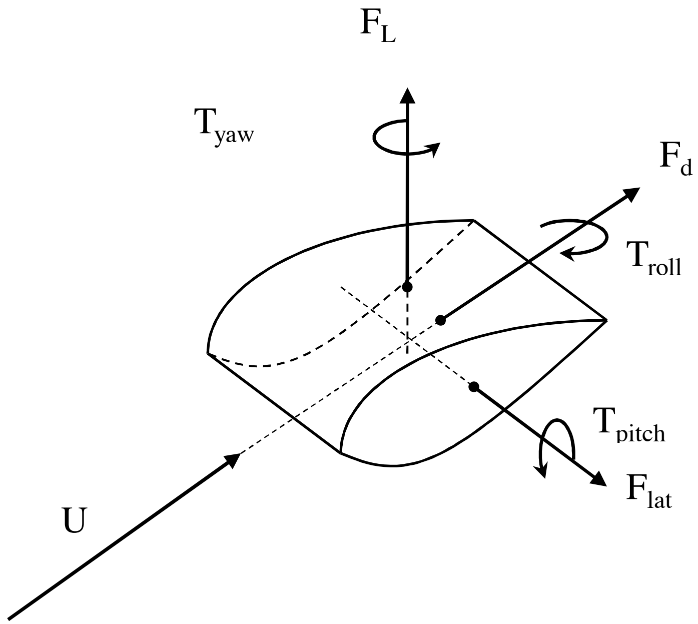
(Taken from Cengel)

3.5.2. Lift force in airfoils (wings)

- ✓ Airfoil:



(Figure taken from Cengel)

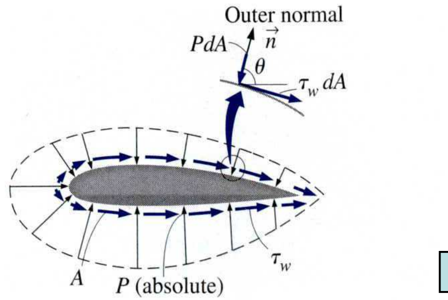


F_d: Drag force (parallel to flow)
F_L: Lift force (normal to flow)
F_{lat}: Lateral force

Roll torque: around flow direction
Yaw torque: around lift direction
Pitch torque: around lateral force direction

3.5.2. Lift force in airfoils (wings)

✓ Source:



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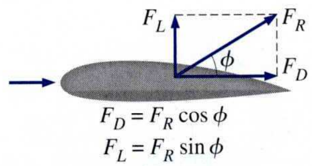
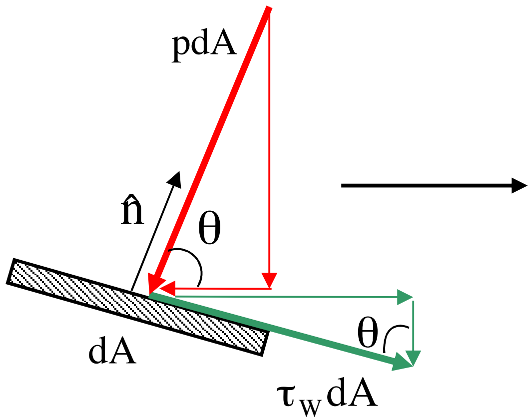


FIGURE 11–5
The pressure and viscous forces acting on a two-dimensional body and the resultant lift and drag forces.

- Differential area element "dA":



Shear force: $\tau_W dA$

Normal force: pdA

- Total forces:

Drag contribution:

$$dF_d = -pdA\cos\theta + \tau_W dA\sin\theta$$

Lift contribution:

$$dF_L = -pdA\sin\theta - \tau_W dA\cos\theta$$

$$F_d = \iint_A \left(-p\cos\theta + \tau_W \sin\theta\right) dA$$

$$F_L = \iint_A \left(-p\sin\theta - \tau_W \cos\theta\right) dA$$

3.5.2. Lift force in airfoils (wings)

✓ Calculation:

$$F_L = \frac{1}{2} C_L \rho A U^2$$

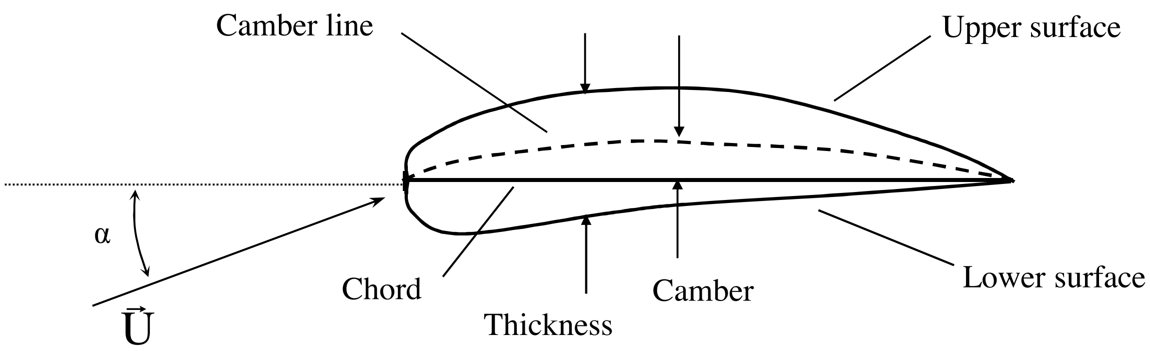
C_L: Lift coefficient (for a specific body as a function of the angle of attack α and Reynolds number (less important))

ρ: fluid density

A: planform area

U: freestream velocity

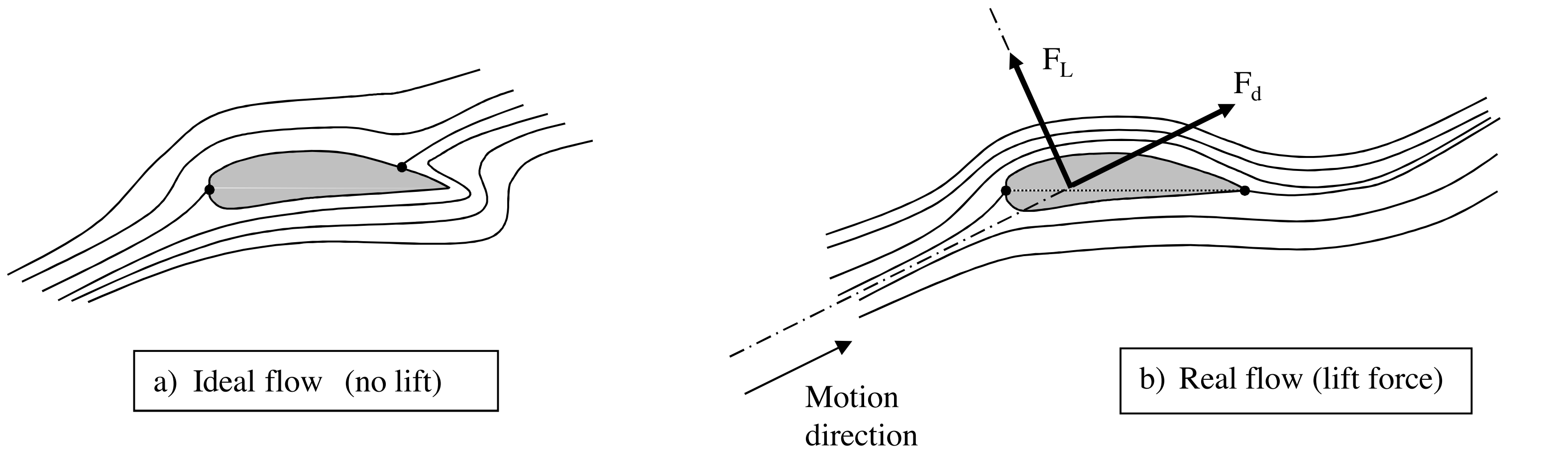
✓ Airfoil terminology:



- **Chord**
- **Camber line**
- **Camber**
- **Angle of attack "α"**
- **Wingspan (or span)**
- **Leading edge**
- **Trailing edge**

3.5.2. Lift force in airfoils (wings)

✓ Source:



Lift force increase:

(1) Curvature of the airfoil (longer length of the upper surface in comparison to the lower surface)

(2) Angle of attack (up to a value of 15º aprox.)

(3) Use of movable leading edge and trailing edge flaps

3.5.2. Lift force in airfoils (wings)

- ✓ Source:

Lift force increase:

(3) Use of flaps

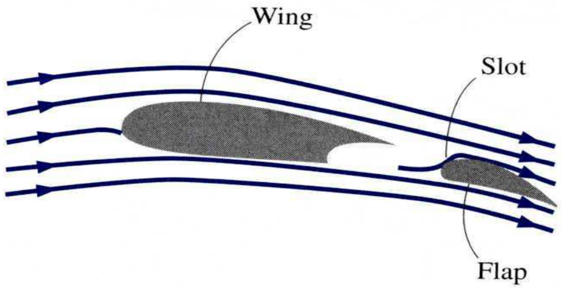
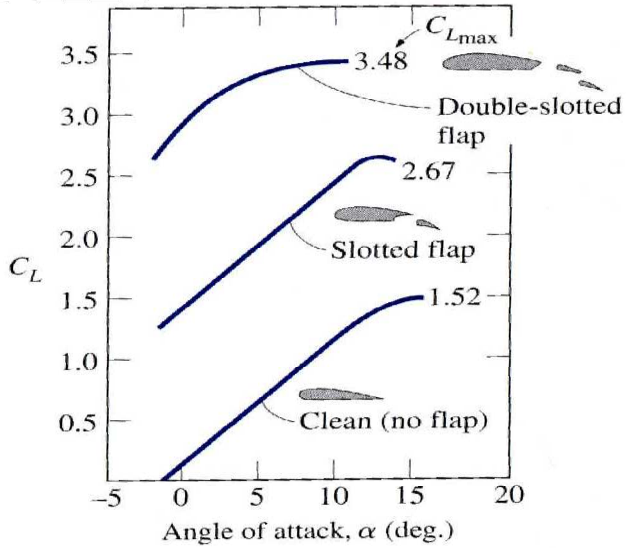
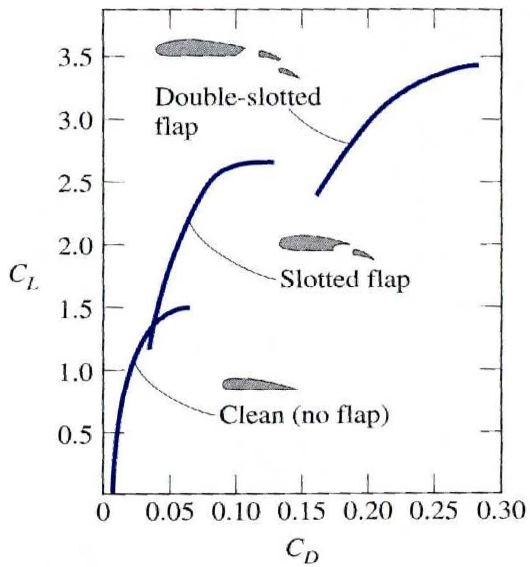


FIGURE 11–46
A flapped airfoil with a slot to prevent the separation of the boundary layer from the upper surface and to increase the lift coefficient.





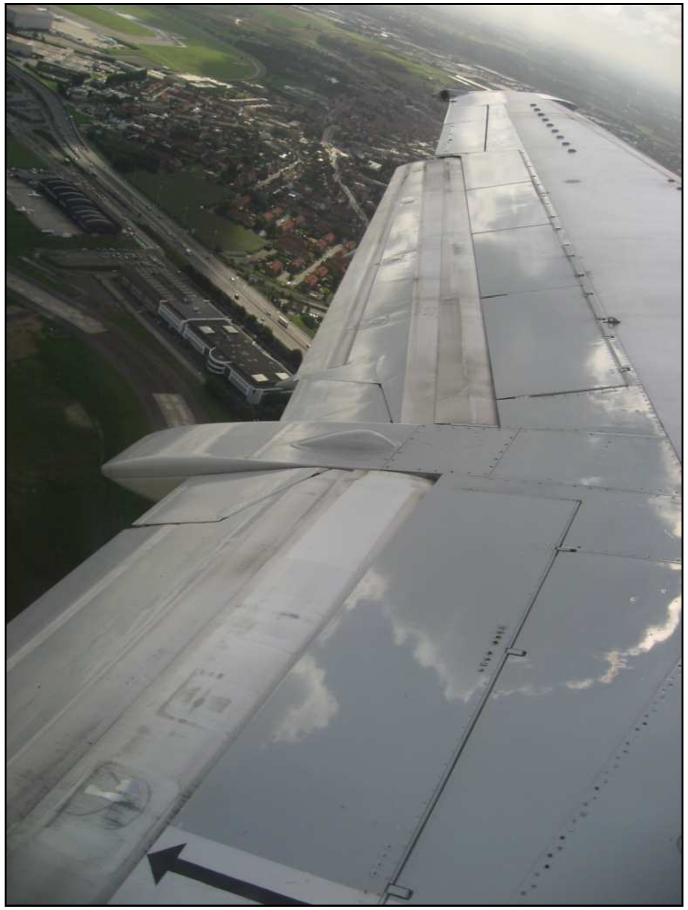
(Figures taken from Cengel)

3.5.2. Lift force in airfoils (wings)

- ✓ Source:

Lift force increase:

(3) Use of flaps



(Photographs G. A. Esteban)

3.5.2. Lift by Magnus effect

- ✓ Cylinder rotation: lift generation

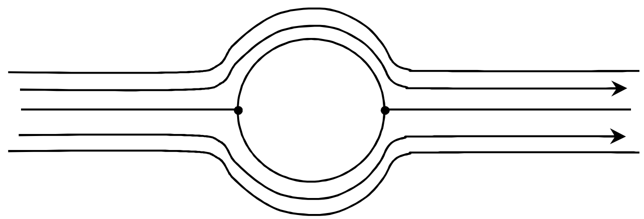
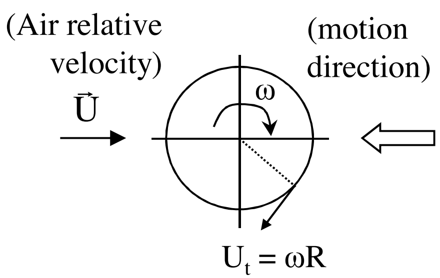


Figure 8.9 Fluid streamlines around a static cylinder



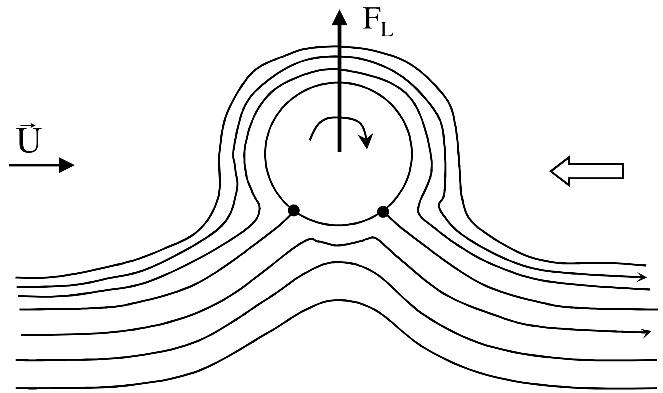
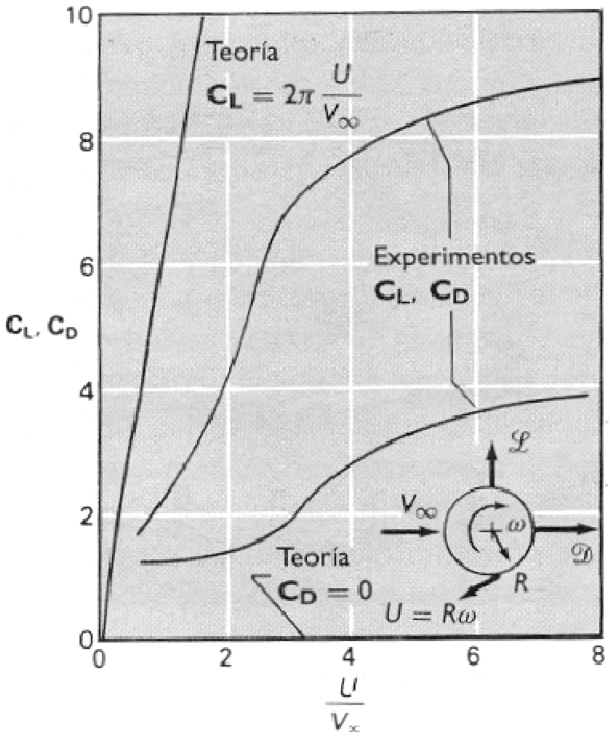


Figure 8.10 Stagnation points modification because of the spinning

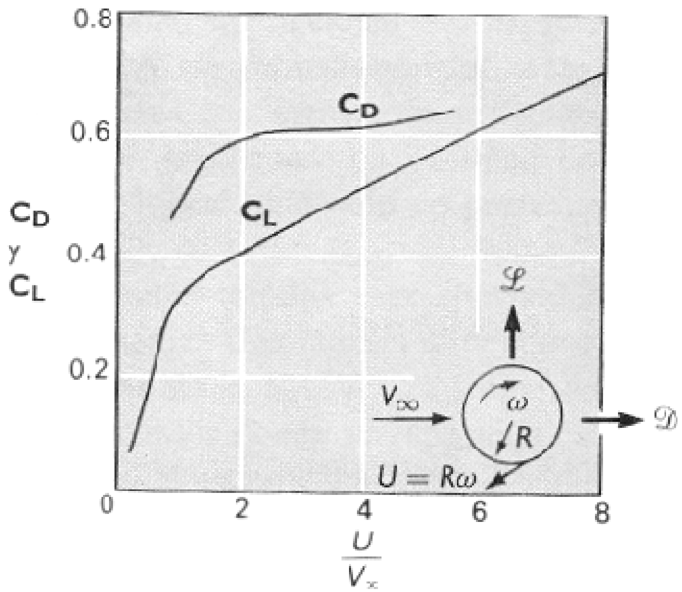
3.5.2. Lift by Magnus effect

✓ Drag and lift coefficients (experimental)

(Figures taken from Gerhart)



SMOOTH CYLINDER

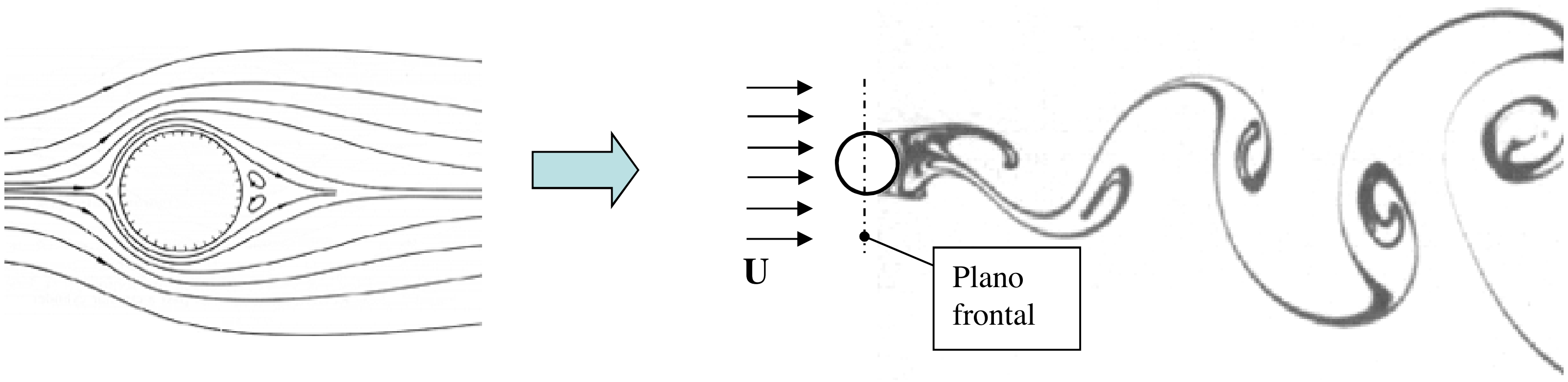


SMOOTH SPHERES

- Theoretical value (cylinder): $$C_L = \frac{2\pi\omega R}{U_\infty}$$

3.5.3. Von-Karman vortex street

- ✓ Phenomenon



- ✓ Dimensional analysis

The vortex emission rate "n" (Hz) o (s^{-1}) depends on free stream flow velocity "U", the cylinder diameter "D", fluid viscosity "μ", fluid density "ρ".

Π groups describing the phenomenon:

Strouhal number: $S = \frac{nD}{U}$

Reynolds number: $Re = \frac{UD\rho}{\mu}$

3.5.3. Von-Karman vortex street

- ✓ Relationship between dimensionless numbers:

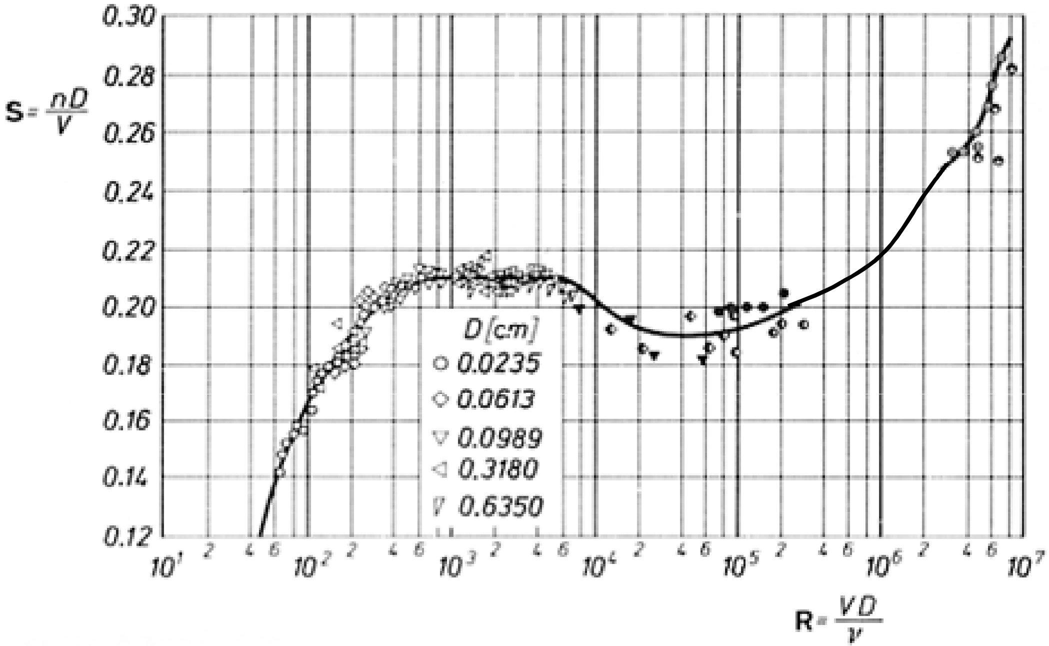


Figure 8.13 Dependence of the vortex emission rate

Figure 8.14 Takoma bridge (Washington) fallen down by “weak” wind 7th of November, 1940.