ADVANCED NUMERICAL METHODS

DEGREE IN INDUSTRIAL TECHNOLOGY ENGINEERING

JUNE 23, 2015

N.B.: The exam will be sat uninterruptedly (without a break) and will be marked over 24 points.

EXERCISE 1 **(6 points)**

A student wants to interpolate some function $f(x)$ in the interval $[1.74, 6.26]$ using 5 nodes. Working with 3 or more significant digits:

a) Calculate the nodes she will use to minimize the maximum absolute error in the interval. (1p)

In case 5 nodes were not enough, the student eventually decides to use 6, and evaluating $f(x)$ on them she obtains the following table (rounded to 3 significant digits):

x_i	1.82	2.4	3.42	4.58	5.6	6.18
$f(x_i)$	0.844	0.309	–0.249	–0.249	0.309	0.844

b) Calculate the table of differences. (1p)

c) Calculate the interpolation polynomial $p_3(x)$ of degree ≤ 3 by the last four nodes. (1p)

d) Estimate $f(5.1)$ evaluating $p_3(x)$ optimally. (1p)

e) Prove the expression $e(z) = f[x_0, x_1, \ldots, x_n, z]\,\Pi(z)$ from the Newton polynomials of degrees n and $n+1$. (1p)

f) Estimate the error made in part d). (0.5p)

g) Knowing that the interpolated function was $f(x) = \cos(\pi x)$, calculate the error actually made in d), compare it with the estimation in f), and comment on the results. (0.5p)

EXERCISE 2 **(6 points)**

Let $I = \int_1^3 f(x)dx$. Working with 5 decimals:

a) Apply the compound midpoint rule to estimate I from the data in this table: (1p)

x_i	1.25	1.75	2.25	2.75
$f(x_i)$	0.5371	0.1900	0.0384	0.0044

b) Knowing that the error term of the compound midpoint rule is

$$E = \frac{b-a}{6} h^2 f''(\xi) \quad \text{for some } \xi \in (a,b)$$

and that $|f''(x)| \leq 0.89617$ $\forall x \in \mathbb{R}$, find an upper bound of the absolute error made in a). (1p)

c) Find how many subintervals will guarantee that the error does not exceed $\varepsilon = 0.005$. (1p)

d) Knowing that the data above correspond to $f(x) = e^{-x^2}(x^2+1)$, explain what kind of Gaussian quadrature you would use to approximate I, and apply it to obtain I with the precision indicated in part c). (Use the nodes and coefficients of the final Appendix.) (2.5p)

e) According to the results of c) and d), what rule guarantees the required precision with less computational cost: the compound midpoint rule or the Gaussian one of d)? Explain. (0.5p)

EXERCISE 3 **(6.5 points)**

The movement of the system of springs of the figure in the Appendix is governed by the system:

$$\begin{cases} m_1 \dfrac{d^2 y_1}{dt^2} = -(k_1 + k_2) y_1 + k_2 y_2 \\ m_2 \dfrac{d^2 y_2}{dt^2} = k_2 y_1 - k_2 y_2 \end{cases}$$

where y_1, y_2 are the displacements of the masses m_1, m_2 from their equilibrium positions, and k_1, k_2 are the springs' elasticity constants.

For $m_1 = m_2 = 1, \quad k_1 = 1, \quad k_2 = 2$

and the initial conditions: $y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 1, \quad y_2'(0) = 0,$

working with 2 decimals:

a) Transform the system of differential equations above into one of order 1. (1p)

b) Estimate the displacements y_1, y_2 at $t = 1$ with the Runge-Kutta method of order 4. (3p)

c) Knowing that the characteristic polynomial of the Jacobian matrix associated to the system obtained in part a) is $p(\lambda) = \lambda^4 + 5\lambda^2 + 2$, and that the absolute stability regions of the RK1, RK2, RK3, RK4 methods are those shown in the figure in the Appendix, is the step size $h = 1$ adequate to solve the problem using RK4? Justify the answer. (2p)

d) Say which of the methods in the figure are adequate, and which are not, to solve the problem with step size $h = 0.5$. (0.5p)

EXERCISE 4 **(5.5 points)**

One wants to estimate $f^{3)}(z)$ using equally-spaced nodes from $x_0 = z$ to its right.

a) What is the least number of nodes one needs to use? (0.5p)

b) Using four nodes (from $x_0 = z$ to $x_3 = z + 3h$), obtain the numerical differentiation formula using Taylor series expansions (without needing to obtain its error term). (2p)

c) Knowing that the error term can be written in the form $E = -3 f^{4)}(\xi)\, h/2 + O(h^2)$ for some $\xi \in [z, z+3h]$, estimate the optimal distance between nodes h_{opt}. Write the result in terms of $M \geq |f^{4)}(\xi)|$ and of $\varepsilon \geq |f_i - \overline{f}_i|$. (1p)

For $f(x) = 2\sin(2x)$, $z = \pi/12$, and for Matlab's usual arithmetic (considering $\varepsilon = 10^{-16}$):

d) Estimate numerical values of h_{opt} and of the expected upper bound of the absolute error. (1p)

e) To calculate these values we have accepted some assumptions that are not exactly satisfied. Say at least one. (0.5p)

f) Since we have accepted assumptions that are not quite satisfied, one may want to check the estimations of part d) experimentally. The final figure in the Appendix has 10^7 points. Each one has as abscissa some random value of h, and as ordinate the absolute error made when estimating $f^{3)}(z)$ using that value of h. Explain if your results in d) are good or not. (0.5p)

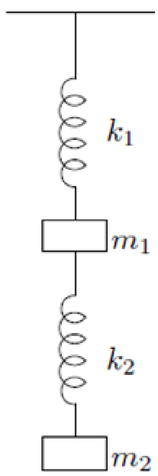
See Appendix in separate sheet

Appendix:

Nodes and weights for part d) of exercise 2:

$n=1$	$x_0=-3^{-1/2}$, $x_1=3^{-1/2}$	$A_0=A_1=1$
$n=2$	$x_0=-0.77460$, $x_1=0$, $x_2=0.77460$	$A_0=A_2=0.55556$, $A_1=0.88889$
$n=3$	$x_0=-0.86114$, $x_1=-0.33998$, $x_2=0.33998$, $x_3=0.86114$	$A_0=A_3=0.34785$, $A_1=A_2=0.65215$
$n=4$	$x_0=-0.90618$, $x_1=-0.53847$, $x_2=0$, $x_3=0.53847$, $x_4=0.90618$	$A_0=A_4=0.23693$, $A_1=A_3=0.47863$, $A_2=0.56889$
$n=5$	$x_0=-0.93247$, $x_1=-0.66121$, $x_2=-0.23862$, $x_3=0.23862$, $x_4=0.66121$, $x_5=0.93247$	$A_0=A_5=0.17132$, $A_1=A_4=0.36076$, $A_2=A_3=0.46791$

Figures of the exercise of Initial Value:



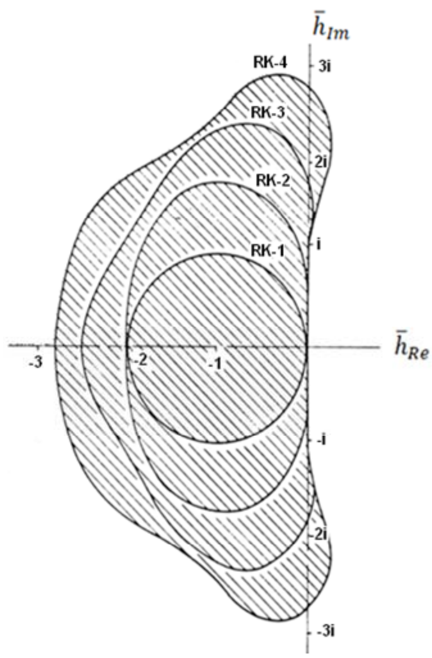
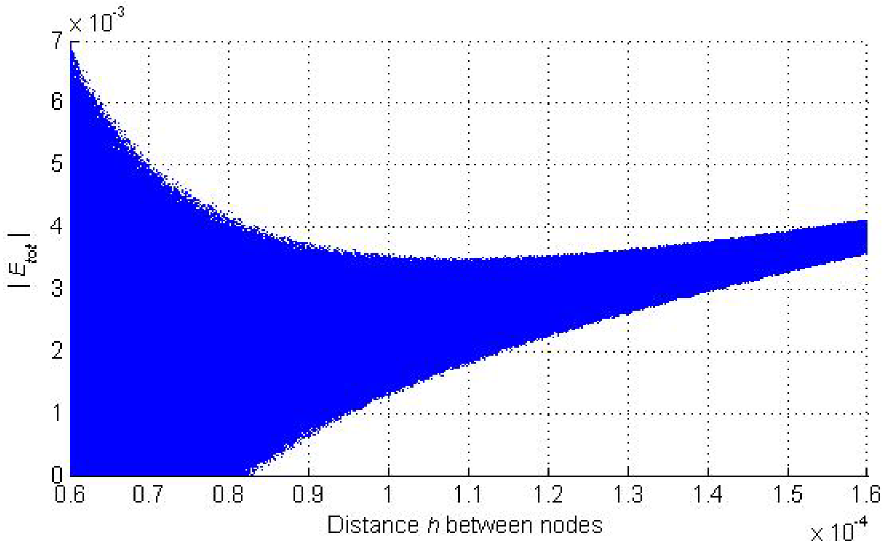


Figure of the exercise of Differentiation:



TOTAL TIME: 3 hours.