

# 5. GAIA

## ZENBAKIZKO METODOAK BERO-EROAPENEAN

### 5.0 - HELBURUAK

2/22

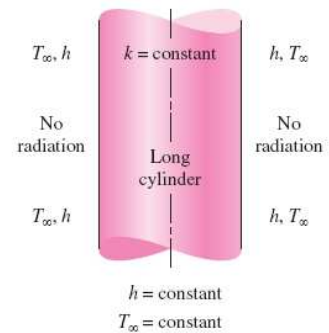
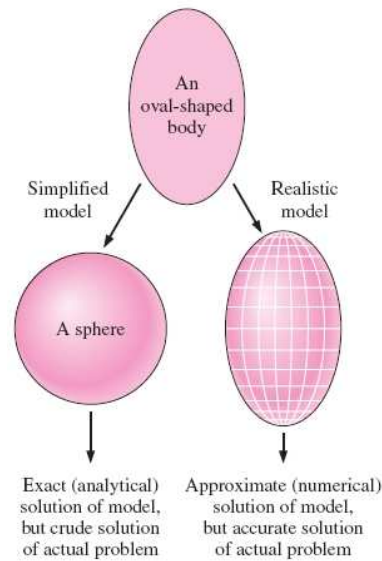
- Eroapen-problemen ebazpen analitikoen mugak eta ordenagailu bidezko **zenbakizko metodo** intentsiboen **beharra** ulertu.
- Deribatuak diferentzia gisa adierazi eta **diferentzia finituko formulazioak** lortu.
- Dimentsio bakarreko edo biko eroapen geldikorreko **problemak numerikoki ebaztea** diferentzia finituen metodoa baliatuta.
- Dimentsio bakarreko edo biko eroapen iragankorreko problemak ebaztea diferentzia finituen metodoa baliatuta.

Ekuazio diferentziala  
+  
Mugaldeko baldintzak



## SOLUZIO ANALITIKOA

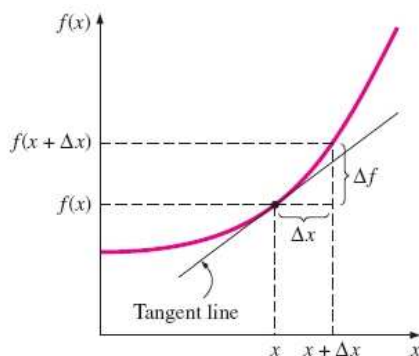
- Mugak
- Eredu hobeak egitea
- Malgutasuna
- Arazoak
- Giza izaera



## 5.2 – EKUAZIO DIFERENTZIALEN DIFERENTZIA FINITUKO FORMULAZIOA

Ekuazio diferentziala → Ekuazio algebraikoa

Deribatuak → Diferentziak



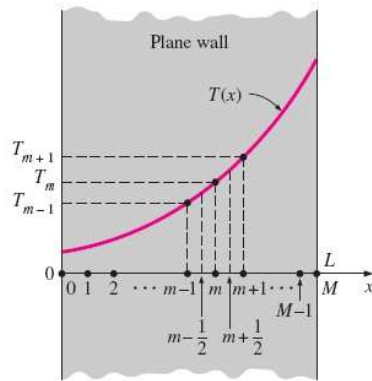
$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{df(x)}{dx} \cong \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Aukera:  $f(x)$  funtzioaren Taylorren garapena

$$f(x + \Delta x) = f(x) + \Delta x \cdot \frac{df(x)}{dx} + \frac{1}{2} \Delta x^2 \cdot \frac{d^2 f(x)}{dx^2} + \dots$$

Errorea  $\Delta x$ -ekiko proportzional



$$\left. \frac{dT}{dx} \right|_{m-\frac{1}{2}} \cong \frac{T_m - T_{m-1}}{\Delta x}$$

$$\left. \frac{dT}{dx} \right|_{m+\frac{1}{2}} \cong \frac{T_{m+1} - T_m}{\Delta x}$$

$$\left. \frac{d^2T}{dx^2} \right|_m \cong \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$

$$\frac{d^2T}{dx^2} + \frac{\dot{e}}{k} = 0 \Rightarrow \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$

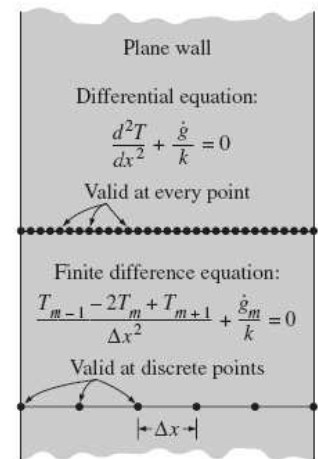
$$m = 1, 2, 3, \dots, M-1$$

$T_0$  eta  $T_M$  ezagunak:

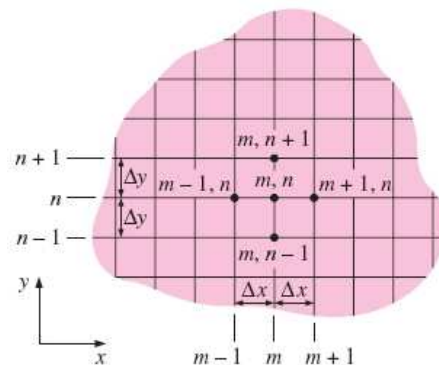
$M - 1$  ekuazio

$M - 1$  ezezagun

$T_0$  eta  $T_M$  ezezagunak:  $\left\{ \begin{array}{l} M - 1 \text{ ekuazio} \\ + \\ 2 \text{ mugaldeko baldintza} \end{array} \right\} M + 1 \text{ ezezagun}$



Bi dimentsiotan:



$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$

$$m = 1, 2, 3, \dots, M - 1$$

$$n = 1, 2, 3, \dots, N - 1$$

## Energia-balantzearen metodoa

$\Delta x = L/M$  lodierako  $M$  eskualde

Posizioa:  $x_m = m \cdot \Delta x$

Temperatura:  $T(x_m) = T_m$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, right}} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_m \cdot V_{\text{element}} = \dot{e}_m \cdot A \cdot \Delta x$$

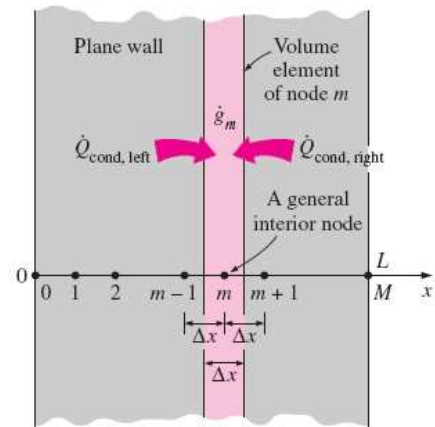
$$\dot{Q}_{\text{cond, left}} = k \cdot A \cdot \frac{T_{m-1} - T_m}{\Delta x}$$

$$\dot{Q}_{\text{cond, right}} = k \cdot A \cdot \frac{T_{m+1} - T_m}{\Delta x}$$

$$k \cdot A \cdot \frac{T_{m-1} - T_m}{\Delta x} + k \cdot A \cdot \frac{T_{m+1} - T_m}{\Delta x} + \dot{e}_m \cdot A \cdot \Delta x = 0$$

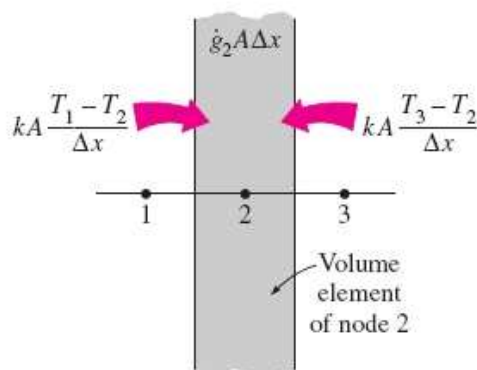
$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$

$$m = 1, 2, 3, \dots, M-1$$



## 5.3 – DIMENTSIO BAKARREKO BERO-EROAPEN GELDIKORRA

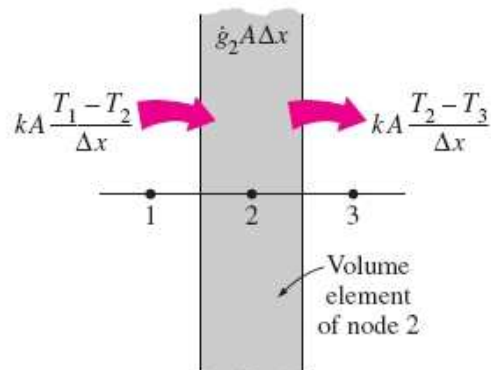
Bolumen elementuaren gainazalean eroapenerako suposatutako norantzak ez du formulazioan eraginik.



$$kA \frac{T_1 - T_2}{\Delta x} + kA \frac{T_3 - T_2}{\Delta x} + \dot{g}_2 A \Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A \Delta x^2 / k = 0$$



$$kA \frac{T_1 - T_2}{\Delta x} - kA \frac{T_2 - T_3}{\Delta x} + \dot{g}_2 A \Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A \Delta x^2 / k = 0$$

**MUGALDEKO BALDINTZAK**

0 nodoa  $\rightarrow x = 0$  posizioa  
 M nodoa  $\rightarrow x = L$  posizioa

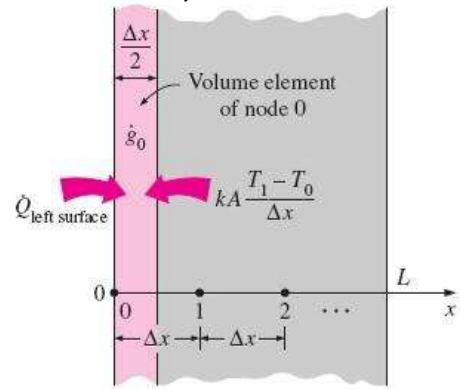
} Zabalera  $\Delta x/2$

Tenperatura zehaztua

$$\begin{cases} T(0) = T_0 \\ T(L) = T_M \end{cases}$$

Konbekzio, erradiazio edo konbekzio eta erradiazio konbinatuen, bero-fluxu zehaztua

$$\dot{Q}_{\text{left surface}} + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$



## 5.3 – DIMENTSIO BAKARREKO BERO-EROAPEN GELDIKORRA

**MUGALDEKO BALDINTZAK**

Bero-fluxu zehaztua

$$\dot{q}_0 \cdot A + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

Kasu partikularra: Mugalde isolatua

$$k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

Konbekzio mugaldea

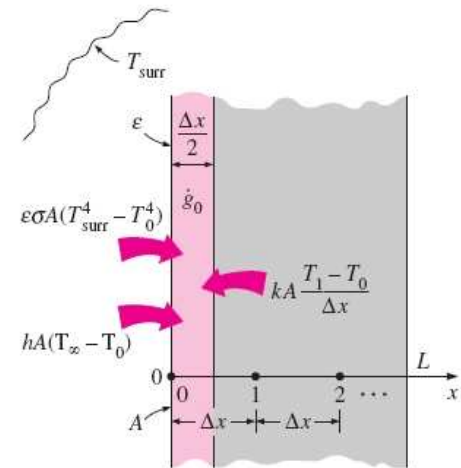
$$h \cdot A \cdot (T_\infty - T_0) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

Erradiazio mugaldea

$$\varepsilon \cdot \sigma \cdot A \cdot (T_{\text{surr}}^4 - T_0^4) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

## MUGALDEKO BALDINTZAK

Konbekzio eta erradiazio konbinatuaren mugaldea



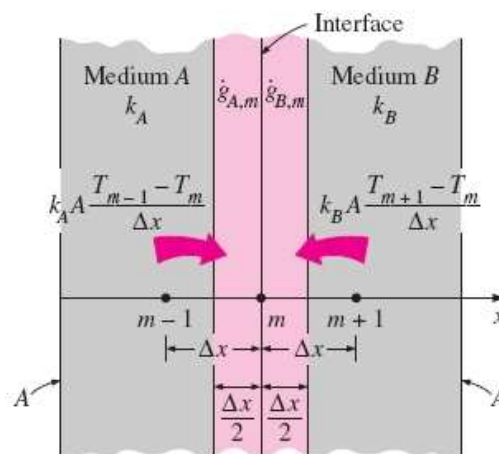
$$h \cdot A \cdot (T_{\infty} - T_0) + \varepsilon \cdot \sigma \cdot A \cdot (T_{surr}^4 - T_0^4) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

$$h_{combined} \cdot A \cdot (T_{\infty} - T_0) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

## 5.3 – DIMENTSIO BAKARREKO BERO-EROAPEN GELDIKORRA

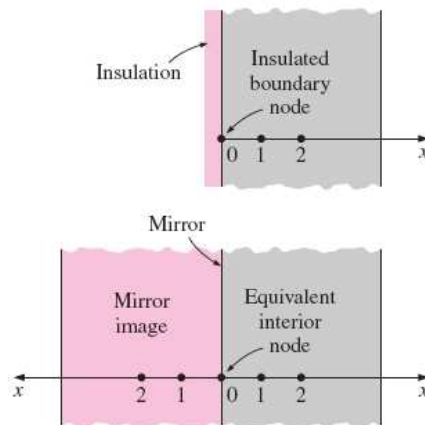
## MUGALDEKO BALDINTZAK

Fasearteko mugaldea



$$k_A \cdot A \cdot \frac{T_{m-1} - T_m}{\Delta x} + k_B \cdot A \cdot \frac{T_{m+1} - T_m}{\Delta x} + \dot{e}_{A,m} \cdot A \cdot \frac{\Delta x}{2} + \dot{e}_{B,m} \cdot A \cdot \frac{\Delta x}{2} = 0$$

## MUGALDE-NODO ISOLATUAK BARNE-NODOTZAT HARTZEA: ISPILU-IRUDIAREN KONTZEPTUA



$$\frac{T_{m-1} - 2 \cdot T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$



$$\frac{T_1 - 2 \cdot T_0 + T_1}{\Delta x^2} + \frac{\dot{e}_0}{k} = 0$$

Energia-balantzearen metodoaren bidez mugalde isolatuaren kasurako aztertutako adierazpen bedina.

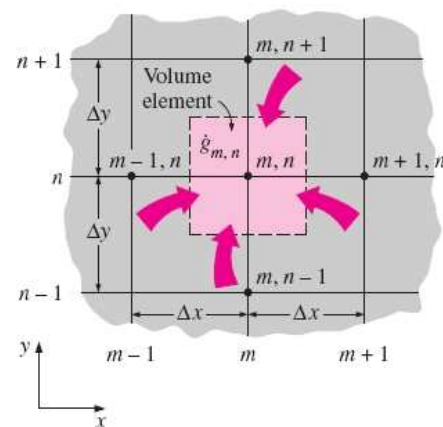
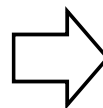
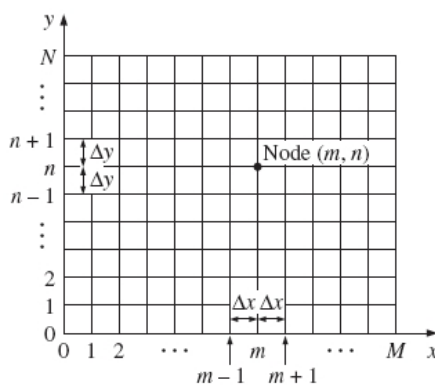
N temperatura nodoetan N ekuazio

Ezabapen-metodoa

Zuzeneko metodoa

Iterazio-metodoa

# 5.4 – BI DIMENTSIOKO BERO-EROAPEN GELDIKORRA



$$\left( \text{Bero-eroapenaren abiadura ezker-, goi-, eskuin- eta behe-gainazalean} \right) + \left( \text{Bero-sorreraren abiadura elementu barruan} \right) = \left( \text{Elementuaren energia-edukiaren aldaketaren abiadura} \right)$$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, top}} + \dot{Q}_{\text{cond, right}} + \dot{Q}_{\text{cond, bottom}} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

$$\begin{aligned}
 & k \cdot \Delta y \cdot \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \cdot \Delta x \cdot \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k \cdot \Delta y \cdot \frac{T_{m+1,n} - T_{m,n}}{\Delta x} \\
 & + k \cdot \Delta x \cdot \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{e}_{m,n} \cdot \Delta x \cdot \Delta y = 0 \\
 & \frac{T_{m-1,n} - 2 \cdot T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2 \cdot T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0 \quad \begin{matrix} m = 1, 2, 3, \dots, M-1 \\ n = 1, 2, 3, \dots, N-1 \end{matrix}
 \end{aligned}$$

Sarea karratu itxura badauka:  $\Delta x = \Delta y = l$

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4 \cdot T_{m,n} + \frac{\dot{e}_{m,n} \cdot l^2}{k} = 0$$

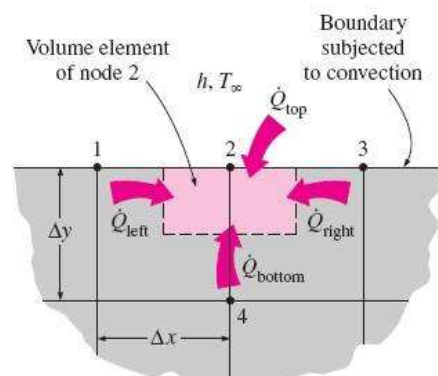
$$T_{left} + T_{top} + T_{right} + T_{bottom} - 4 \cdot T_{node} + \frac{\dot{e}_{node} \cdot l^2}{k} = 0$$

$$T_{node} = \frac{T_{left} + T_{top} + T_{right} + T_{bottom}}{4}$$

TERMOTEKNIA

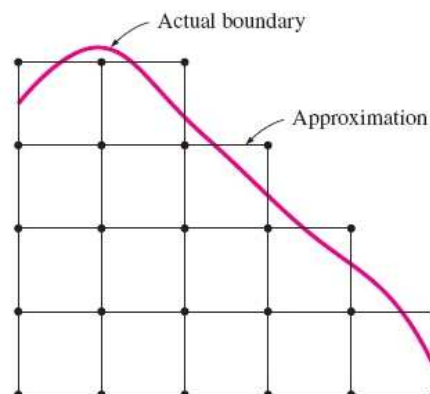
## 5.4 – BI DIMENTSIOKO BERO-EROAPEN GELDIKORRA

## Mugalde-nodoak



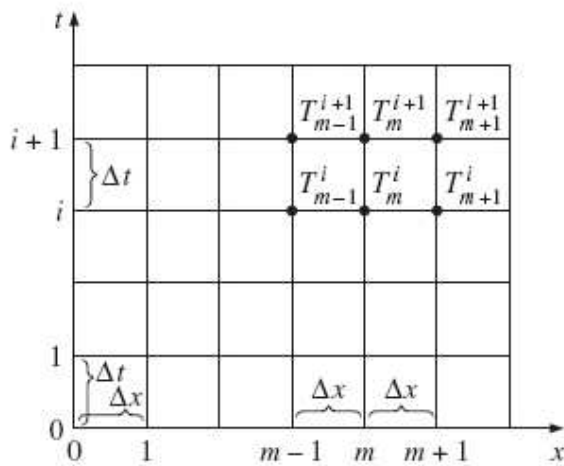
## 5.3 ADIBIDEA

## Mugalde irregularrak



TERMOTEKNIA





$i$  = denbora-urratzen indizea edo zenbatzailea

$i = 0$ ; Hasierako baldintza

$t_i = i \cdot \Delta t$

$T_m^i$  =  $m$  nodo eta  $i$  denbora-urratseko tenperatura

$$\left( \text{Bolumen-elementuaren gainazal guztietatik barrurantz } \Delta t \text{ tartean transferitutako beroa} \right) + \left( \text{Bolumen-elementuaren barnean } \Delta t \text{ tartean sortutako beroa} \right) = \left( \text{Bolumen-elementuan } \Delta t \text{ tartean gertatutako energia-edukiaren aldaketa} \right)$$

$$\sum_{\text{All sides}} \dot{Q} + \dot{E}_{\text{gen,element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = \rho \cdot V_{\text{elem}} \cdot c_p \frac{\Delta T}{\Delta t} = \rho \cdot V_{\text{elem}} \cdot c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

## 5.5 – BERO-EROAPEN IRAGANKORRA

Metodo esplizitua:

$$\sum_{\text{All sides}} \dot{Q}^i + \dot{E}_{\text{gen,element}}^i = \rho \cdot V_{\text{elem}} \cdot c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$



Ebazteko erreza.



Egonkortasuna mantentzeko  $\Delta t$  mugatuta.

Metodo inplizitua:

$$\sum_{\text{All sides}} \dot{Q}^{i+1} + \dot{E}_{\text{gen,element}}^{i+1} = \rho \cdot V_{\text{elem}} \cdot c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

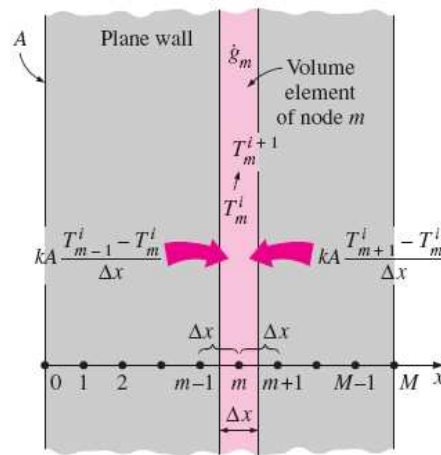


$\Delta t$  ez dago mugatuta.



Nodo-tenperatura guztiak aldi berean ebatzi behar dira.

## Horma laua



$$k \cdot A \cdot \frac{T_{m-1}^i - T_m^i}{\Delta x} + k \cdot A \cdot \frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{q}_m \cdot A \cdot \Delta x = \rho \cdot A \cdot \Delta x \cdot c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

$$T_{m-1}^i - 2 \cdot T_m^i + T_{m+1}^i + \frac{\dot{q}_m \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \cdot \Delta t} (T_m^{i+1} - T_m^i)$$

Egoera egonkorreko formulazioa

$$\tau = \frac{\alpha \cdot \Delta t}{\Delta x^2}$$

## 5.5 – BERO-EROAPEN IRAGANKORRA

## Horma laua

Metodo esplizitua:

$$T_{m-1}^i - 2 \cdot T_m^i + T_{m+1}^i + \frac{\dot{q}_m \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$T_m^{i+1} = \tau \cdot (T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau) \cdot T_m^i + \tau \frac{\dot{q}_m \Delta x^2}{k}$$

Egonkortasun-irizpidea

$$\tau = \frac{\alpha \cdot \Delta t}{\Delta x^2} \leq \frac{1}{2} \Rightarrow \Delta t \leq \frac{1}{2} \frac{\Delta x^2}{\alpha}$$

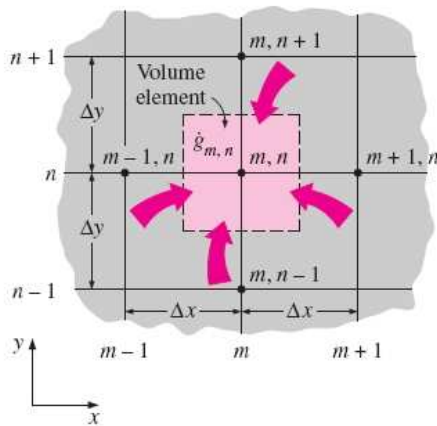
Metodo inplizitua:

$$T_{m-1}^{i+1} - 2 \cdot T_m^{i+1} + T_{m+1}^{i+1} + \frac{\dot{q}_m^{i+1} \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$\tau \cdot T_{m-1}^{i+1} - (1 + 2\tau) \cdot T_m^{i+1} + \tau \cdot T_{m+1}^{i+1} + \tau \frac{\dot{q}_m^{i+1} \Delta x^2}{k} + T_m^i = 0$$

M-1 barne-nodoentzako formulazio baliogarria

## Bi dimentsioko bero-eroapen iragankorra



$$k \cdot \Delta y \cdot \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \cdot \Delta x \cdot \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k \cdot \Delta y \cdot \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k \cdot \Delta x \cdot \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{e}_{m,n} \cdot \Delta x \cdot \Delta y = \rho \cdot \Delta x \cdot \Delta y \cdot c_p \cdot \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Sarea karratu itxura badu:  $\Delta x = \Delta y = l$

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4 \cdot T_{m,n} + \frac{\dot{e}_{m,n} \cdot l^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$\underbrace{T_{left} + T_{top} + T_{right} + T_{bottom} - 4 \cdot T_{node} + \frac{\dot{e}_{node} \cdot l^2}{k}} = \frac{T_{node}^{i+1} - T_{node}^i}{\tau}$$

Egoera egonkorreko formulazioa

## 5.5 – BERO-EROAPEN IRAGANKORRA

## Bi dimentsioko bero-eroapen iragankorra

Metodo esplizitua

$$T_{left}^i + T_{top}^i + T_{right}^i + T_{bottom}^i - 4 \cdot T_{node}^i + \frac{\dot{e}_{node}^i \cdot l^2}{k} = \frac{T_{node}^{i+1} - T_{node}^i}{\tau}$$

$$T_{node}^{i+1} = \tau \cdot (T_{left}^i + T_{top}^i + T_{right}^i + T_{bottom}^i) + (1 - 4 \cdot \tau) \cdot T_{node}^i + \tau \cdot \frac{\dot{e}_{node}^i \cdot l^2}{k}$$

$$T_{node}^{i+1} = \frac{T_{left}^i + T_{top}^i + T_{right}^i + T_{bottom}^i}{4}$$

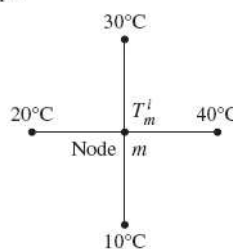
Egonkortasun-irizpidea

$$\tau = \frac{\alpha \cdot \Delta t}{l^2} \leq \frac{1}{4}$$

Erabilitako irudien iturria:

ÇENCCEL, Y.A. TRANSFERENCIA DE CALOR Y MASA, Un enfoque práctico. McGraw-Hil.3 Edición. 2007

Time step  $i$ :



Time step  $i + 1$ :

