CHAPTER 7. THE LINEAR MOMENTUM THEOREM

1. The momentum theorem. 1^{st} Euler theorem
2. Applications of the momentum theorem

1. Linear momentum theorem

- 2nd Newton's law (rigid body of mass "m")

$$\sum \vec{F} = \frac{d\vec{M}}{dt}$$

$$\vec{M} = m\vec{U}$$

- 2nd Newton's law (continuum, system)

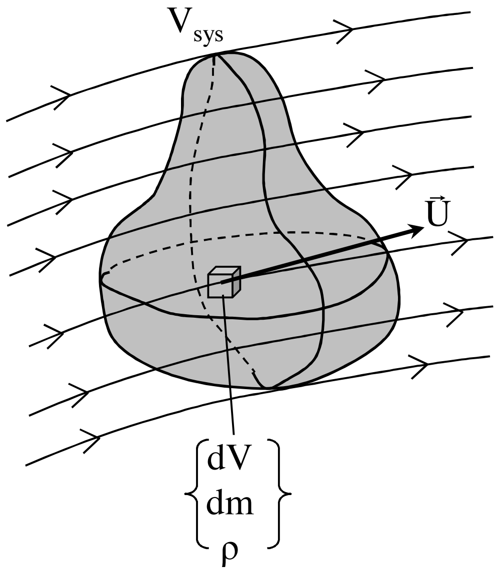


Figure 7.1 Momentum of a fluid system

$$\sum_{sys} \vec{F} = \frac{d\vec{M}_{sys}}{dt}$$

$$d\vec{M} = dm\vec{U} = (\rho dV)\vec{U}$$

$$\vec{M}_{sys} = \iiint_{V_{sys}} \rho \vec{U} dV$$

1. Linear momentum theorem

$$\sum_{sys} \vec{F} = \frac{d\vec{M}_{sys}}{dt}$$

- ✓ 2nd term. Transport theorem

$$B_{sys} = \iiint_{V_{sys}} b\rho dV \qquad \frac{dB_{sys}}{dt} = \frac{d}{dt}\iiint_{V_c} \rho b dV + \oiint_{A_c} \rho b \vec{U}_r d\vec{A}$$

$$\vec{M}_{sys} = \iiint_{V_{sys}} \rho \vec{U} dV \qquad \frac{d\vec{M}_{sys}}{dt} = \frac{d}{dt}\iiint_{V_c} \rho \vec{U} dV + \oiint_{A_c} \rho \vec{U} \vec{U}_r d\vec{A}$$

$$\Updownarrow$$

$$\frac{d\vec{M}_{sys}}{dt} = \frac{d\vec{M}_{Vc}}{dt} + \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in}$$

1. Linear momentum theorem

$$\sum_{sys} \vec{F} = \frac{d\vec{M}_{sys}}{dt}$$

✓ 1st term. Forces

- Intrinsic $$\vec{P} = \oiint_{Ac} \vec{T} dA$$

- External field $$\vec{G} = \iiint_{Vc} \rho \vec{F} dV$$

✓ Integral form of the linear momentum theorem

$$\vec{P} + \vec{G} = \frac{d\vec{M}_{Vc}}{dt} + \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in}$$

$$\oiint_{Ac} \vec{T} dA + \iiint_{Vc} \rho \vec{F} dV = \frac{d}{dt} \iiint_{V_c} \rho \vec{U} dV + \oiint_{A_c} (\rho \vec{U}) \vec{U}_r d\vec{A}$$

1. Linear momentum theorem

$$\oint\!\!\oint_{A_c} \vec{T}dA + \iiint_{V_c} \rho\vec{F}dV = \frac{d}{dt}\iiint_{V_c} \rho\vec{U}dV + \oint\!\!\oint_{A_c} (\rho\vec{U})\vec{U}_r d\vec{A}$$

- ✓ Steady-state regime

$$\frac{d}{dt}\iiint_{V_c} \rho\vec{U}dV = 0$$

- ✓ Fixed and rigid control volume

$$\vec{U} = \vec{U}_r$$

- ✓ Integral form of the linear momentum theorem

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in}$$

$$\oint\!\!\oint_{A_c} \vec{T}dA + \iiint_{V_c} \rho\vec{F}dV = \oint\!\!\oint_{A_c} (\rho\vec{U})\vec{U}d\vec{A}$$

1. First Euler theorem

✓ Application of the linear momentum theorem to a streamtube

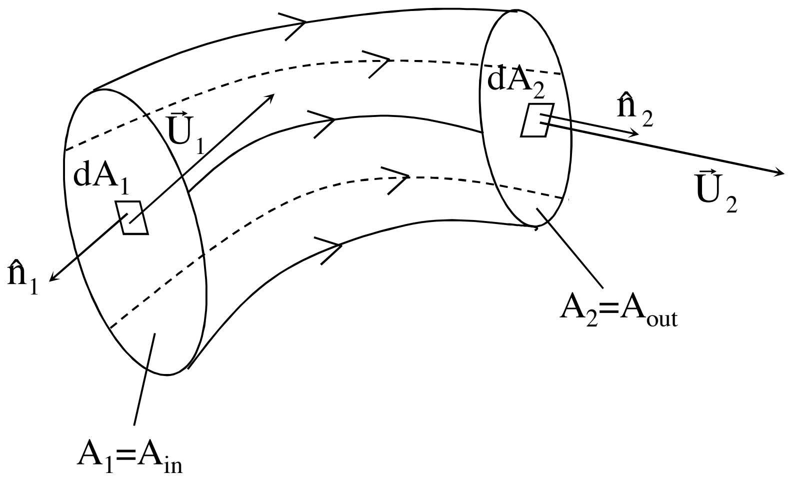


Figure 7.2 Linear momentum theorem applied to a streamtube in steady-state regime. 1st Euler theorem

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in}$$

⇩

$$\vec{P} + \vec{G} = q_m (\vec{U}_2 - \vec{U}_1)$$

1. First Euler theorem

✓ Hydrodynamic thrust forces

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in} \quad \Rightarrow \quad \vec{P} + \vec{G} + (\dot{\vec{M}}_{in}) + (-\dot{\vec{M}}_{out}) = 0$$

- Hydrodynamic action force $(\dot{\vec{M}}_{in}) = q_m \vec{U}_1$
- Hydrodynamic reaction force $(-\dot{\vec{M}}_{out}) = -q_m \vec{U}_2$

✓ Momentum-flux correction factor

$$\dot{M} = \iint_A \rho U^2 dA \neq \rho U_{avg}{}^2 A \quad \Rightarrow \quad \dot{M} = \beta \cdot (\rho U_{avg}{}^2 A) = \beta \cdot q_m U_{avg}$$

$$\beta = \frac{\iint U^2 dA}{U_{avg}{}^2 A}$$

✓ 1st Euler theorem: $\vec{P} + \vec{G} = q_m (\beta_2 \vec{U}_2 - \beta_1 \vec{U}_1)$

2. Applications of the linear momentum theorem

1. Applications of the linear momentum theorem I:
   - ✓ Propulsion systems
2. Applications of the linear momentum theorem II:
   - ✓ Reaction of an incompressible fluid control volume on a guiding channel
   - ✓ Forces by jets on obstacles. Static, moving and succession of obstacles
3. Applications of the linear momentum theorem III:
   - ✓ Sudden expansion (Borda-Carnot)
   - ✓ Hydraulic jump

2. Applications of the linear momentum theorem

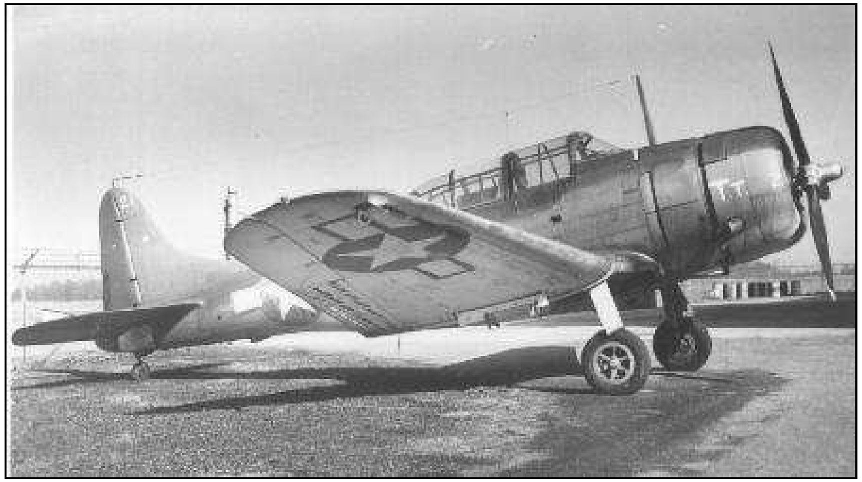
2.1. Applications of the linear momentum theorem I:

Propulsion systems:

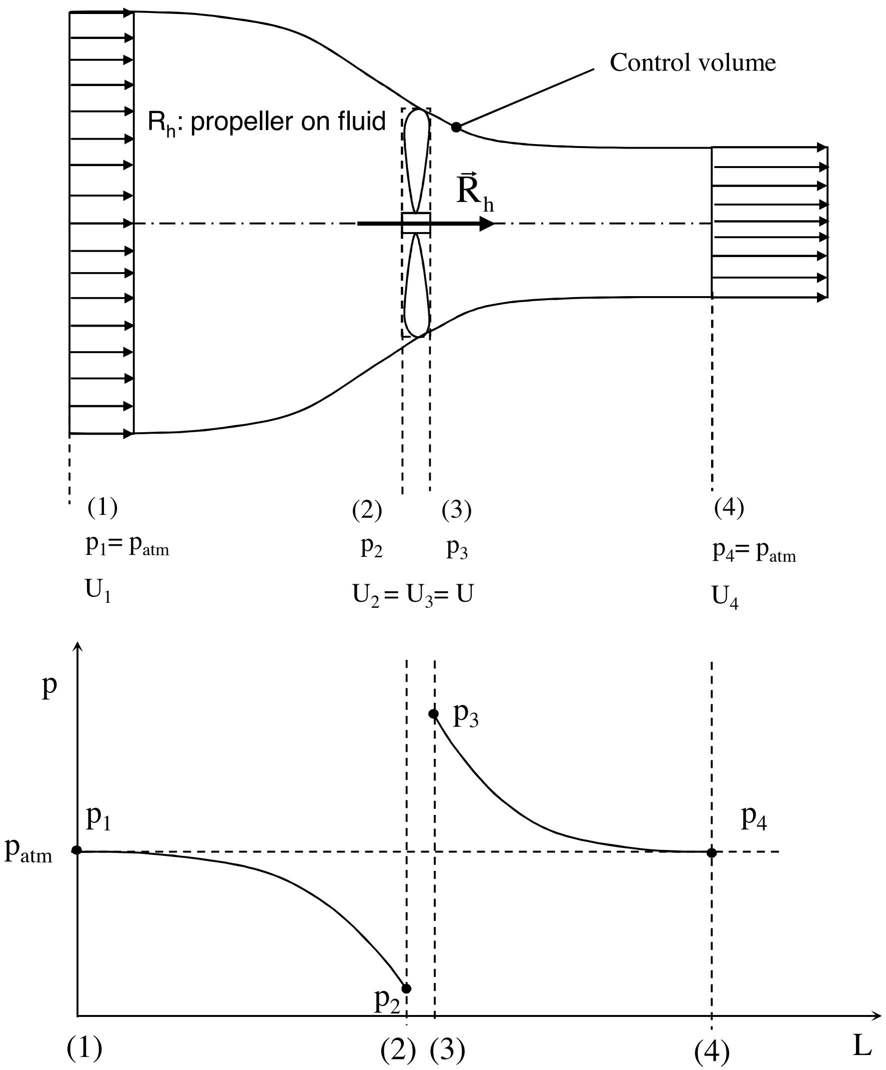
- ✓ Propellers
- ✓ Turbojets
- ✓ Rockets

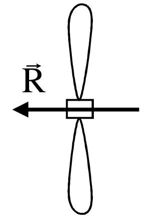
2.1. Propulsion systems: propellers

- ✓ Concept



✓ Fluid flow. Pressure evolution





R: Fluid on propeller (propulsion)

Figure 7.4 Pressure evolution in the propulsion by a propeller

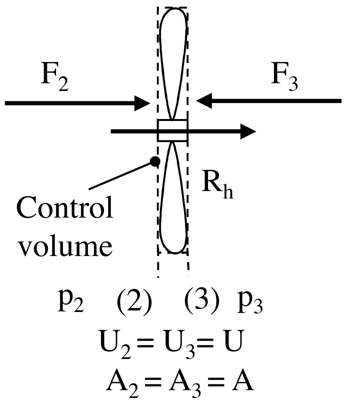
- Mathematical analysis

- 1st Euler theorem: control volume between sections (1) and (4)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in} = q_m(\vec{U}_4 - \vec{U}_1)$$

$$R_h = \rho U A (U_4 - U_1)$$

- 1st Euler theorem: control volume between sections (2) and (3)



$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in} = q_m(\vec{U}_3 - \vec{U}_2)$$

$$R_h = A(p_3 - p_2)$$

✓ Mathematical analysis

- Bernoulli eq. between sections (1) and (2): $$\frac{U_1^2}{2g} + \frac{p_1}{\gamma} = \frac{U_2^2}{2g} + \frac{p_2}{\gamma}$$
- Bernoulli eq. between sections (3) and (4): $$\frac{U_3^2}{2g} + \frac{p_3}{\gamma} = \frac{U_4^2}{2g} + \frac{p_4}{\gamma}$$

$$\frac{U_1^2}{2g} + \frac{p_3}{\rho g} = \frac{U_4^2}{2g} + \frac{p_2}{\rho g}$$

$$\boxed{p_3 - p_2 = \frac{1}{2}\rho(U_4^2 - U_1^2)}$$

$R_h = A(p_3 - p_2)$

$$\Rightarrow \quad R_h = \frac{1}{2}\rho A(U_4^2 - U_1^2) \qquad R_h = \rho U A\,(U_4 - U_1)$$

$$\Downarrow$$

$$\boxed{U = \frac{U_4 + U_1}{2}}$$

(velocity through the propeller)

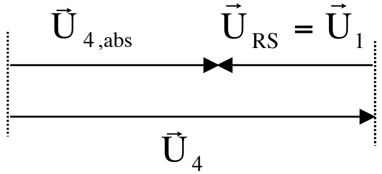
2.1. Propulsion systems: propellers

✓ Mathematical analysis

- Efficiency of the propeller

$$\eta = \frac{Useful\_power}{Propeller\_power} = \frac{P_u}{P_p} = \frac{R_h U_1}{R_h U} = \frac{U_1}{U}$$

$$\boxed{\eta = \frac{U_1}{U}}$$



$$\vec{U}_{4,abs} = \vec{U}_{RS} + \vec{U}_4$$

$$\boxed{U_{4,abs} = U_4 - U_1}$$

$$U = \frac{U_4 + U_1}{2} = \frac{(U_1 + U_{4,abs}) + U_1}{2} = U_1 + \frac{U_{4,abs}}{2}$$

$$\boxed{\eta = \frac{U_1}{U_1 + \frac{U_{4,abs}}{2}}}$$

2.1. Windmill. Wind turbine

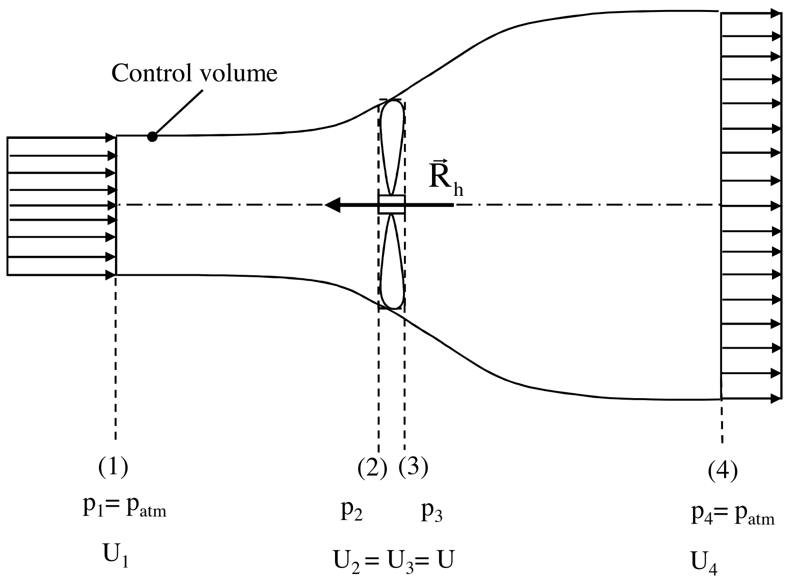
- ✓ Concept

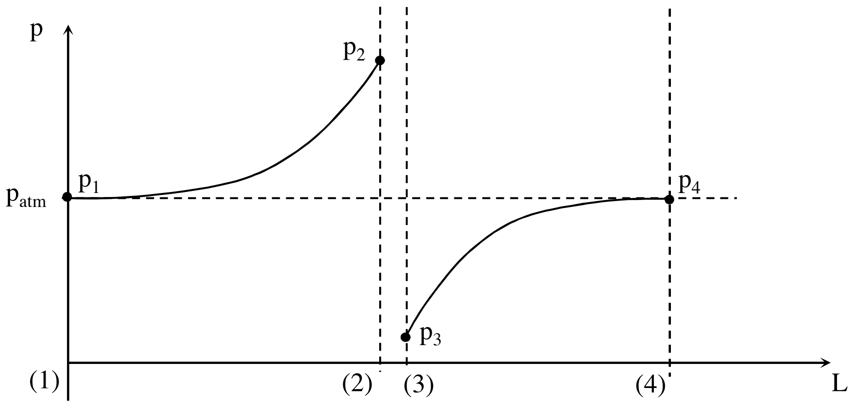
2.1. Windmill. Wind turbine

- ✓ Concept

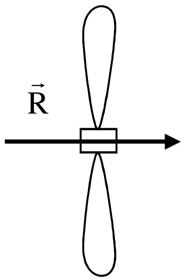
2.1. Windmill. Wind turbine

✓ Fluid flow. Pressure evolution





R_h: windmill on fluid
R: fluid on windmill



$$\vec{R} = -\vec{R}_h$$

2.1. Windmill. Wind turbine

- ✓ Mathematical analysis (similar to propeller)

$$R_h = \rho U A (U_1 - U_4)$$

$$R_h = A(p_2 - p_3)$$

$$p_2 - p_3 = \frac{1}{2}\rho(U_1^2 - U_4^2)$$

$$R_h = \frac{1}{2}\rho A(U_1^2 - U_4^2)$$

$$U = (U_4 + U_1)/2$$

2.1. Windmill. Wind turbine

- ✓ Mathematical analysis
  - Efficiency of the wind turbine

USEFUL POWER

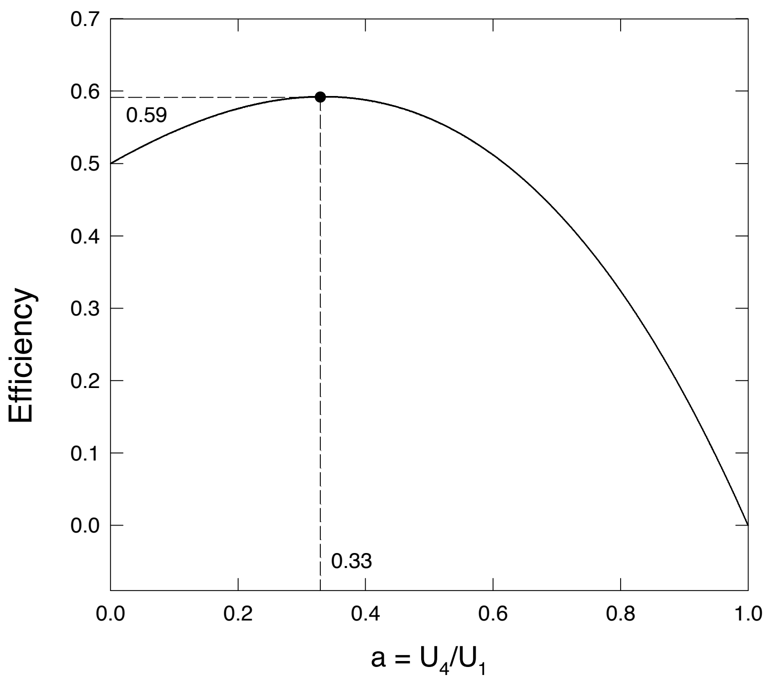
$$P_u = R_h U = \frac{1}{2} q_m U_1^2 - \frac{1}{2} q_m U_4^2 = \frac{1}{2} \rho A (U_1^2 - U_4^2) U$$

AVAILABLE POWER

$$P = \frac{1}{2} (\rho A U_1) U_1^2 = \frac{1}{2} \rho A U_1^3$$

$$\eta = \frac{P_u}{P} = \frac{1}{2} \left(1 - \left(\frac{U_4}{U_1} \right)^2 \right) \left(1 + \frac{U_4}{U_1} \right)$$

(Betz limit)



2.1. Propulsion systems: turbojet

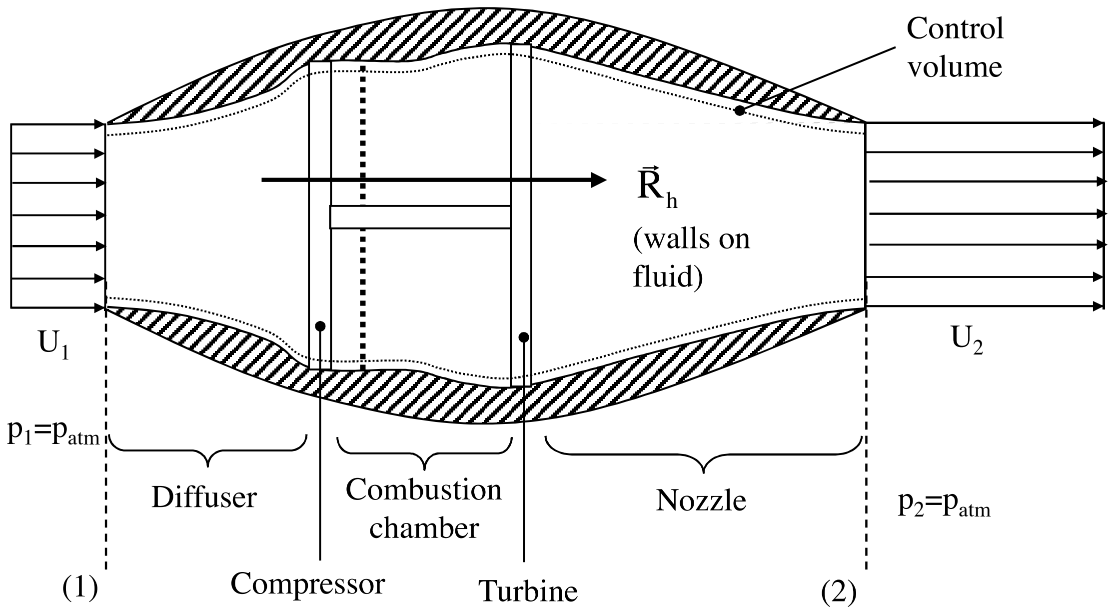
- ✓ Concept

2.1. Propulsion systems: turbojet

✓ Concept

2.1. Propulsion systems: turbojet

✓ Fluid flow



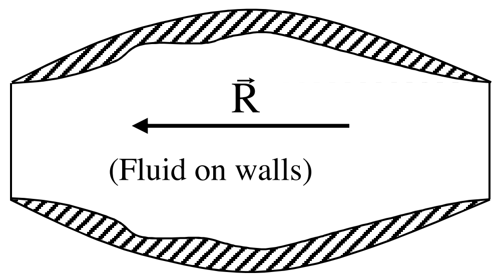


Figure 7.9 Propulsion by a turbojet. Scheme

2.1. Propulsion systems: turbojet

✓ Mathematical analysis

- 1st Euler theorem: control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in} = q_m(\vec{U}_2 - \vec{U}_1)$$

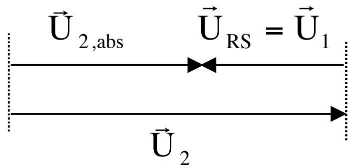
$$R_h = q_m(U_2 - U_1) = R$$

- Efficiency of the turboject

$$P_{useful} = R_h U_1 = q_m(U_2 - U_1)U_1$$

$$P_{losses} = (1/2)q_m U_{2,abs}^2$$

$$P_{losses} = (1/2)q_m(U_2 - U_1)^2$$



$$\vec{U}_{2,abs} = \vec{U}_{RS} + \vec{U}_2$$

$$U_{2,abs} = U_2 - U_1$$

$$\eta = \frac{P_u}{P_t} = \frac{P_u}{P_u + P_{loss}} = \frac{U_1}{U_1 + \frac{U_{2,abs}}{2}}$$

2.1. Propulsion systems: turbojet

✓ Mathematical analysis. Specific consumption of fuel

$$K = \frac{q_f}{q_m}$$

- 1st Euler theorem: control volume between sections (1) and (2)

$$q_m' = q_m + q_f = (1+K)q_m$$

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in}$$

$$R_h = q_m' U_2 - q_m U_1 = q_m[U_2(1+K) - U_1]$$

- Efficiency of the turbojet

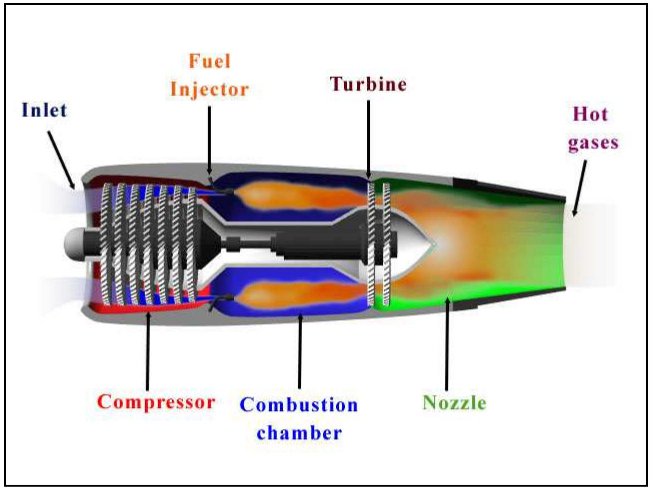
$$P_{useful} = R_h U_1 = q_m[U_2(1+K) - U_1]U_1$$

$$P_{losses} = (1/2) q_m' U_{2,abs}^2 = (1/2) q_m (1+K)(U_2 - U_1)^2$$

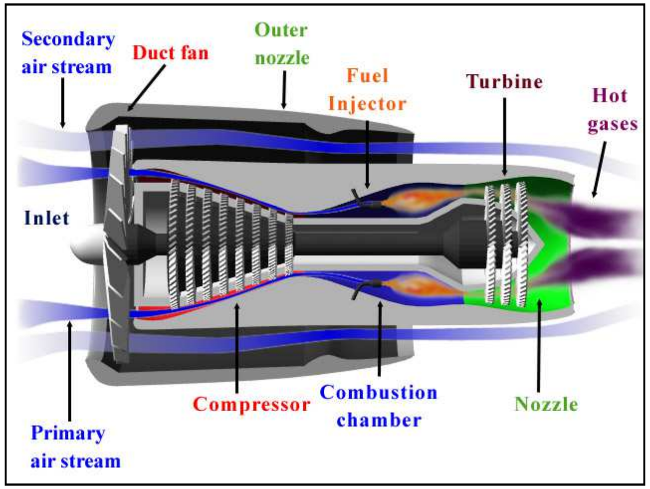
$$\eta = \frac{P_u}{P_t} = \frac{P_u}{P_u + P_{loss}} = \frac{1}{1 + \frac{(1+K)(U_2 - U_1)^2}{2(U_2(1+K) - U_1)U_1}}$$

2.1. Propulsion systems: turbojet

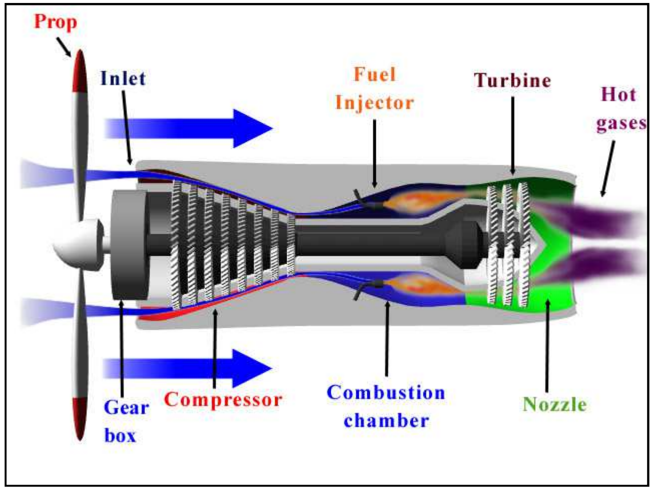
✓ Concept



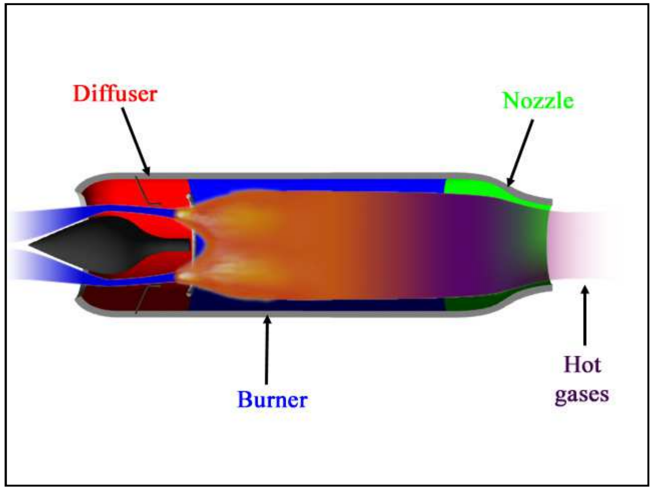
"TURBOJET"



"TURBOFAN"



"TURBOPROPELLER"



"RAMJET"

2.1. Propulsion systems: rockets

✓ Concept

2.1. Propulsion systems: rockets

- ✓ Fluid flow

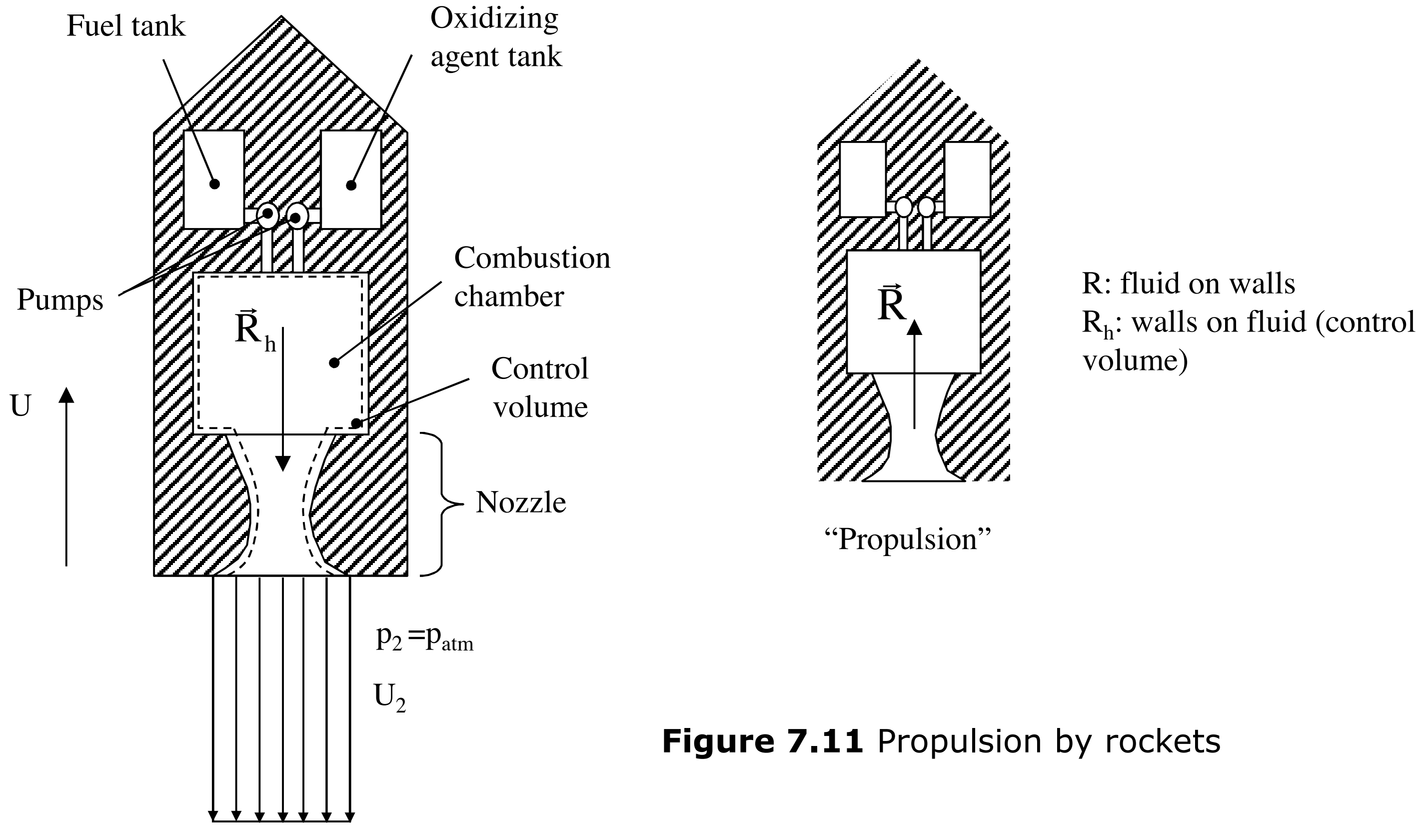


Figure 7.11 Propulsion by rockets

✓ Mathematical analysis

- 1^{st} Euler theorem: control volume

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in} = q_m(\vec{U}_2 - \vec{U}_1)$$

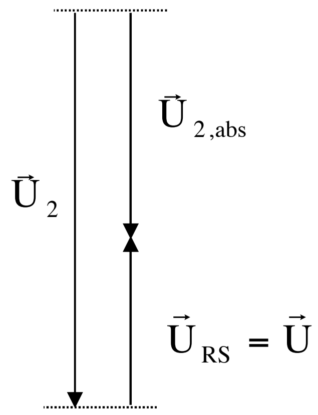
$$R_h = q_m(U_2) = \rho_2 A_2 U_2^2 = R$$

- Efficiency of the rocket

$$P_{useful} = R_h U = q_m U_2 U$$

$$P_{losses} = (1/2) q_m U_{2,abs}^2 = (1/2) q_m (U_2 - U)^2$$

$$\eta = \frac{P_u}{P_t} = \frac{P_u}{P_u + P_{loss}} = \frac{2(U_2/U)}{1 + (U_2/U)^2}$$



$$\vec{U}_{2,abs} = \vec{U}_{RS} + \vec{U}_2$$

$$U_{2,abs} = U_2 - U$$

2. Applications of the linear momentum theorem

2.2. Applications of the linear momentum theorem II:

1. Reaction by an incompressible fluid control volume on a guiding channel

2. Forces by jets on obstacles. Static, moving and succession of obstacles:
    - ✓ Normal jet on a vertical wall
    - ✓ Inclined plane plate
    - ✓ Conical obstacle
    - ✓ Blade

2.2. Reaction of an incompressible fluid control volume on a guiding channel

✓ Concept

(Pipe elbow in the siphon of Deusto: diameter 1,3 m, Max. Flow: 3050 L/s)

2.2. Reaction of an incompressible fluid control volume on a guiding channel

- ✓ Fluid flow

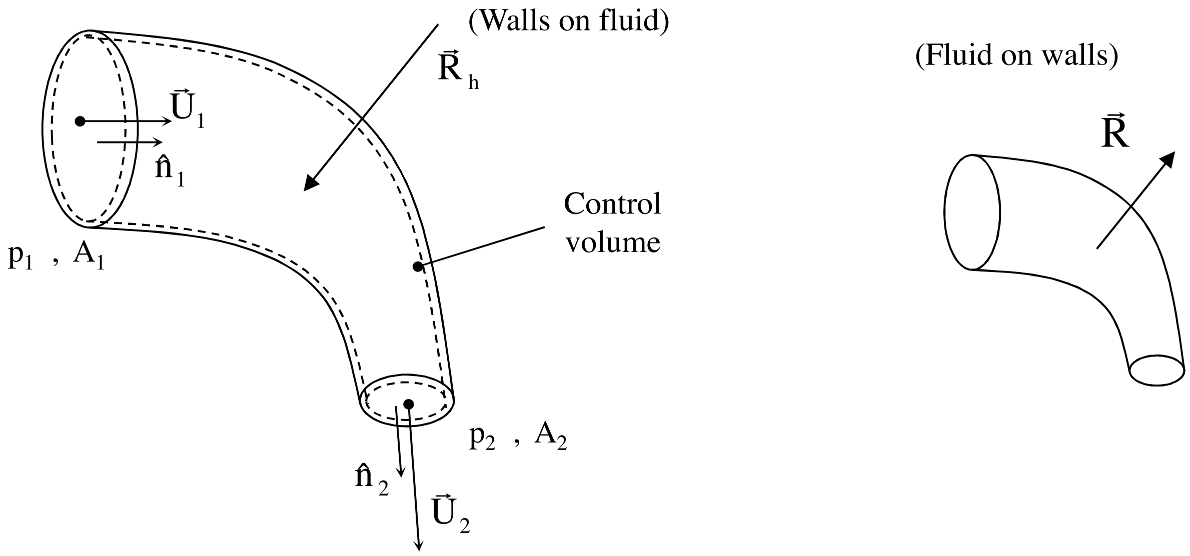


Figure 7.12 Reaction of an incompressible fluid on a guiding channel

2.2. Reaction of an incompressible fluid control volume on a guiding channel

- ✓ Mathematical analysis

  - 1st Euler theorem: control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in} = q_m(\vec{U}_2 - \vec{U}_1)$$

$$\boxed{\vec{R} = \hat{n}_1(A_1 p_1 + q_m U_1) - \hat{n}_2(A_2 p_2 + q_m U_2)}$$

$$\Downarrow$$

$$\boxed{\vec{R} = \hat{n}_1 A_1 p_1 - \hat{n}_2 A_2 p_2}$$

2.2. Forces by jets on obstacles

- ✓ Forces by jets on obstacles
  - Static
  - Moving
  - Succession of obstacles
    - ✓ Normal jet on vertical wall
    - ✓ Inclined plane plate
    - ✓ Conical obstacle
    - ✓ Blade

✓ JET ON VERTICAL WALL

- Fluid flow

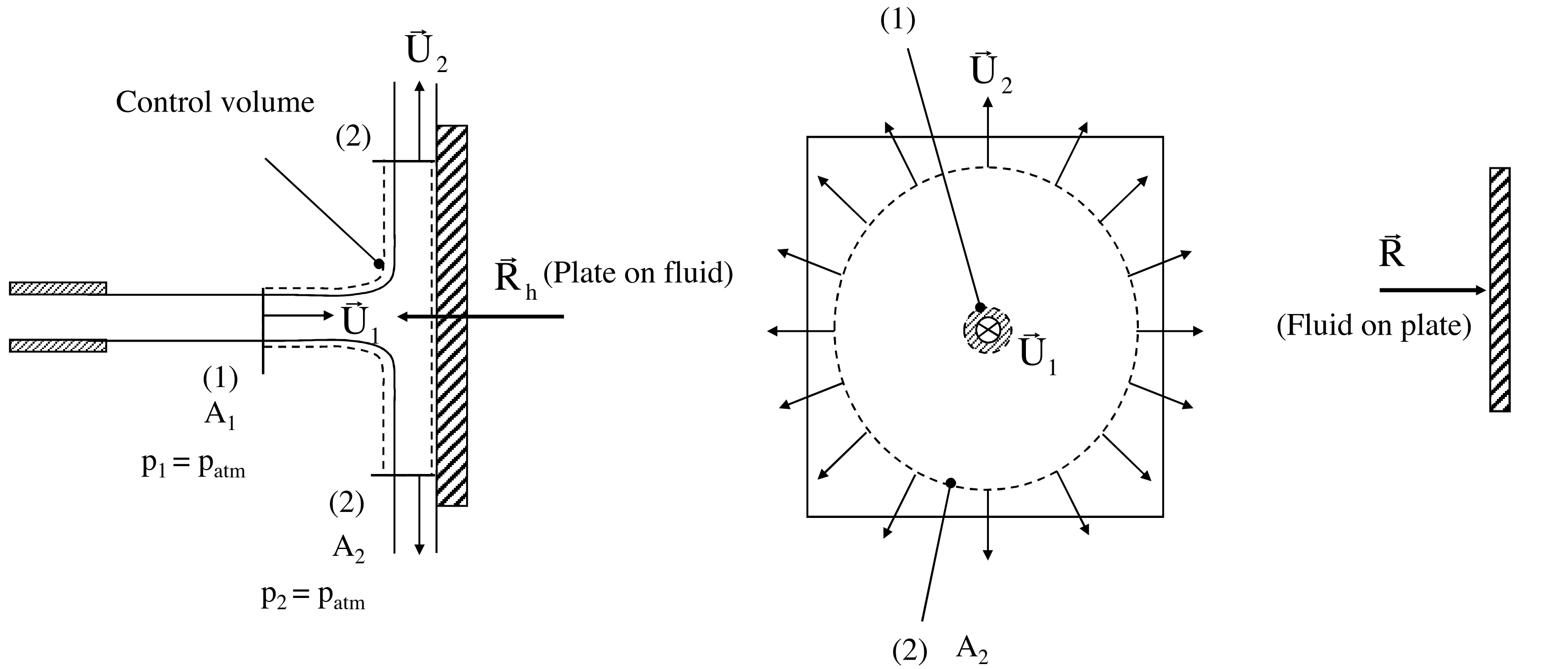


Figure 7.13 Jet on vertical wall

2.2. Forces by jets on obstacles: Static case

- ✓ JET ON VERTICAL WALL

  - Mathematical analysis

    - ✓ 1st Euler theorem: control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1 \qquad \boxed{\vec{R}_h = \dot{\vec{M}}_2 - \dot{\vec{M}}_1}$$

    - ✓ Bernoulli between sections (1) and (2)

$$z_1+U_1^2/2g + p_1/\gamma = z_2+U_2^2/2g + p_2/\gamma \;\Rightarrow\; U_1^2 = U_2^2 \;\Rightarrow\; \boxed{\left|\vec{U}_1\right| = \left|\vec{U}_2\right| = Ct.}$$

$$\dot{\vec{M}}_2 = \iint_{A_2} \vec{U}_2(\rho\vec{U}_2)d\vec{A}_2 = 0$$

$$\boxed{\vec{R} = q_m\vec{U}_1}$$

$$\boxed{R=\rho A_1 U_1^2}$$

✓ JET ON INCLINED PLANE PLATE

- Fluid flow

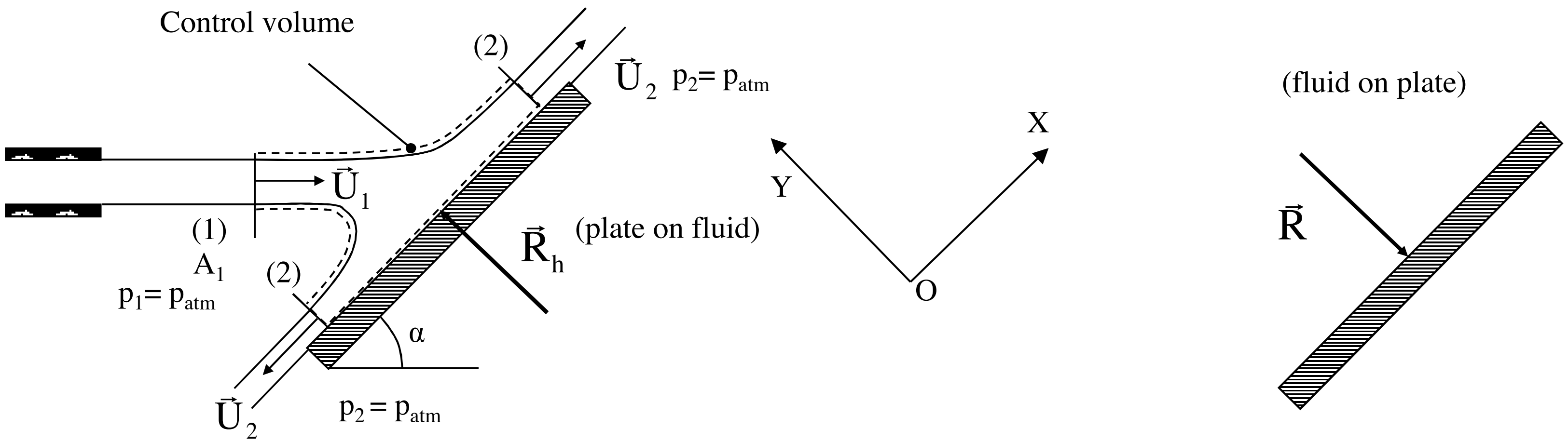


Figure 7.14 Jet on inclined plate

2.2. Forces by jets on obstacles: Static case

- ✓ JET ON INCLINED PLANE PLATE
  - Mathematical analysis (similar)
    - ✓ 1st Euler theorem: control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1$$

- ✓ Bernoulli between sections (1) and (2)

$$z_1+U_1^2/2g + p_1/\gamma = z_2+U_2^2/2g + p_2/\gamma \Rightarrow U_1^2 = U_2^2 \Rightarrow \boxed{|\vec{U}_1| = |\vec{U}_2| = Ct.}$$

<u>IN COMPONENTS</u>

$$\boxed{R_h = \dot{M}_{2y} - \dot{M}_{1y}}$$

Component OY:

$$\boxed{R_h = q_m U_1 \sin\alpha = \rho A_1 U_1^2 \sin\alpha}$$

Component OX:

$$\boxed{\dot{M}_{2x} = q_m U_1 \cos\alpha}$$

✓ JET ON SYMMETRICAL CONICAL OBSTACLE

- Fluid flow

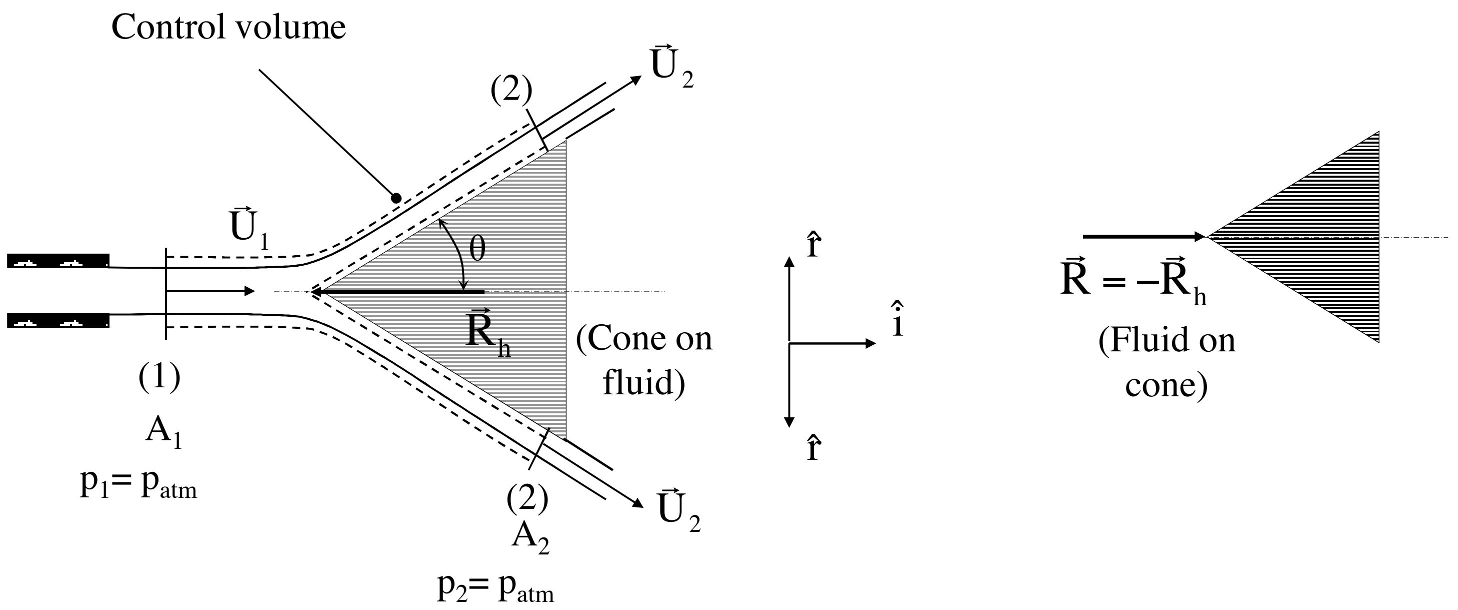


Figure 7.15 Jet on conical obstacle

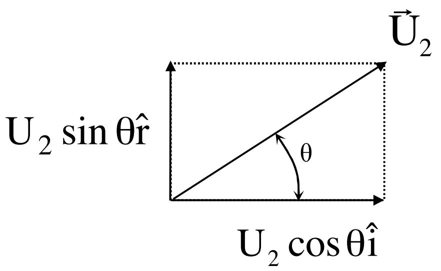
- ✓ JET ON SYMMETRICAL CONICAL OBSTACLE
  - Mathematical analysis
    - ✓ 1st Euler theorem: control volume between sections (1) and (2)

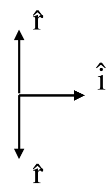
$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1$$

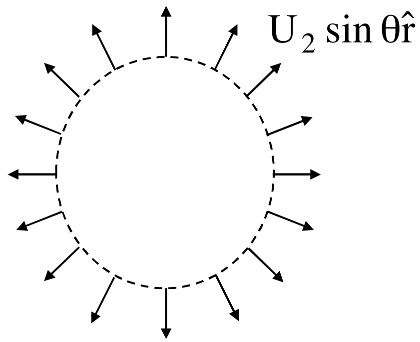
$$\boxed{\vec{R}_h = \dot{\vec{M}}_2 - \dot{\vec{M}}_1}$$

- ✓ Bernoulli between sections (1) and (2)

$$z_1 + U_1^2/2g + p_1/\gamma = z_2 + U_2^2/2g + p_2/\gamma \Rightarrow U_1^2 = U_2^2 \Rightarrow \boxed{|\vec{U}_1| = |\vec{U}_2| = \text{Cte.}}$$







$$\boxed{\dot{\vec{M}}_2 = U_2 \cos\theta \hat{i} \iint_{A_2} (\rho U_2) dA_2 = q_m \vec{U}_1 \cos\theta}$$

$$\boxed{\dot{\vec{M}}_1 = q_m \vec{U}_1}$$

$$\boxed{\vec{R} = q_m \vec{U}_1 (1 - \cos\theta)} \qquad \boxed{R = \rho A_1 U_1^2 (1 - \cos\theta)}$$

✓ JET ON A BLADE

- Fluid flow

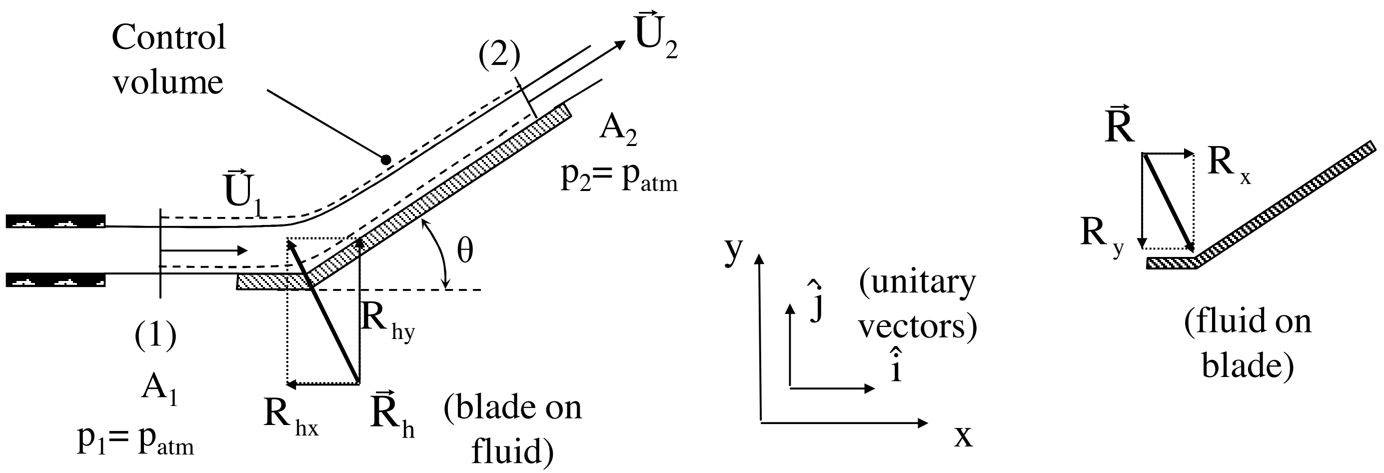


Figure 7.17 Jet on a blade

2.2. Forces by jets on obstacles: Static case

✓ JET ON A BLADE

- Mathematical analysis

✓ 1st Euler theorem: control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1$$

✓ Bernoulli between sections (1) and (2)

$z_1 + U_1^2/2g + p_1/\gamma = z_2 + U_2^2/2g + p_2/\gamma \Rightarrow U_1^2 = U_2^2 \Rightarrow \left|\vec{U}_1\right| = \left|\vec{U}_2\right| = Ct.$

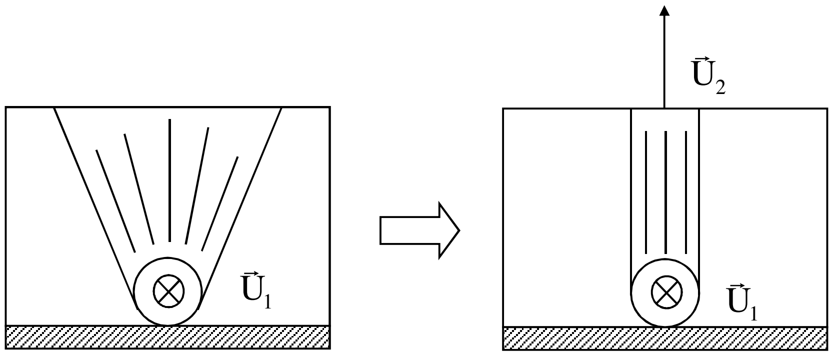


Figure 7.18 Lateral dispersion of the jet at the outlet of the blade

2.2. Forces by jets on obstacles: Static case

- JET ON A BLADE

  - Mathematical analysis

    - 1st Euler theorem: control volume between sections (1) and (2)

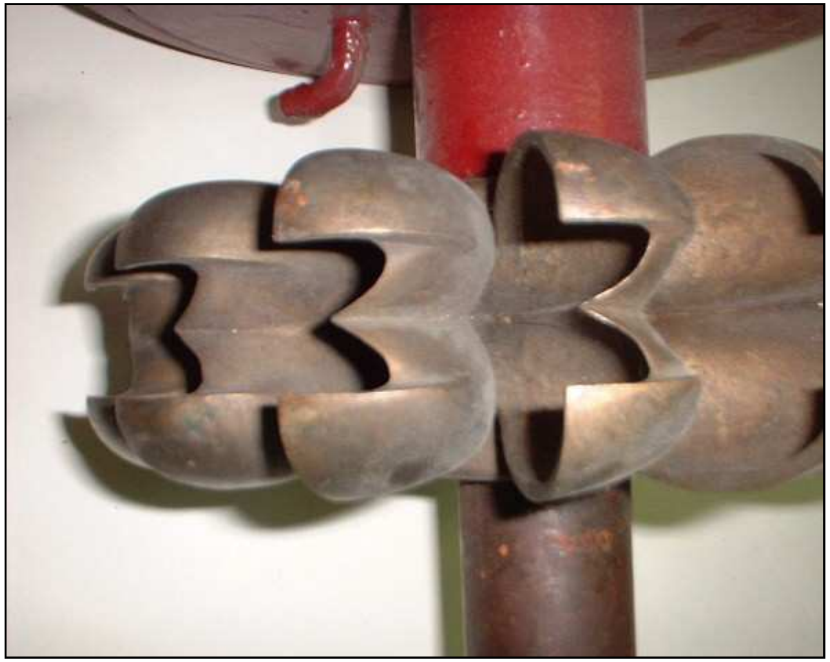
$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1 \qquad \boxed{\vec{R}_h = \dot{\vec{M}}_2 - \dot{\vec{M}}_1}$$

$$\boxed{\vec{R} = q_m U_1[(1 - \cos\theta)\hat{i} - \sin\theta\hat{j}]}$$

$$\begin{cases} \boxed{R_x = q_m U_1 (1 - \cos\theta) = \rho A_1 U_1^2 (1 - \cos\theta)} \\ \boxed{R_y = q_m U_1 \sin\theta = \rho A_1 U_1^2 \sin\theta} \end{cases}$$

2.2. Forces by jets on obstacles: Static case

- ✓ JET ON A BLADE



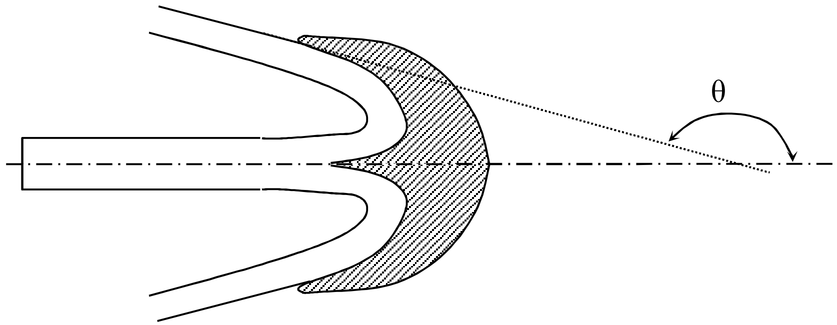


Figure 7.20 Sketch of a blade (bucket) of a Pelton turbine

$$\vec{R} = q_m U_1 [(1-\cos\theta)\hat{i} - \sin\theta \hat{j}]$$

$$\begin{cases} R_x = q_m U_1 (1-\cos\theta) = \rho A_1 U_1^2 (1-\cos\theta) \\ R_y = q_m U_1 \sin\theta = \rho A_1 U_1^2 \sin\theta \end{cases}$$

2.2. Forces by jets on obstacles: Moving case

✓ JET ON A VERTICAL WALL

- Fluid flow

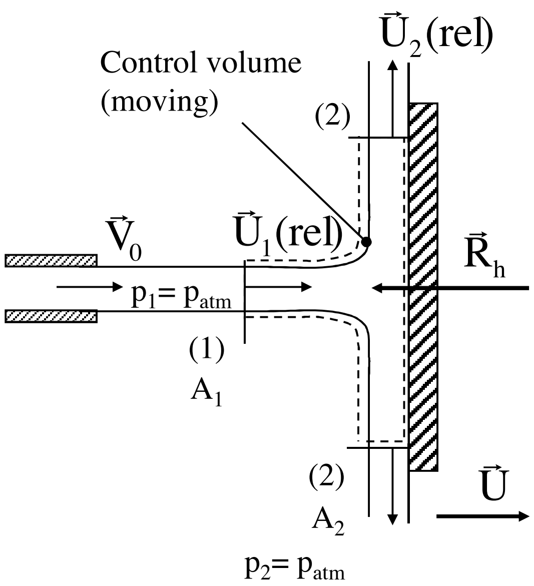
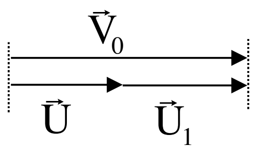


Figure 7.13 Jet on a vertical wall

$$\vec{U}_{abs} = \vec{U}_{RS} + \vec{U}_{rel}$$



$V_0 = U + U_1$

$U_1 = V_0 - U$

2.2. Forces by jets on obstacles: Moving case

✓ JET ON A VERTICAL WALL

- Mathematical analysis (similar)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1$$

$$q_m' = \rho A_1 U_1 = \rho A_1 (V_0 - U)$$

$$\vec{R} = q_m' \vec{U}_1$$

$$R = \rho A_1 (V_0 - U)^2$$

2.2. Forces by jets on obstacles: Moving case

✓ JET ON A SYMMETRICAL CONICAL OBSTACLE

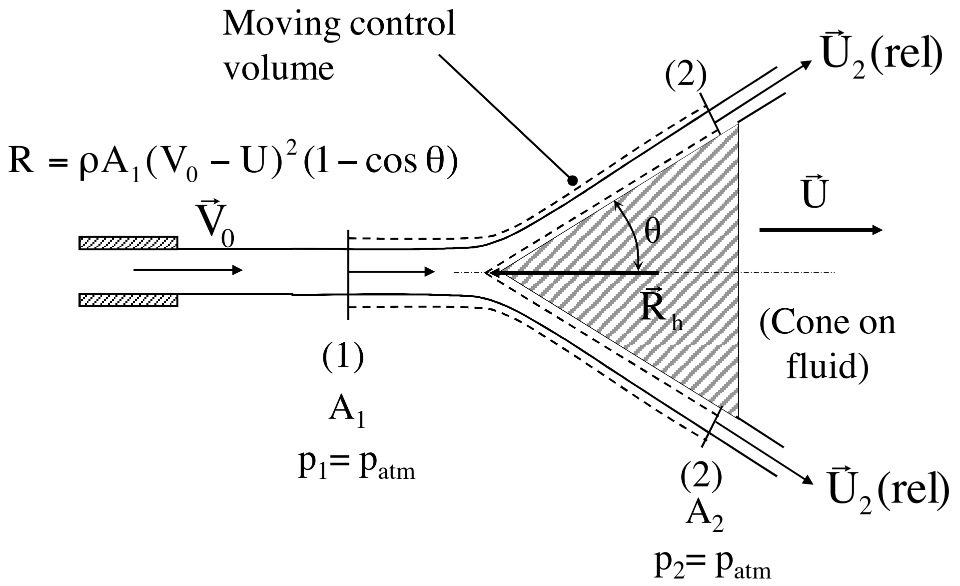


Figure 7.22 Jet on moving symmetrical conical obstacle

$$\vec{R} = q_m' \vec{U}_1 (1 - \cos\theta)$$

$$R = \rho A_1 (V_0 - U)^2 (1 - \cos\theta)$$

2.2. Forces by jets on obstacles: Moving case

- ✓ JET ON A BLADE

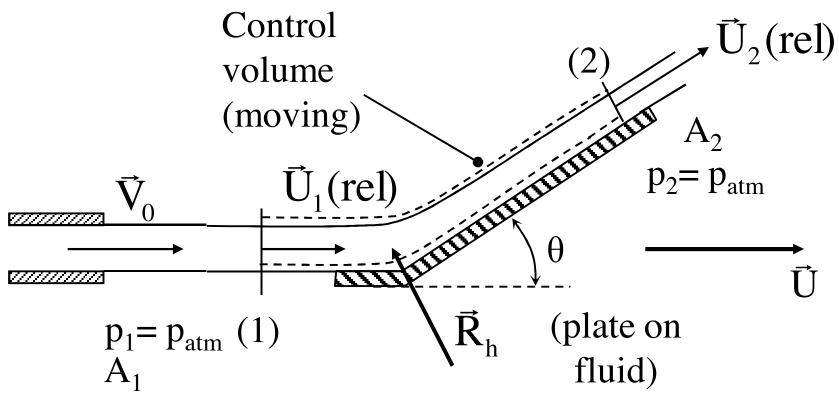


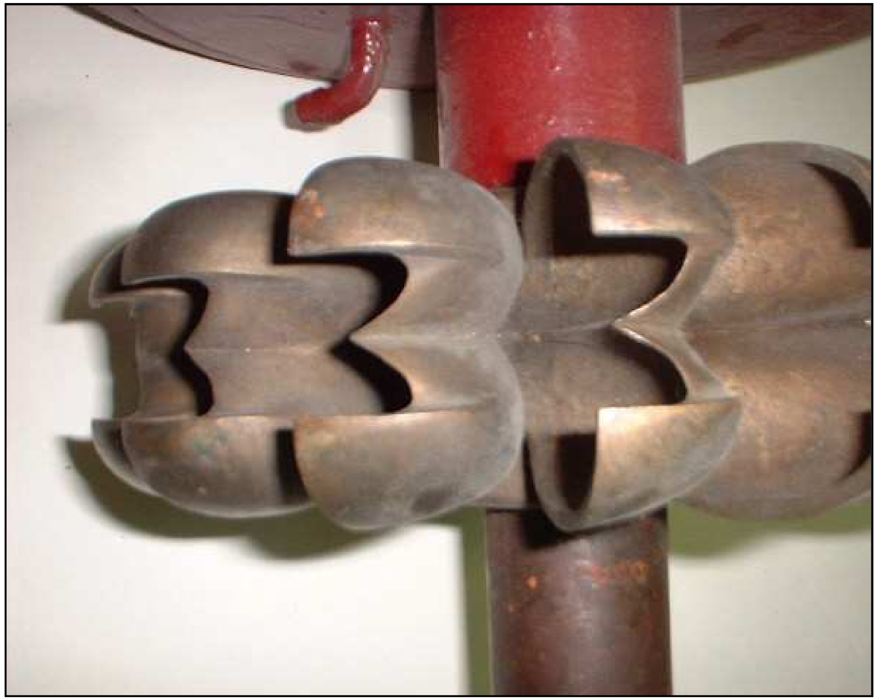
Figure 7.23 Jet on a moving blade

$$\vec{R} = q_m' U_1[(1-\cos\theta)\hat{i} - \sin\theta\hat{j}]$$

$$\begin{cases} R_x = \rho A_1 (V_0 - U)^2 (1-\cos\theta) \\ R_y = \rho A_1 (V_0 - U)^2 \sin\theta \end{cases}$$

2.2. Forces by jets on obstacles: Succession of obstacles

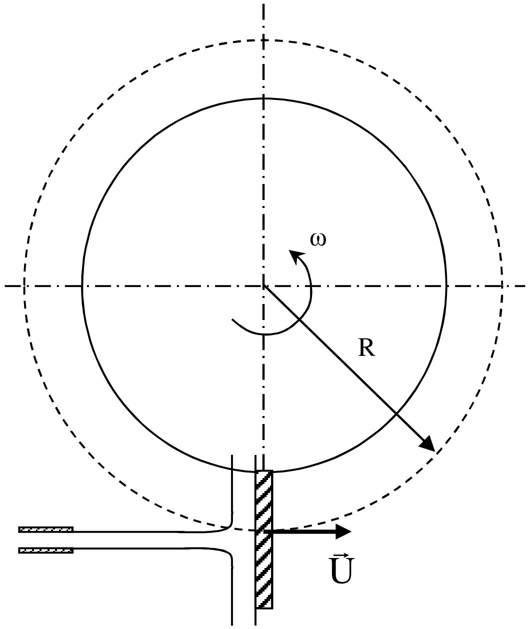
✓ JET ON A SUCCESSION OF BLADES ASSEMBLED ON A WHEEL



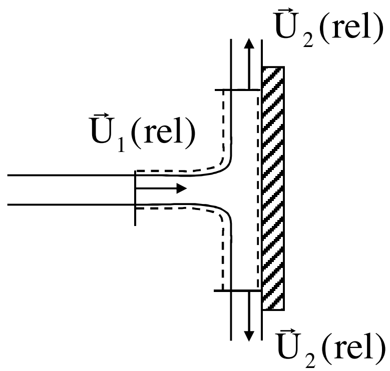
2.2. Forces by jets on obstacles: Succession of obstacles

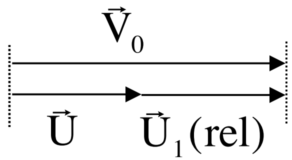
✓ JET ON A SUCCESSION OF VERTICAL PLANE PLATES ASSEMBLED ON A WHEEL

- Fluid flow



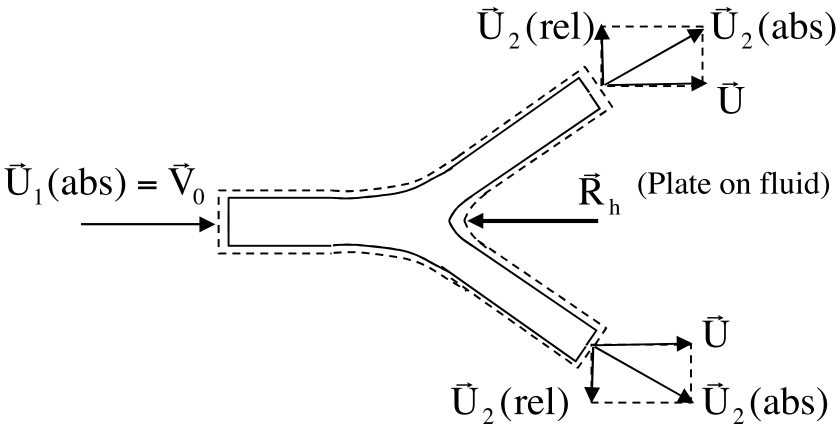
Observer accompanying the blade

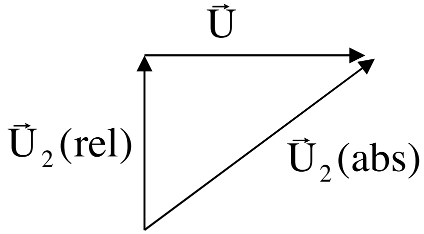




INLET

Fixed observer





OUTLET (top)

2.2. Forces by jets on obstacles: Succession of obstacles

- JET ON A SUCCESSION OF VERTICAL PLANE PLATES ASSEMBLED ON A WHEEL

  - Mathematical analysis

    - 1st Euler theorem: FIXED control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1$$

$$\boxed{\vec{R}_h = \dot{\vec{M}}_2 - \dot{\vec{M}}_1}$$

$$\boxed{\dot{\vec{M}}_1 = q_m \vec{V}_0 = q_m V_0 \hat{i}}$$

$$\boxed{\dot{\vec{M}}_2 = \iint_{A_2} \vec{U}_{2,abs} (\rho \vec{U}_{2,abs}) d\vec{A}_2 = q_m \vec{U}}$$

$$\boxed{\vec{R} = -\vec{R}_h = q_m (\vec{V}_0 - \vec{U})}$$

$$\Rightarrow \boxed{R = \underbrace{\rho A_0 V_0}_{(*)} \overbrace{(V_0 - U)}^{(*)}}$$

(*) Relative velocity

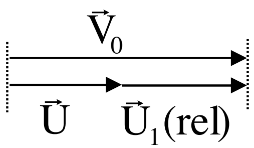
(*) Mass flow rate calculated with absolute velocity

2.2. Forces by jets on obstacles: Succession of obstacles

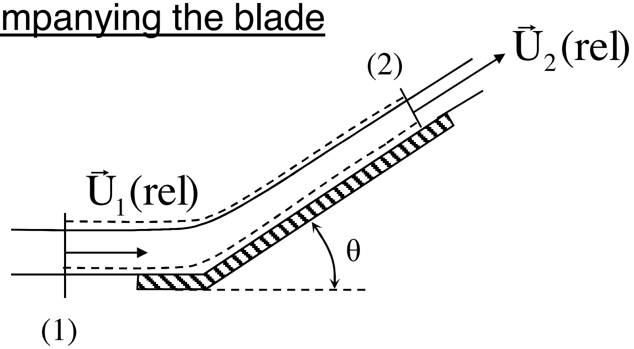
✓ JET ON A SUCCESSION OF INCLINED BLADES ASSEMBLED ON A WHEEL

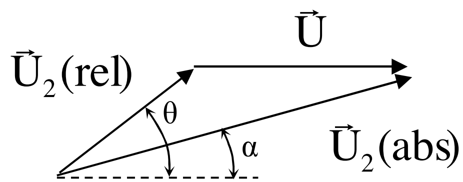
- Fluid flow

Observer accompanying the blade

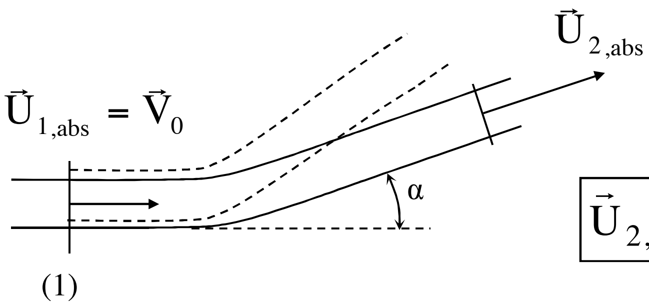


INLET





OUTLET



$$\vec{U}_{2,abs} = (U_2 \cos\theta + U)\hat{i} + U_2 \sin\theta \hat{j}$$

Fixed observer

2.2. Forces by jets on obstacles: Succession of obstacles

✓ JET ON A SUCCESSION OF INCLINED BLADES ASSEMBLED ON A WHEEL

- Mathematical analysis

✓ Bernoulli, MOVING control volume between sections (1) and (2)

$$\left|\vec{U}_{1,rel}\right| = \left|\vec{U}_{2,rel}\right|$$

✓ 1st Euler theorem: FIXED control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1$$

$$\vec{R}_h = \dot{\vec{M}}_2 - \dot{\vec{M}}_1$$

$$\dot{\vec{M}}_1 = q_m \vec{V}_0 = q_m V_0 \hat{i}$$

$$\dot{\vec{M}}_2 = \iint_{A_2} \vec{U}_{2,abs} (\rho \vec{U}_{2,abs}) d\vec{A}_2 = q_m [(U_2 \cos\theta + U)\hat{i} + U_2 \sin\theta \hat{j}]$$

$$\vec{R} = q_m (V_0 - U)[(1 - \cos\theta)\hat{i} - \sin\theta \hat{j}]$$

$$\begin{cases} R_x = \rho A_0 V_0 (V_0 - U)(1 - \cos\theta) \\ R_y = \rho A_0 V_0 (V_0 - U) \sin\theta \end{cases}$$

2.2. Forces by jets on obstacles: Succession of obstacles

✓ JET ON A SUCCESSION OF INCLINED BLADES ASSEMBLED ON A WHEEL

- Mathematical analysis

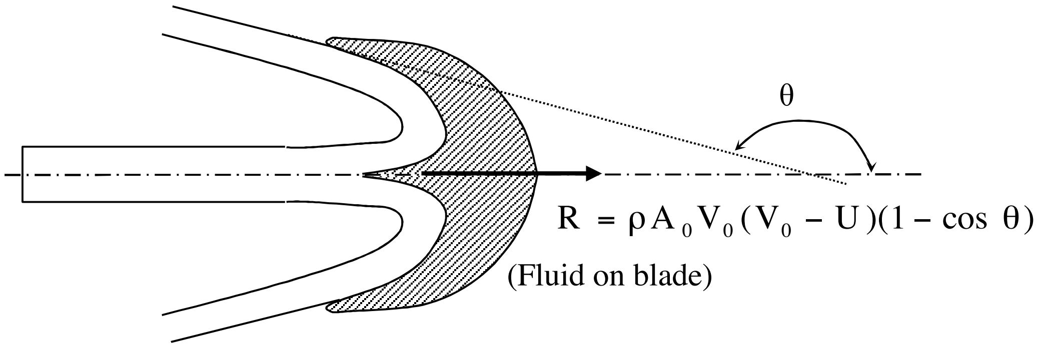


Figure 7.29 Bucket-shaped blade in a turbine of Pelton type

✓ Maximum power

$$P_{useful} = RU = \rho A_0 V_0 (V_0 - U)(1-\cos\theta)U$$

$$\frac{dP}{dU} = 0 \Rightarrow U = \frac{V_0}{2}$$

✓ Efficiency

$$\eta = \frac{P_{useful}}{P_{jet}} = \frac{RU}{\frac{1}{2} q_m V_0^2} = \frac{2(V_0 - U)(1-\cos\theta)U}{V_0^2}$$

This value is maximized with $U=V_0/2$ and $\theta=180°$ where $\eta=1$

2. Applications of the linear momentum theorem

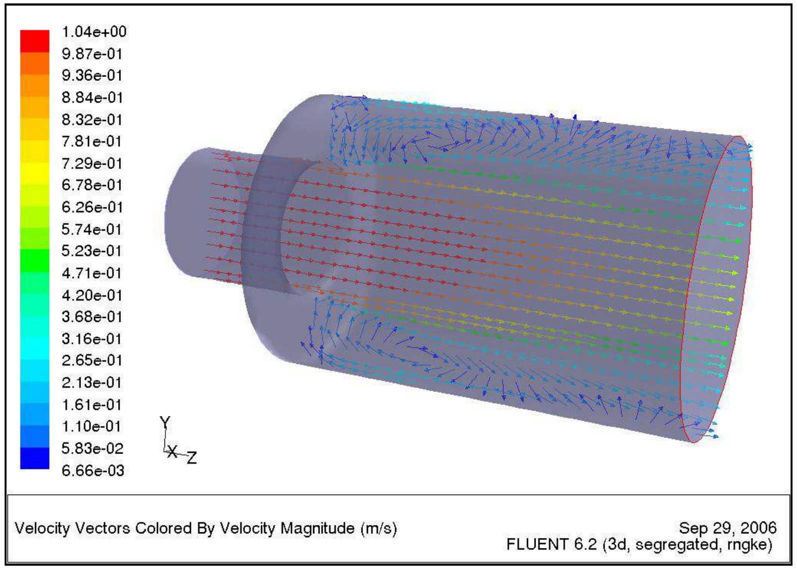
2.3. Applications of the linear momentum theorem III:
Other applications

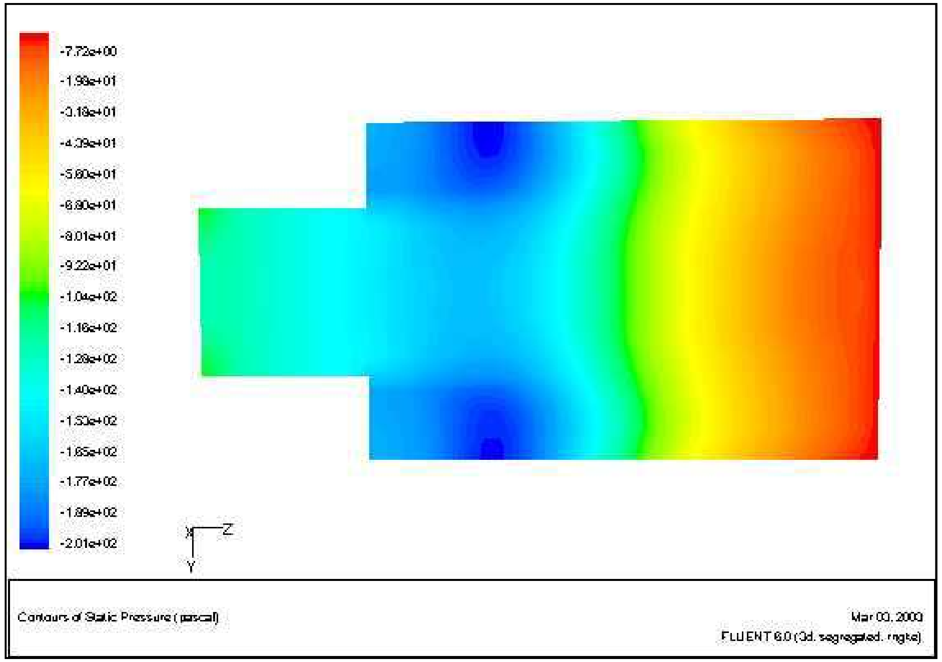
1. Sudden expansion (Borda-Carnot)

2. Hydraulic jump

2.3. Sudden expansion (Borda-Carnot)

✓ Concept





2.3. Sudden expansion (Borda-Carnot)

- ✓ Fluid flow

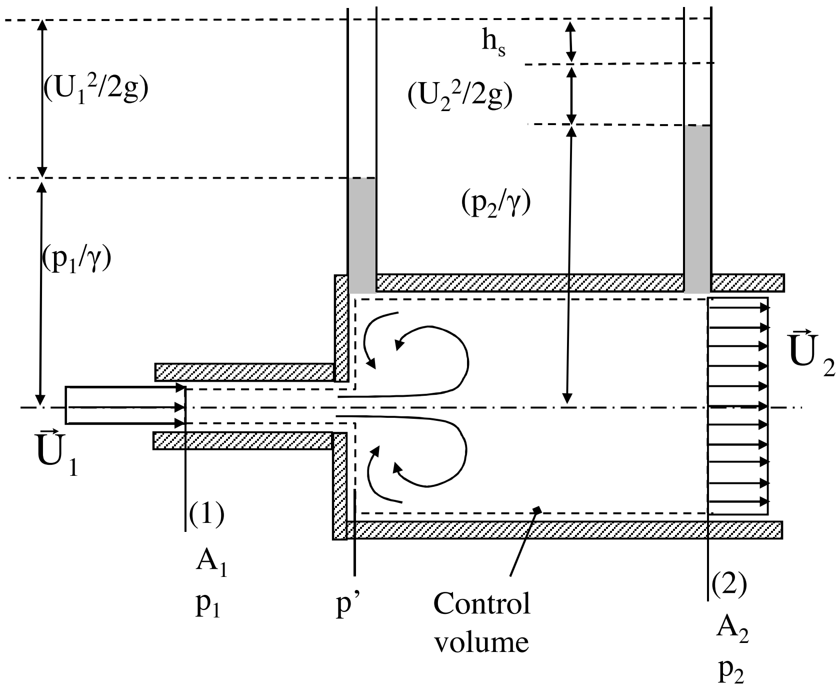


Figure 7.30 Energy losses in a sudden expansion

2.3. Sudden expansion (Borda-Carnot)

✓ Mathematical analysis

- 1st Euler theorem: control volume between sections (1) and (2)

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1 = q_m(\vec{U}_2 - \vec{U}_1)$$

$$(p_2 - p_1) = \frac{q_m}{A_2}(U_1 - U_2)$$

- Energy equation: between (1) and (2)

$$z_1 + \frac{p_1}{\gamma} + \frac{U_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{U_2^2}{2g} + h_s + h_f$$

$$h_s = \frac{(U_1 - U_2)^2}{2g}$$

- Continuity equation

$$q_m = \rho A_1 U_1 = \rho A_2 U_2$$

$$h_s = \frac{U_1^2}{2g}\left[1 - \beta^2\right]^2 = K_1 \frac{U_1^2}{2g}$$

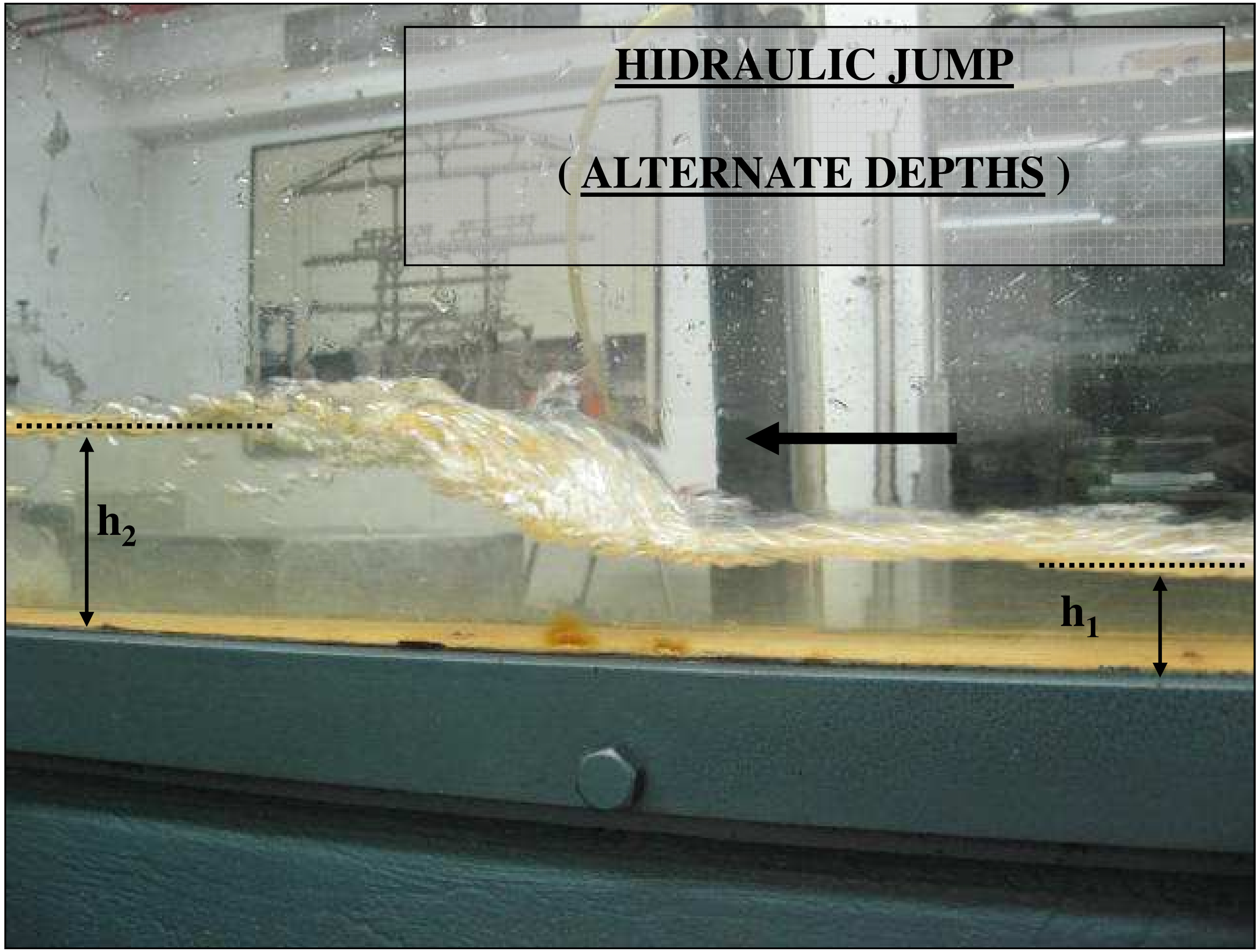
$$\beta = \frac{D_1}{D_2}$$

$$h_s = K_2 \frac{U_2^2}{2g}$$

$$K_2 = \frac{K_1}{\beta^4} = \left[\frac{1 - \beta^2}{\beta^2}\right]^2$$

2.3. Hydraulic jump

- ✓ Concept



- ✓ Fluid flow

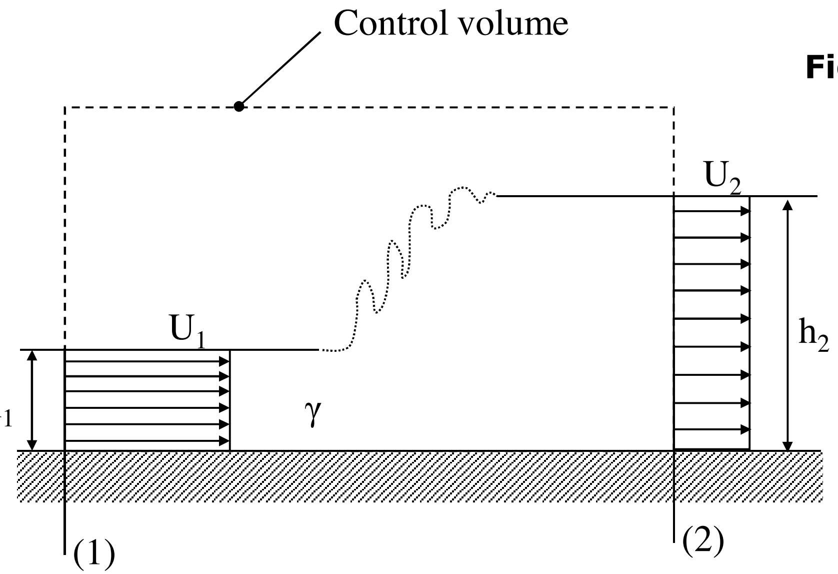
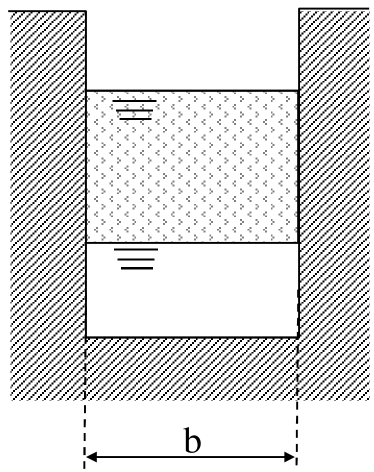


Figure 7.34 Sketch of a hydraulic jump



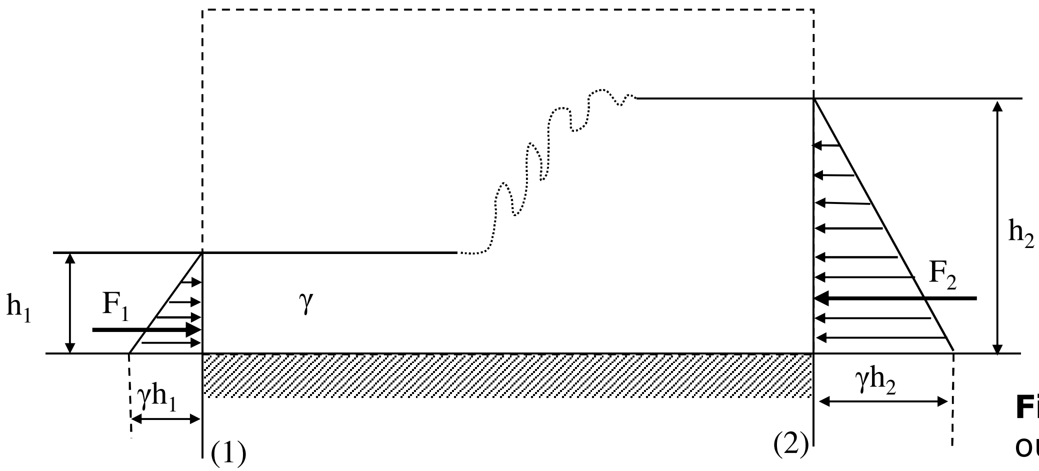


Figure 7.35 Pressures at the inlet and outlet sections of the control volume

2.3. Hydraulic jump

✓ Mathematical analysis: relationship between depths

- 1st Euler theorem

$$\vec{P} + \vec{G} = \dot{\vec{M}}_2 - \dot{\vec{M}}_1 = q_m(\vec{U}_2 - \vec{U}_1)$$

$$\frac{1}{2}\gamma h_1^2 b - \frac{1}{2}\gamma h_2^2 b = q_m(U_2 - U_1) = \rho(h_1 b)U_1(U_2 - U_1)$$

- Continuity equation

$$q_m = \rho(h_1 b)U_1 = \rho(h_2 b)U_2$$

$$U_2 = \frac{h_1}{h_2}U_1$$

$$\frac{1}{2}g(h_1^2 - h_2^2) = h_1 U_1^2\left(\frac{h_1}{h_2} - 1\right) \quad (*)$$

$$h_2^2 + (h_1)h_2 - \frac{2h_1U_1^2}{g} = 0$$

$$h_2 = \frac{h_1}{2}\left[\sqrt{1 + \frac{8U_1^2}{gh_1}} - 1\right]$$

Relationship between "alternate depths"

2.3. Hydraulic jump

✓ Mathematical analysis: Minor energy loss

- Energy equation: between surface points (1) and (2)

$$h_1 + \frac{p_1}{\gamma} + \frac{U_1^2}{2g} = h_2 + \frac{p_2}{\gamma} + \frac{U_2^2}{2g} + h_s \qquad \frac{p_1}{\gamma} = \frac{p_2}{\gamma} = 0$$

(atmospheric pressure)

- Continuity equation

$$U_2 = \frac{h_1}{h_2} U_1 \;\Rightarrow\; h_s = h_1 - h_2 + \frac{U_1^2 - U_2^2}{2g} = h_1 - h_2 + \frac{U_1^2}{2g}\left(1 - \frac{h_1^2}{h_2^2}\right)$$

(*) $$\frac{1}{4}(h_1 + h_2)\frac{h_2}{h_1} = \frac{U_1^2}{2g} \;\Rightarrow\; h_s = \frac{1}{4h_1h_2}(h_2^3 + 3h_1^2h_2 - 3h_1h_2^2 - h_1^3)$$

$$h_s = \frac{(h_2 - h_1)^3}{4h_1h_2}$$

"Minor energy loss" or "energy loss due to turbulent dissipation in the wave" as a function of the alternate depths

2.3. Hydraulic jump

✓ Graphical analysis

- 1st Euler theorem

$$\frac{1}{2}\gamma h_1^2 + \frac{\rho q^2}{h_1} = \frac{1}{2}\gamma h_2^2 + \frac{\rho q^2}{h_2}$$

$$q = \frac{Q}{b}$$

$$F + \dot{M} = \frac{1}{2}\gamma h^2 + \frac{\rho q^2}{h} = Cte.$$

"Thrust function"

$$\{F + \dot{M}\} = f(h)$$

- Energy equation

Section (1)

$$E_1 = h_1 + \frac{q^2}{2gh_1^2}$$

Section (2)

$$E_2 = h_2 + \frac{q^2}{2gh_2^2}$$

$$E = h + \frac{q^2}{2gh^2}$$

E=f(h)

$$E_1 - E_2 = h_s$$

- Graphical analysis

$$h_C = \left(\frac{q^2}{g}\right)^{1/3}$$

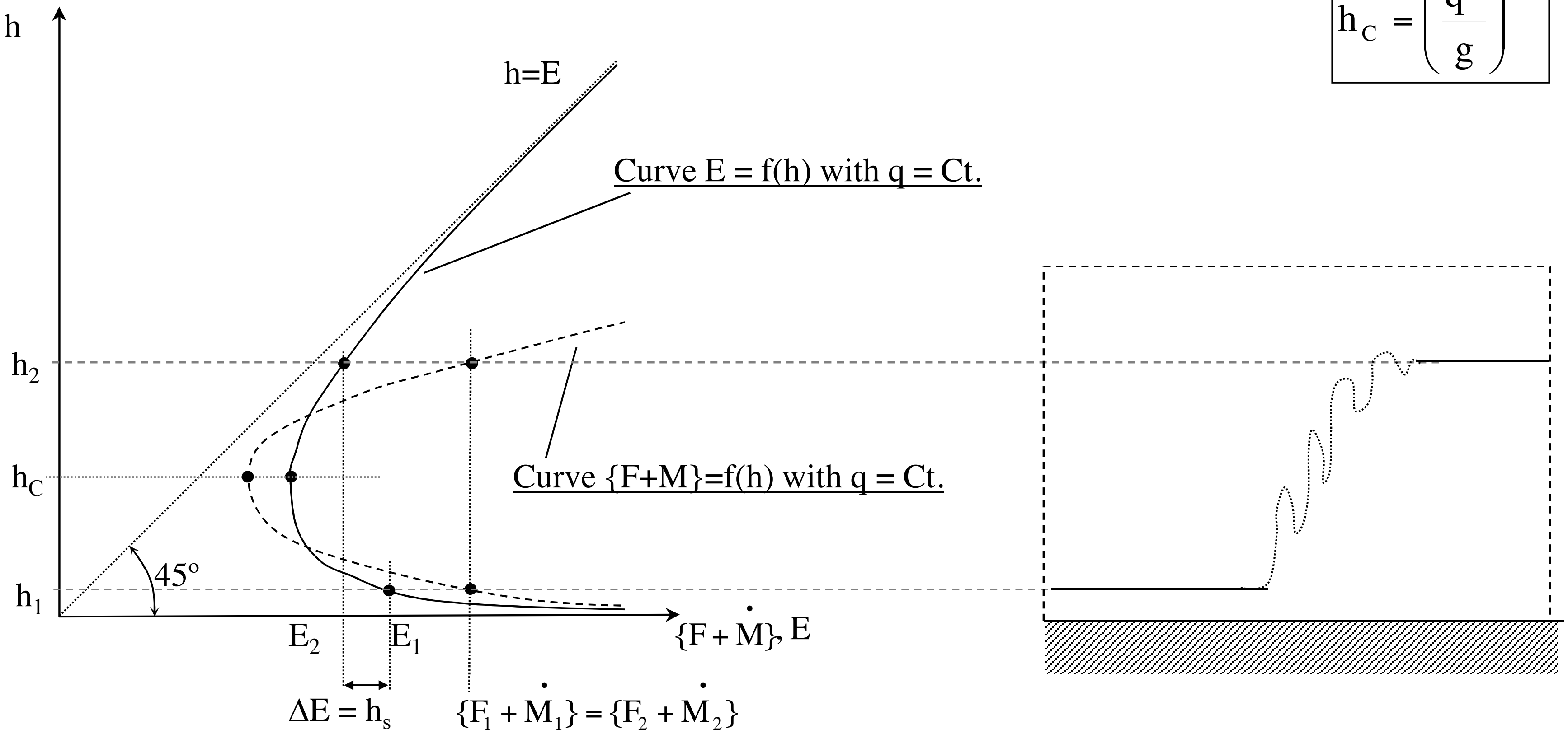


Figure 7.36 Graphical analysis of hydraulic jump