CHAPTER 6. INVISCID COMPRESSIBLE FLOW

1. Bernoulli equation for gases. Saint-Venant equation
2. Different behaviour of compressible fluid and incompressible fluid in isentropic flow
3. Concepts: generator state, stagnation state, critical state, limit velocity
4. Hugoniot’s theorems. Application to the design of nozzles and diffusers

1. Bernoulli equation for gases: Saint - Venant

1. Isentropic flow. Saint-Venant equation
2. Speed of sound
3. Saint-Venant equation, dimensionless form

1.1. Isentropic flow. Saint-Venant equation

- ✓ Hypotheses
- ✓ Differential energy equation of the steady-state flow of perfect gases

- Bernoulli: $$\int_1^2 \frac{dp}{\rho} + \frac{1}{2}(U_2^{\,2} - U_1^{\,2}) + g(z_2 - z_1) = 0$$

"**Bernoulli equation in a compressible fluid**" (steady-state regime) valid for a streamline.

$$\frac{U^2}{2} + gz + \int \frac{dp}{\rho} = Cte \quad \Rightarrow \quad \rho U dU + dp = 0$$

- ✓ Saint – Venant equation:

$$\left.\begin{array}{l} \rho U dU + dp = 0 \\ p = C\rho^k \end{array}\right\} \Rightarrow \frac{U^2}{2g} + \frac{k}{k-1}\frac{p}{\gamma} = Cte$$

Saint-Venant equation (1st form)

$$\frac{U^2}{2g} + \frac{k}{k-1} R'T = Cte$$

Saint-Venant equation (2nd form)

$$\frac{U^2}{2g} + c_p' T = Cte$$

Saint-Venant equation (3rd form)

$$c_p'(m/K) = \frac{c_p(J/k\cdot mol))}{(A\cdot 10^{-3})(kg/mol)\cdot g(m/s^2)}$$

1.1. Isentropic flow. Saint-Venant equation

✓ Saint – Venant equation, between 2 points

$$\frac{U^2}{2g} + \frac{k}{k-1}\frac{p}{\gamma} = Cte$$

$$p = C\rho^k$$

⇨

$$\frac{U_1^2 - U_2^2}{2g} = \frac{k}{k-1}\frac{p_1}{\gamma_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

- Alternative expressions:

$$\frac{U_1^2 - U_2^2}{2g} = \frac{k}{k-1}\frac{p_1}{\gamma_1}\left[\left(\frac{\gamma_2}{\gamma_1}\right)^{k-1} - 1\right]$$

$$\frac{U_1^2 - U_2^2}{2g} = \frac{k}{k-1}R'(T_2 - T_1)$$

$$\frac{U_1^2 - U_2^2}{2g} = c_p'(T_2 - T_1)$$

1.2. Speed of sound

- ✓ Definition
- ✓ Newton's formula for the speed of sound: $c^2 = \frac{dp}{d\rho} = \frac{E_v}{\rho}$
- ✓ Example

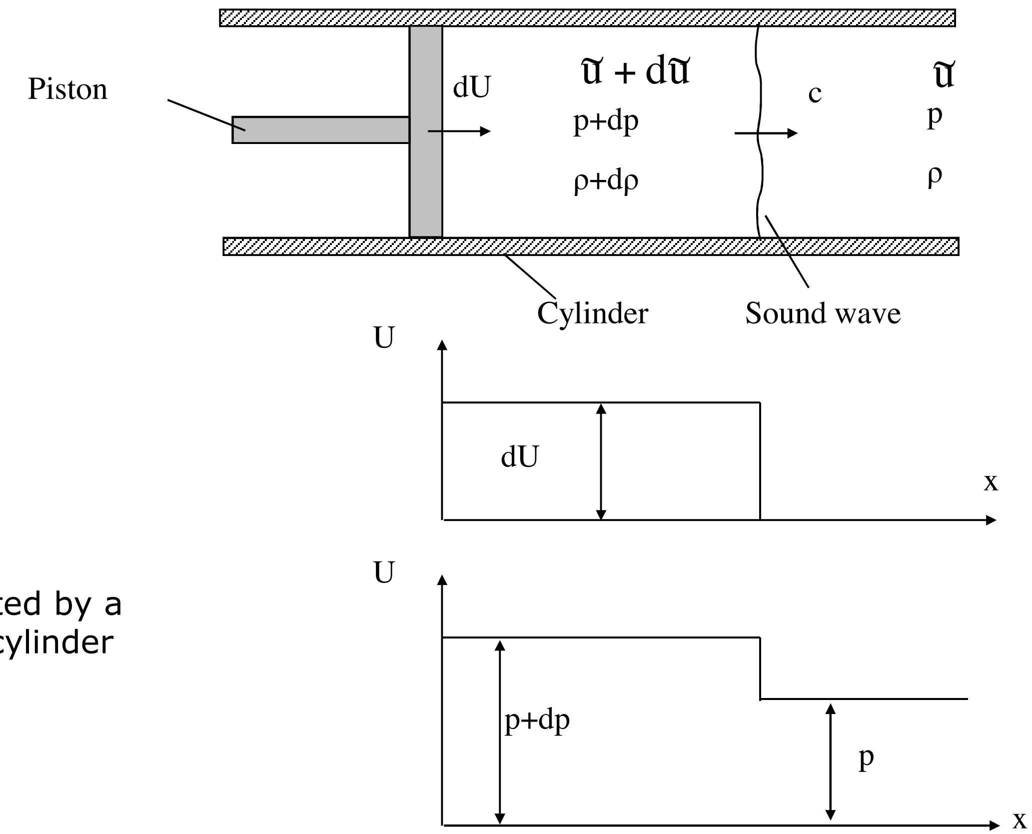


Figure 6.1 Disturbance generated by a piston on a static fluid inside a cylinder

1.2. Speed of sound

✓ Mach number:

$$M = \frac{U}{c}$$

- Flow regime classification:

  - Subsonic regime: $M < 1$, $U < c$
  - Sonic regime or critical regime: $M = 1$, $U = c$
  - Supersonic regime: $M > 1$, $U > c$

1.3. Saint-Venant equation, dimensionless form

- ✓ Saint – Venant equation, as a function of Mach No.

$$\frac{U_1^2 - U_2^2}{2g} = \frac{k}{k-1}\frac{p_1}{\gamma_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

$$\frac{1}{2}\frac{U^2}{c^2} = \frac{1}{k-1}\left[\left(\frac{p_S}{p}\right)^{\frac{k-1}{k}} - 1\right]$$

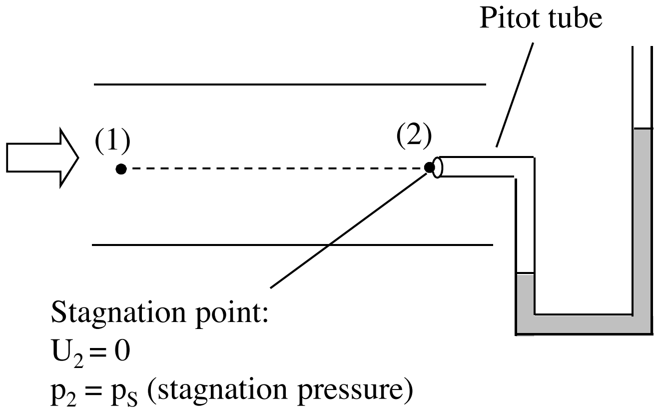


Figure 6.3 Pitot tube in a fluid flow

$$\frac{p_S}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{\frac{k}{k-1}}$$

"Saint-Venant equation as a function of the Mach No."

1.3. Saint-Venant equation, dimensionless form

$$\frac{p_S}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}}$$

NEWTON BINOMIAL:

$$(a+x)^\alpha = a^\alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} \left(\frac{x}{a}\right)^n = a^\alpha \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} \left(\frac{x}{a}\right)^n$$

$$\binom{\alpha}{n} = \frac{\alpha!}{(\alpha-n)!n!}$$

$$\alpha = \frac{k}{k-1} \qquad a = 1 \qquad x = \frac{k-1}{2} M^2$$

$$\Rightarrow \quad \frac{p_S}{p} = 1 + \frac{k}{2} M^2 + \frac{k}{8} M^4 + \frac{k(2-k)}{48} M^6 + \ldots$$

$$\Rightarrow \quad \frac{p_S - p}{\frac{1}{2} \rho U^2} = 1 + \frac{M^2}{4} + \frac{2-k}{24} M^4 + \ldots$$

“Saint – Venant equation, dimensionless form”

2. Different behaviour of compressible fluid and incompressible fluid in isentropic flow

$$\frac{p_S - p}{\frac{1}{2}\rho U^2} = 1 + \frac{M^2}{4} + \frac{2-k}{24}M^4 + \dots$$

Saint – Venant equation, in dimensionless form

$$\frac{p_S - p}{\frac{1}{2}\rho U^2} = 1$$

Bernoulli equation, in dimensionless form

M	$\frac{p_S - p}{\frac{1}{2}\rho U^2}$	
0	1	Incompressible case
0,1	1,003	Equivalent to incompressible
0,2	1,010	
0,3	1,023	Compressible case
0,4	1,041	

Conclusion: a gas can be considered as incompressible fluid for a Mach number of $M<0.2$ by admitting an error smaller than 1% in the dimensionless form of the Saint-Venant equation

3. Concepts: generator state, stagnation state, critical state, limit velocity

- ✓ Generator state (0)
- ✓ Stagnation state (S)

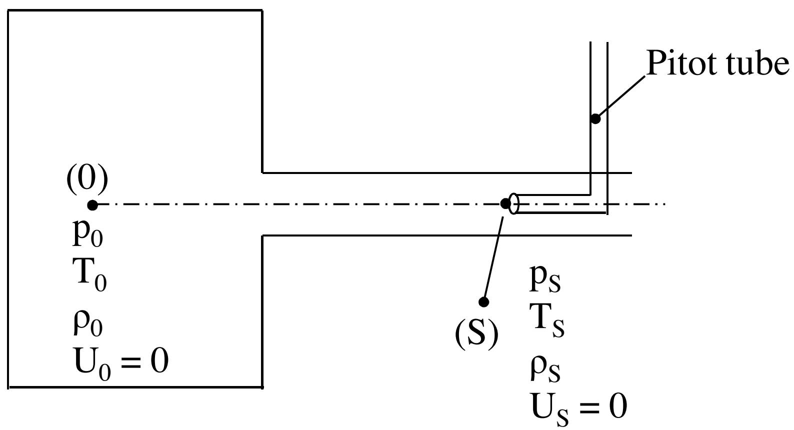


Figure 6.4 Generator state and stagnation point

3. Concepts: generator state, stagnation state, critical state, limit velocity

- ✓ Perfect fluid, isentropic flow

Stagnation state (S) = Generator state (0)

$$\left\{ \begin{array}{l} p_0 = p_S \\ \gamma_0 = \gamma_S \\ T_0 = T_S \end{array} \right\}$$

- ✓ Demonstration

  - Saint – Venant (0)-(S)
  - Ideal gas

  ⇒ $T_0 = T_S$

  - Isentropic flow
  - Ideal gas

  ⇒ $\gamma_0 = \gamma_S$ $\quad$ $p_0 = p_S$

3. Concepts: generator state, stagnation state, critical state, limit velocity

✓ Generator state, Mach functions

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}}$$

$$\frac{\gamma_0}{\gamma} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}}$$

$$\frac{T_0}{T} = \left[1 + \frac{k-1}{2} M^2\right]$$

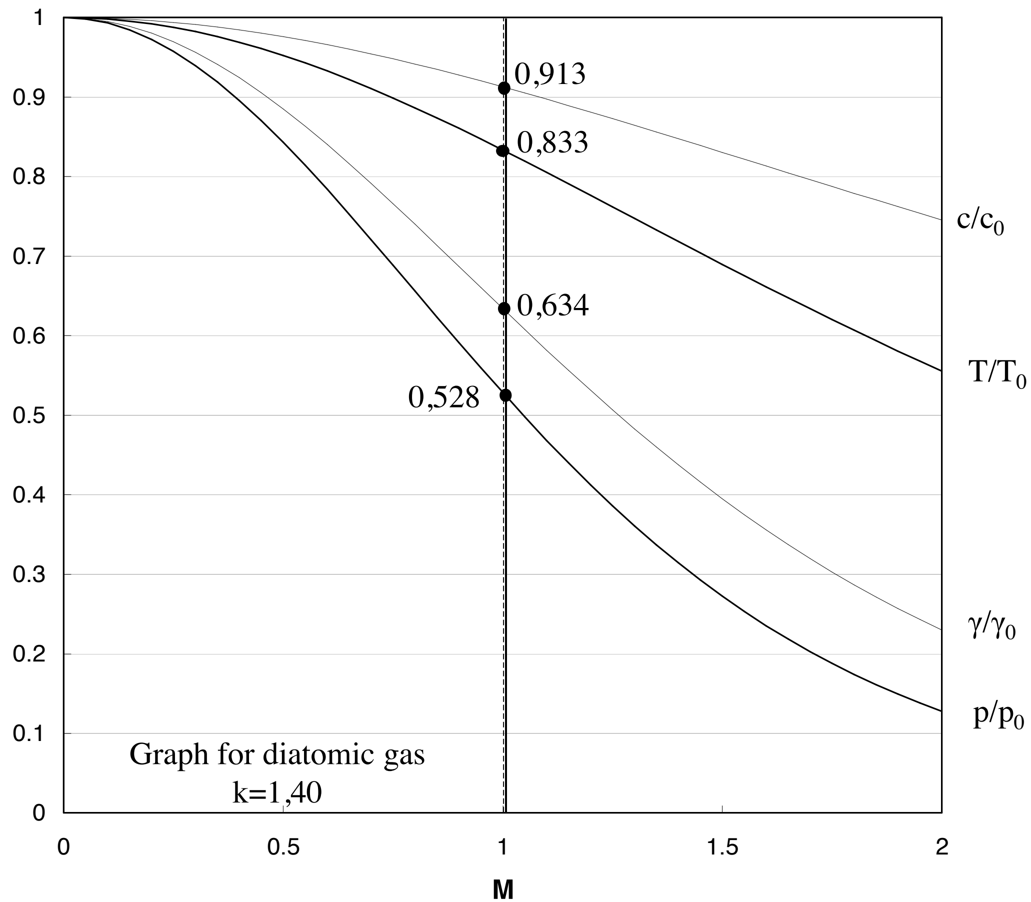


Figure 6.5 Characteristic variables of the flow with respect to the generator state, as a function of Mach no.

$$\left[1 + \frac{k-1}{2} M^2\right] = \frac{T_0}{T} = \left(\frac{p_0}{p}\right)^{\frac{k-1}{k}} = \left(\frac{\gamma_0}{\gamma}\right)^{k-1}$$

3. Concepts: generator state, stagnation state, critical state, limit velocity

- ✓ Critical state

Diatomic gas (k=1,4)

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}} \quad \Rightarrow \quad \frac{p_0}{p_C} = \left[\frac{k+1}{2}\right]^{\frac{k}{k-1}} \quad \Rightarrow \quad \frac{T_C}{T_0} = 0{,}833$$

$$\frac{\gamma_0}{\gamma} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}} \quad \Rightarrow \quad \frac{\gamma_0}{\gamma_C} = \left[\frac{k+1}{2}\right]^{\frac{1}{k-1}} \quad \Rightarrow \quad \frac{\gamma_C}{\gamma_0} = 0{,}634$$

$$\frac{T_0}{T} = \left[1 + \frac{k-1}{2} M^2\right] \quad \Rightarrow \quad \frac{T_0}{T_C} = \left[\frac{k+1}{2}\right] \quad \Rightarrow \quad \frac{p_C}{p_0} = 0{,}528$$

3. Concepts: generator state, stagnation state, critical state, limit velocity

- ✓ Limit velocity

  - Saint – Venant (0)-(lim)

$$U_{lim} = c_0 \sqrt{\frac{2}{k-1}}$$

3. Experimental characterization of a gas flow

- Variables

1- Mach no.: “M”
2- Temperature: “T”
3- Speed of sound: “c”
4- Fluid velocity: “U”
5- Fluid density: “ρ”
6- Mass flow rate: “q_m”

- Measurements

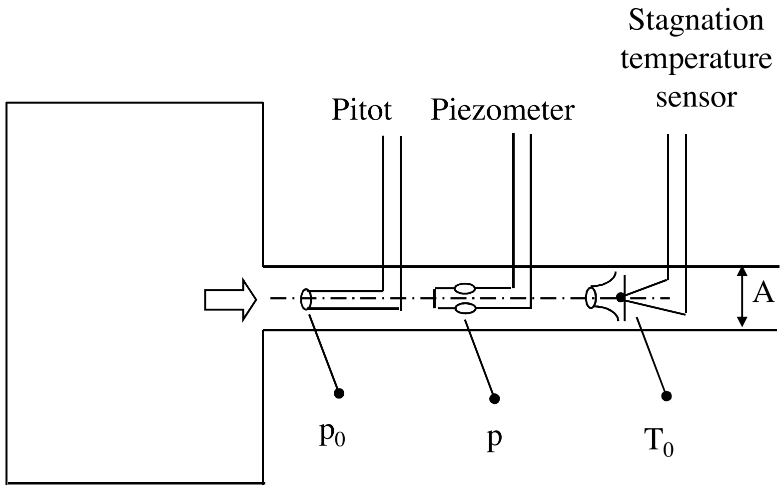


Figure 6.6 Characterization of a gas flow

3. Experimental characterization of a gas flow

✓ Scheme of calculation

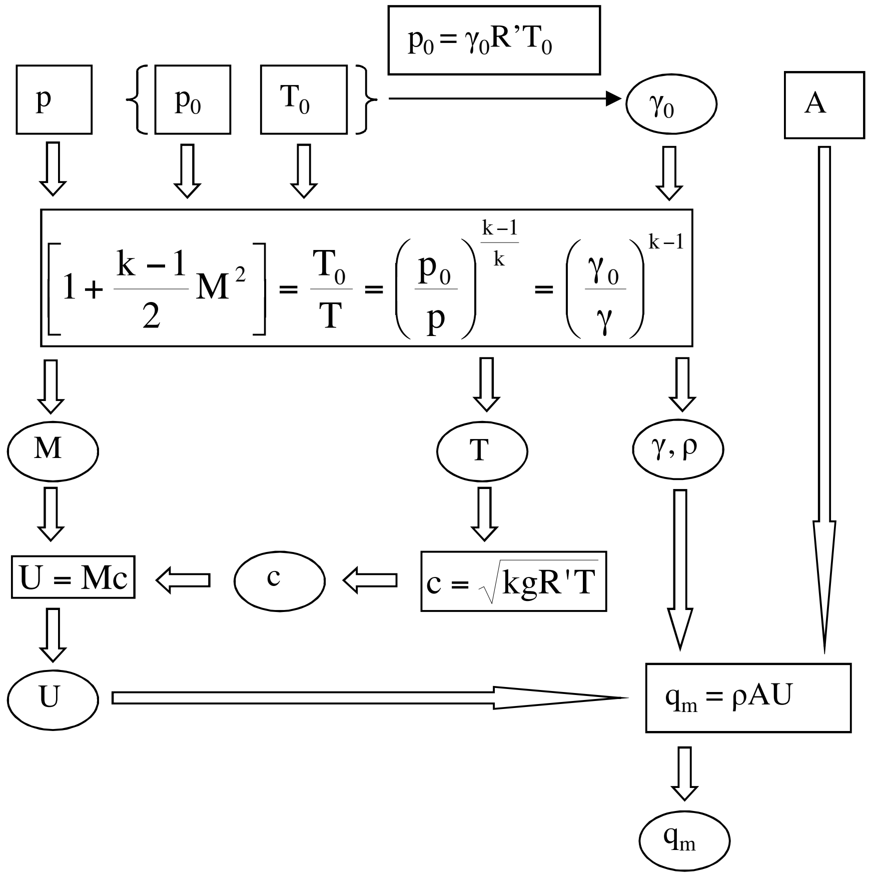


Figure 6.7 Scheme of calculation in a standard problem with a gas flow

4. Hugoniot's theorems

- ✓ Formulation of the theorems (4)
- ✓ Hugoniot's equations (2)

$$\frac{dA}{A} = \frac{dU}{U}(M^2 - 1)$$

$$\frac{dU}{U} = -\frac{1}{kM^2}\frac{dp}{p}$$

- ✓ Proof of the equations

  - Continuity equation: $q_m = Cte = \rho AU$ ⇒ $0 = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dU}{U}$
  - Energy equation: $\rho U dU + dp = 0$
  - Newton's formula for the speed of sound: $c^2 = \frac{dp}{d\rho}$

⇒ $$\frac{dA}{A} = \frac{dU}{U}(M^2 - 1)$$

EQUATION 1

4. Hugoniot's theorems

✓ Proof of the equations

- Energy equation $\rho U dU + dp = 0$
- Newton's formula for the speed of sound $c^2 = \frac{kp}{\rho}$

$$\Rightarrow \frac{dU}{U} = -\frac{1}{kM^2}\frac{dp}{p}$$

EQUATION 2

4. Hugoniot's theorems

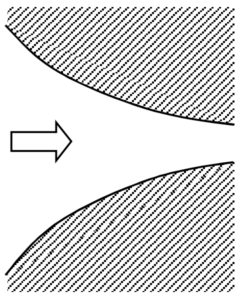
- ✓ Physical interpretation of the equations

$$\frac{dA}{A} = \frac{dU}{U}(M^2 - 1)$$

EQUATION 1

- Theorem 1: Subsonic regime: M < 1

$(M^2\text{-}1)$	dU	dA	
(-)	(+)	(-)	CASE A
(-)	(-)	(+)	CASE B

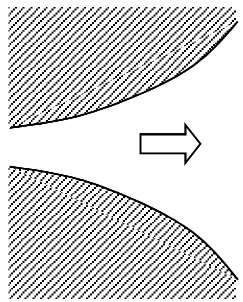


$dA < 0\ ,\ dU > 0$

Velocity increase
Cross section decrease

$dA > 0\ ,\ dU < 0$

Velocity decrease
Cross section increase



CASE A: Convergent nozzle

CASO B: Divergent diffuser

4. Hugoniot's theorems

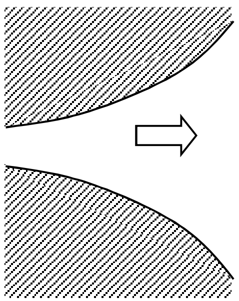
✓ Physical interpretation of the equations

$$\frac{dA}{A} = \frac{dU}{U}(M^2 - 1)$$

EQUATION 1

- Theorem 2: Supersonic regime: M > 1

$(M^2\text{-}1)$	dU	dA	
(+)	(+)	(+)	CASE C
(+)	(-)	(-)	CASE D

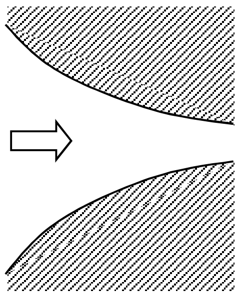


$dA > 0\ ,\ dU > 0$

Velocity increase
Cross section increase

$dA < 0\ ,\ dU < 0$

Velocity decrease
Cross section decrease



CASE C: Divergent nozzle

CASE D: Convergent diffuser

4. Hugoniot's theorems

- ✓ Interpretation of the equations

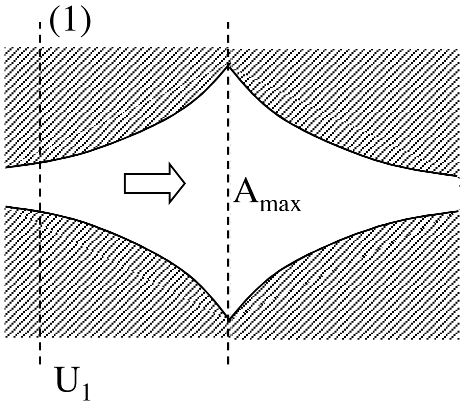
$$\frac{dA}{A} = \frac{dU}{U}(M^2 - 1)$$

EQUATION 1

- Theorem 3: Sonic regime: M = 1

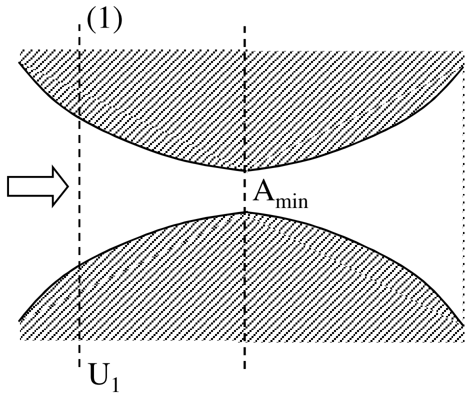
$$\frac{dA}{A} = 0$$

<u>Case:</u> A_{max}



IMPOSSIBLE

<u>Case:</u> A_{min}



POSSIBLE

4. Hugoniot's theorems

✓ Physical interpretation of the equations

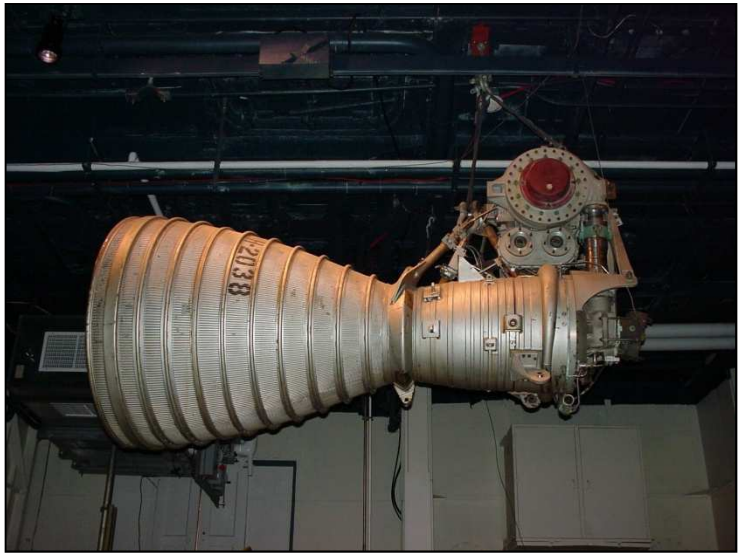
$$\frac{dU}{U} = -\frac{1}{kM^2}\frac{dp}{p}$$

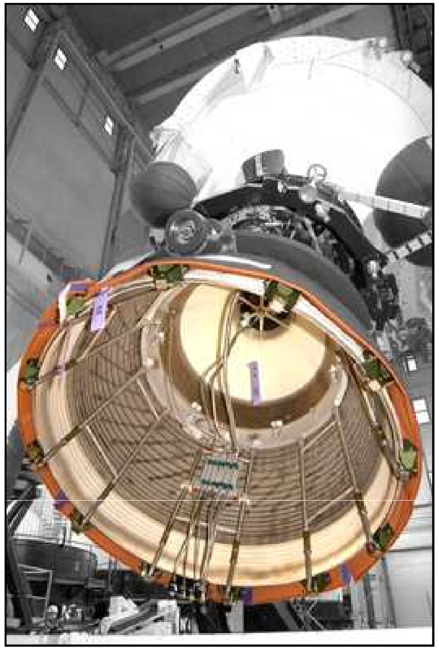
EQUATION 2

- Theorem 4: if dU >0 then dp < 0 and if dU < 0 then dp > 0

4. Design of nozzles and diffusers

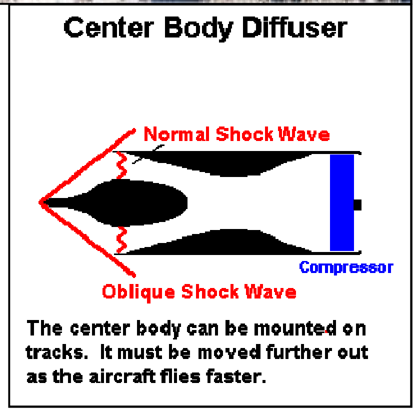
✓ Concept: Nozzle

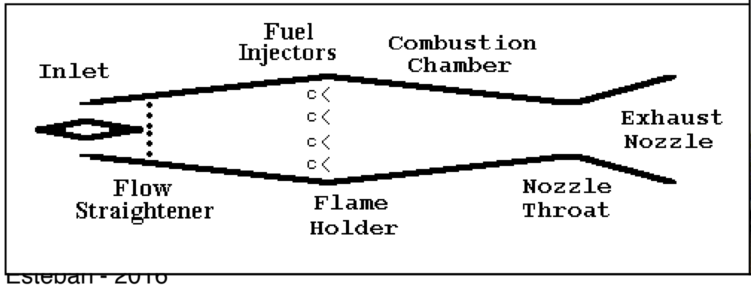


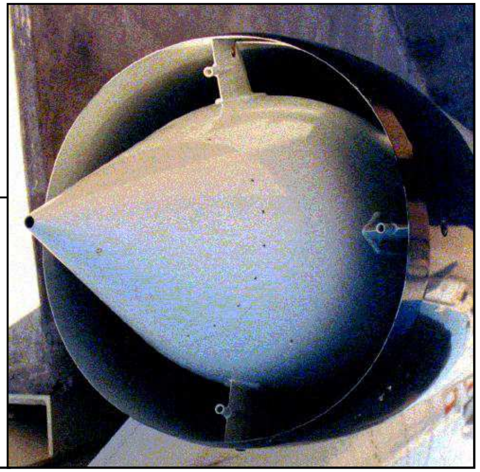


4. Design of nozzles and diffusers

- ✓ Concept: Diffuser







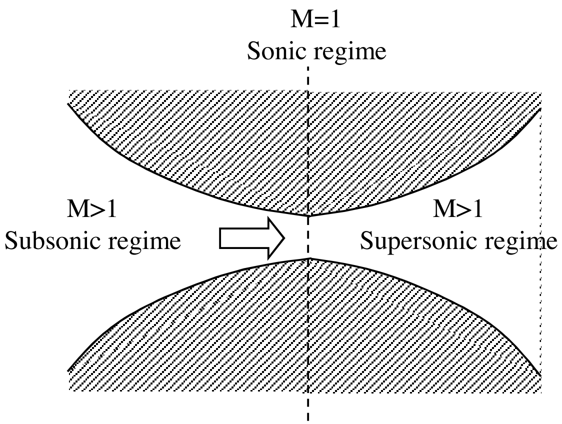
4. Design of nozzles and diffusers

- ✓ Nozzle

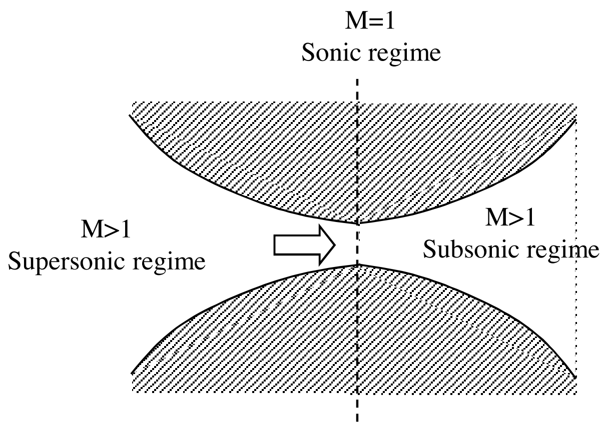
		(M^2-1)	dU	dA	dp	dT	dρ	
Subsonic regime	M < 1	(-)	(+)	(-)	(-)	(-)	(-)	CASE A (convergent)
Supersonic regime	M > 1	(+)	(+)	(+)	(-)	(-)	(-)	CASE C (divergent)

- ✓ Diffuser

		(M^2-1)	dU	dA	dp	dT	dρ	
Subsonic regime	M < 1	(-)	(-)	(+)	(+)	(+)	(+)	CASE B (divergent)
Supersonic regime	M > 1	(+)	(-)	(-)	(+)	(+)	(+)	CASE D (convergent)



Convergent-divergent nozzle (Laval)



Convergent-divergent diffuser