8. Gaia

Zenbaki konplexuak

8.1 Zenbaki konplexuak. Forma binomikoa

Definizioa. Zenbaki konplexuak zenbaki errealen bikote ordenatuak dira. Zenbaki konplexuen multzoa adierazteko $\mathbb{C}$ ikurra erabili ohi da,

$$\mathbb{C} = \{(x, y) \; : \; x, y \in \mathbb{R}\}.$$

Beraz, multzotzat hartuta, $\mathbb{C}$ eta $\mathbb{R}^2$ multzo bera dira. Aldea da zenbaki konplexuen arteko biderketa definituko dugula.

Definizioa. $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in \mathbb{C}$ badira,

$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \; y_1 + y_2), \\ z_1 \cdot z_2 &= (x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, \; x_1y_2 + x_2y_1). \end{aligned}$$

Eragiketa horien propietate aljebraikoak kontuan hartuta, $\mathbb{C}$ gorputz trukakorra (edo abeldarra) da.

Zenbaki errealak konplexuen azpimultzotzat har daitezke. Horretarako, $(a, 0)$ bikotea a zenbaki errealarekin identifikatuko dugu, $(a, 0) \equiv a$. Bereziki, $(1, 0) \equiv 1$. Identifikazioa justifikatuta dago, eragiketekin bateragarria delako:

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{eta} \quad (a, 0) \cdot (b, 0) = (ab, 0).$$

Goiko eragiketen ondorioz, $(x, y) \in \mathbb{C}$ bada,

$$(x, y) = (x, 0) \cdot (1, 0) + (y, 0) \cdot (0, 1) = x \cdot 1 + y \cdot (0, 1).$$

Gainera, $(0, 1) = i$ deitzen badugu, $z = (x, y)$ zenbaki konplexuaren *forma binomikoa* lortzen dugu,

$$z = x + yi.$$

Definizioa. $z = x + yi$ zenbaki konplexuaren *parte erreala* x da eta *parte irudikaria* y da. $x = \operatorname{Re} z$ eta $y = \operatorname{Im} z$ notazioa erabiliko dugu. i *unitate irudikaria* dela diogu.

Biderketaren definiziotik,

$$i^2 = (0,1) \cdot (0,1) = -1.$$

Propietate hori izan behar dugu gogoan zenbaki konplexuen biderketa definitzeko. Hona hemen eragiketa aljebraikoen itxura, forma binomikoa erabiliz: $z_1 = x_1 + y_1 i$ eta $z_2 = x_2 + y_2 i$ badira,

$$\begin{aligned} z_1 + z_2 &= (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i, \\ z_1 \cdot z_2 &= (x_1 + y_1 i) \cdot (x_2 + y_2 i) = x_1x_2 + x_1y_2 i + x_2 y_1 i + y_1 y_2 i^2 \\ &= x_1x_2 - y_1y_2 + (x_1y_2 + y_1x_2)i. \end{aligned}$$

Adibidea. Izan bitez $z_1 = 3 - i$, $z_2 = 7 + 2i$.

$$\begin{aligned} z_1 + z_2 &= (3 - i) + (7 + 2i) = 10 + i, \\ z_1 \cdot z_2 &= (3 - i) \cdot (7 + 2i) = 21 + 6i - 7i - 2i^2 = 21 - i + 2 = 23 - i. \end{aligned}$$

Definizioa. Izan bedi $z = x + yi \in \mathbb{C}$. z-ren *konjugatua* honela definitzen da:

$$\overline{z} = x - yi.$$

Adibideak. $\overline{3 - i} = 3 + i$, $\overline{2 + 5i} = 2 - 5i$, $\overline{-2} = -2$, $\overline{-3i} = 3i$.

Proposizioa 8.1 (Konjugatuaren propietateak). *Izan bitez* $z, w \in \mathbb{C}$.

(i) $\overline{\overline{z}} = z$.

(ii) $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ *eta* $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$.

(iii) $\overline{z + w} = \overline{z} + \overline{w}$ *eta* $\overline{zw} = \overline{z}\,\overline{w}$.

Froga. Izan bitez $z = x + iy$, $w = u + iv$. Orduan,

(i) $\overline{z} = \overline{x + yi} = x - yi$, beraz, $\overline{\overline{z}} = \overline{x - yi} = x - (-y)i = x + iy = z$.

(ii) $\dfrac{z + \overline{z}}{2} = \dfrac{x + yi + x - yi}{2} = \dfrac{2x}{2} = x$ eta $\dfrac{z - \overline{z}}{2i} = \dfrac{x + yi - (x - yi)}{2i} = \dfrac{2yi}{2i} = y$.

(iii) Alde batetik, $\overline{z + w} = \overline{x + u + (y + v)i} = x + u - (y + v)i$, eta bestetik $\overline{z} + \overline{w} = x - yi + u - vi = (x + y) - (y + v)i$.

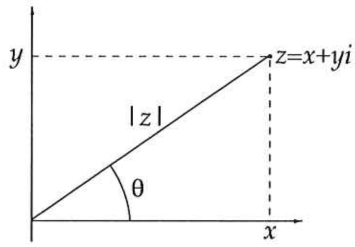
Gainera, $\overline{zw} = \overline{xu - yv + (xv + yu)i} = xu - yv - (xv + yu)i$ eta konjugatuen biderkadura $\overline{z}\,\overline{w} = (x - yi)(u - vi) = xu - yv + (-xv - yu)i$ da. □

8.2 Zenbaki konplexuak. Forma polarra

Esan dugun bezala, multzo modura $\mathbb{C}$ eta $\mathbb{R}^2$ multzo bera dira, beraz zenbaki konplexuak planoko puntuekin identifikatzen dira. Planoko puntuen koordenatu polarrak ere erabili daitezke zenbaki konplexuak adierazteko.

Definizioa. Izan bedi $z \in \mathbb{C}$. Jatorritik z punturainoko distantzia z-ren *modulua* da eta $|z|$ ikurraren bidez adierazten da.

Izan bedi $z \in \mathbb{C} - \{0\}$. Jatorrian hasi eta z puntutik pasatzen den zuzenerdiak ardatz erreal positiboarekin osatzen duen angelua da z-ren *argumentua*. Angelua ardatzetik hsten da neurtzen, erlojuaren orratzen aurkako noranzkoan.



Argumentua ez da bakarra. Baldin θ z-ren argumentua bada, $\theta + 2k\pi$ ere z-ren argumentua da, $k \in \mathbb{Z}$ guztietarako.

Argumentu guztietatik $(-\pi, \pi]$ tartean dagoenari z-ren *argumentu nagusia* deritzo eta $\arg z$ ikurraren bidez adierazten da. $\operatorname{Arg} z$ notazioak z-ren argumentu guztiek osatzen duten multzoa adieraziko dugu, hots,

$$\operatorname{Arg} z = \{\arg z + 2k\pi \ : \ k \in \mathbb{Z}\}.$$

Bestalde, z-ren argumentu bat adierazteko ere $\operatorname{Arg} z$ erabili daiteke, zehaztu gabe zein argumentu hartzen den.

Adibideak. $|i| = 1$; $\arg i = \dfrac{\pi}{2}$, $\operatorname{Arg} i = \{\dfrac{\pi}{2} + 2k\pi \ : \ k \in \mathbb{Z}\}$.

$|1| = 1$; $\arg 1 = 0$, $\operatorname{Arg} 1 = \{2k\pi \ : \ k \in \mathbb{Z}\}$.

$|-1-i| = \sqrt{2}$; $\arg(-1-i) = -\dfrac{3\pi}{4}$, $\operatorname{Arg}(-1-i) = \{-\dfrac{3\pi}{4} + 2k\pi \ : \ k \in \mathbb{Z}\} = \{\dfrac{5\pi}{4} + 2k\pi \ : \ k \in \mathbb{Z}\}$.

Zenbakiaren forma binomikoa $z = x + yi$ baldin bada, orduan

$$|z| = \sqrt{x^2 + y^2}, \quad \tan(\operatorname{Arg} z) = \frac{y}{x}.$$

Bestalde, $|z| = r$ eta $\theta \in \operatorname{Arg} z$ baldin badira

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Oharra. Arku tangente funtzio errealaren irudia $(-\pi/2, \pi/2)$ tartean dago, horregatik $\arctan y/x$ bakarrik izango da z-ren argumentua puntua lehen edo laugarren koadrantean badago.

Definizioa. Izan bedi $z \in \mathbb{C} - \{0\}$. Baldin $|z| = r$ eta $\theta \in \operatorname{Arg} z$ badira, koordenatu kartesiar eta polarren arteko erlazioa kontuan izanda, z-ren forma polarra eman daiteke

$$z = r(\cos\theta + i\sin\theta).$$

Proposizioa 8.2 (Moduluaren propietateak). *Izan bitez $z, w \in \mathbb{C}$.*

(i) $z \cdot \overline{z} = |z|^2$, *beraz* $z^{-1} = \dfrac{\overline{z}}{|z|^2}$, $\forall z \neq 0$.

(ii) $|z| = |-z| = |\overline{z}|$.

(iii) $|\operatorname{Re} z| \leq |z|$, $|\operatorname{Im} z| \leq |z|$, *eta* $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$.

(iv) $|zw| = |z|\,|w|$.

(v) $|z + w| \leq |z| + |w|$.

(vi) $||z| - |w|| \leq |z - w|$.

Froga. Izan bitez $z = x + yi$ eta $w = u + vi$.

(i) $z\overline{z} = (x + yi)(x - yi) = x^2 - xyi + xyi - y^2i^2 = x^2 + y^2 = |z|^2$.

(ii) $|-z| = |-x - yi| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$ eta modu berean, $|\overline{z}| = |x - yi| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$.

(iii) $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |z|$ eta $|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} = |z|$.
Bestalde, $x^2 + y^2 \leq x^2 + y^2 + 2|x||y| = (|x| + |y|)^2$, beraz, $|z| \leq |x| + |y|$.

(iv) (i) atala erabiliz, $|zw|^2 = zw\overline{zw} = zw\overline{z}\,\overline{w} = z\overline{z}w\overline{w} = |z|^2|w|^2$. Hots, $|zw| = |z||w|$.

(v) Berriro (i) atala erabiliz, $|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2$, non azken berdintzan konjugatuaren bigarren propietatea erabili baitugu. Orain, proposizio honen (iii) eta (iv) atalen arabera, $\operatorname{Re}(z\overline{w}) \leq |\operatorname{Re}(z\overline{w})| \leq |z\overline{w}| = |z||\overline{w}| = |z||w|$, beraz,

$$|z + w|^2 \leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2 \implies |z + w| \leq |z| + |w|.$$

(vi) Aurreko atala erabiliz,

$$|z| = |(z - w) + w| \leq |z - w| + |w| \implies |z| - |w| \leq |z - w|$$
$$|w| = |(w - z) + z| \leq |w - z| + |z| \implies |w| - |z| \leq |w - z|$$

Hau da, $-|z - w| \leq |z| - |w| \leq |z - w|$ eta ondorioz, $\big||z| - |w|\big| \leq |z - w|$. □

Oharra. Proposizioaren (i) atala erabiltzen da zatidura baten forma binomikoa lortzeko:

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2} = \frac{\operatorname{Re} z\overline{w}}{|w|^2} + \frac{\operatorname{Im} z\overline{w}}{|w|^2}i.$$

Adibidea. $\dfrac{2+i}{3-i} = \dfrac{(2+i)(3+i)}{(3-i)(3+i)} = \dfrac{6+2i+3i+i^2}{9+1} = \dfrac{5+5i}{10} = \dfrac{1+i}{2}$.

Proposizioa 8.3 (Argumentuaren propietateak). *Izan bitez* $z, w \in \mathbb{C} - \{0\}$.

(i) $\arg z = 0 \iff z \in \mathbb{R}^+$, $\arg z = \pi \iff z \in \mathbb{R}^-$.

(ii) $\arg \overline{z} = -\arg z$, $z \notin \mathbb{R}^-$ *bada.*

(iii) $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$, *baina* $\arg(zw) \neq \arg z + \arg w$ *izan daiteke.*

(iv) $\arg(z^{-1}) = -\arg z$, $z \notin \mathbb{R}^-$ *bada.*

Froga. Ikusi dugun bezala, $z = |z|\big(\cos(\operatorname{Arg} z) + i\sin(\operatorname{Arg} z)\big)$. Hau erabiliko dugu goiko propietateak frogatzeko.

(i) Argumentuaren definiziotik berehala ateratzen da.

(ii) Funtzio trigonometrikoen bakoititasuna/bikoititasuna erabiliz,

$$\overline{z} = |z|\cos\theta - i|z|\sin\theta = |z|\big(\cos(-\theta) + i\sin(-\theta)\big),$$

eta $\theta \in (-\pi, \pi)$ bada, $-\theta \in (-\pi, \pi)$, beraz $-\theta = -\arg z = \arg \overline{z}$.

(iii) Baldin $z = |z|(\cos\theta + i\sin\theta)$ eta $w = |w|(\cos\varphi + i\sin\varphi)$ badira,

$$\begin{aligned} zw =& |z|(\cos\theta + i\sin\theta)|w|(\cos\varphi + i\sin\varphi) \\ =& |z||w|\big((\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\big) \\ =& |zw|\big(\cos(\theta+\varphi) + i\sin(\theta+\varphi)\big) \end{aligned}$$

beraz, $\theta \in \operatorname{Arg} z$ eta $\varphi \in \operatorname{Arg} w$ badira, $\theta + \varphi \in \operatorname{Arg}(zw)$. Ezin dena ziurtatu da $\theta, \varphi \in (-\pi, \pi]$ direnean, $\theta + \varphi$ ere tarte horretan egongo denik.

(iv) $z^{-1} = \dfrac{\overline{z}}{|z|^2}$ denez, $\arg(z^{-1}) = \arg(\overline{z}) = -\arg z$ da. □

Definizioa (Eulerren formula). $t \in \mathbb{R}$ guztietarako,

$$\boxed{e^{it} = \cos t + i\sin t.}$$

Argumentuaren propietateen (iii) atalak erakusten du funtzio horrek esponentzialaren onarrizko propietatea betetzen duela: $e^{it_1}e^{it_2} = e^{i(t_1+t_2)}$.

Eulerren formulan $t = \pi$ hartuz, $e^{\pi i} = -1$ edo,

$$\boxed{e^{\pi i} + 1 = 0.}$$

Azken formula horretan, matematikako zenbakirik ospetsuenak (0, 1, e, π eta i), oinarrizko eragiketa aritmetikoa (batuketa), eta oinarrizko erlazioa (berdintza) agertzen dira.

Definizioa. $|z| = r$ eta $\theta \in \operatorname{Arg} z$ badira, z-ren forma esponentziala honako hau da:

$$z = re^{i\theta}.$$

Eulerren formulako funtzioak esponentzialaren oinarrizko propietatea betetzen duenez, forma esponentziala oso egokia da biderkadurak, alderantzizkoak eta berreturak kalkulatzeko. Izan bitez $z = re^{i\theta}$, $w = \rho e^{i\varphi}$. Orduan,

(i) $z \cdot w = re^{i\theta}\rho e^{i\varphi} = r\rho e^{i(\theta+\varphi)}$,

(ii) $z^{-1} = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r}e^{-i\theta}$,

(iii) $\dfrac{z}{w} = \dfrac{re^{i\theta}}{\rho e^{i\varphi}} = \dfrac{r}{\rho}e^{i(\theta-\varphi)}$,

(iv) $z^n = (re^{i\theta})^n = r^n e^{in\theta}$.

Adibidea. $(1+i)^3$ kalkulatzeko erabili daitezke forma binomikoa zein esponentziala:

$$\begin{aligned}(1+i)^3 &= 1^3 + 3\cdot 1^2 \cdot i + 3 \cdot 1 \cdot i^2 + i^3 = -2+2i,\\ (1+i)^3 &= \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^3 = 2^{3/2}e^{\frac{3\pi}{4}i} = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -2+2i.\end{aligned}$$

Baina berretzailea altuagoa bada, forma binomikoa ez da aproposa. Adibidez,

$$(1-i)^{18} = \binom{18}{0}1^{18}(-i)^0 + \binom{18}{1}1^{17}(-i)^1 + \binom{18}{2}1^{16}(-i)^2 + \cdots + \binom{18}{18}1^0(-i)^{18}$$

eta kalkulua oso luzea da. Aldiz, forma esponentziala erabiliz,

$$(1-i)^{18} = \left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^{18} = (\sqrt{2})^{18}e^{-18\frac{\pi}{4}i} = 2^9 e^{-\frac{9\pi}{2}i} = -512i.$$

De Moivreren formula. $z = e^{i\theta}$ hartuz,

$e^{-\frac{9\pi}{2}i} = \cos\left(-\frac{9\pi}{2}\right) + i\sin\left(-\frac{9\pi}{2}\right) = -i$ 0 -1

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Adibidea. $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$ beraz, parte errealak eta parte irudikariak berdinduz,

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= \cos(2\theta),\\ 2\cos\theta\sin\theta &= \sin(2\theta).\end{aligned}$$

8.3 Zenbaki konplexuen n-erroak

Proposizioa 8.4. *Izan bitez $z \in \mathbb{C} - \{0\}$ eta $n \in \mathbb{N}$. Orduan z-k n n-erro desberdin ditu, hau da n zenbaki desberdin existitzen dira, $w_1, \ldots, w_n \in \mathbb{C}$ zeinetarako $w_i^n = z$ baita $i = 1, \ldots, n$ guztietarako.*

Froga. Izan bitez $z = re^{i\theta}$, $\theta = \arg z$ izanik, eta $w = \rho e^{i\varphi}$, non $w^n = z$ baita. Orduan, $re^{i\theta} = \rho^n e^{in\varphi}$ izan dadin,

$$r = \rho^n \quad \text{eta} \quad n\varphi = \theta + 2k\pi$$

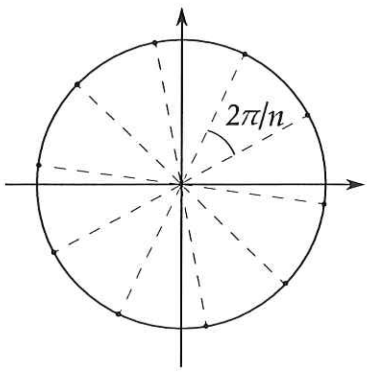
$k \in \mathbb{Z}$ baterako. Hortik, $\rho = \sqrt[n]{r}$ eta k-ren arabera, zenbait aukera ditugu φ-rako. k-ri balioak emanez,

$$\begin{aligned}
w_0 &= \sqrt[n]{r} e^{\frac{\theta}{n} i} \\
w_1 &= \sqrt[n]{r} e^{\frac{\theta + 2\pi}{n} i} \\
&\ldots\ldots \\
w_{n-1} &= \sqrt[n]{r} e^{\frac{\theta + 2(n-1)\pi}{n} i} \\
w_n &= \sqrt[n]{r} e^{\frac{\theta + 2n\pi}{n} i} = \sqrt[n]{r} e^{(\frac{\theta}{n} + 2\pi) i} = w_0,
\end{aligned}$$

eta hortik aurrera errepikatu egiten dira lortutako balioak. Beraz, $w_0, \ldots, w_{n-1}$ dira z-ren n-erroak. Hots,

$$\sqrt[n]{z} = \sqrt[n]{|z|} e^{\frac{\arg z + 2k\pi}{n} i}, \quad k = 0, 1, \ldots, n-1. \qquad \square$$

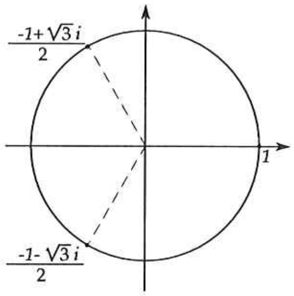
Frogan ikusi dugun bezala, $z \in \mathbb{C} - \{0\}$ zenbaki konplexuaren n-erro guztien modulua berdina da, $\sqrt[n]{|z|}$, eta ondorioz z-ren n-erroak jatorrian zentratuta dagoen eta $\sqrt[n]{|z|}$ erradioa duen zirkunferentzian daude. Gainera ondoz-ondoko erroen arteko angelua $\dfrac{2\pi}{n}$ anplitudekoa denez, erroak n aldeko poligono erregular baten erpinak dira.



Adibidea. $z = 1$ zenbakiaren erro kubikoak.

$z = 1 = e^{0i} = e^{2\pi i} = e^{4\pi i}$, beraz,

$$\begin{aligned} w_1 &= \sqrt{1}e^{\frac{0}{3}i} = e^{0i} = 1, \\ w_2 &= \sqrt{1}e^{\frac{2\pi}{3}i} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \\ w_3 &= \sqrt{1}e^{\frac{4\pi}{3}i} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i. \end{aligned}$$



Oharra. Zenbaki konplexuen erro karratuak kalkulatu daitezke forma binomikoa erabiliz. Izan bitez $z = a + bi \in \mathbb{C}$ eta $w = x + yi \in \mathbb{C}$ z-ren erro karratua.

$b = 0$ bada, $z = a \in \mathbb{R}$ dugu eta z-ren bi erro karratuak $w_1 = \sqrt{a}$ eta $w_2 = -\sqrt{a}$ dira $a > 0$ bada; edo $w_1 = \sqrt{-a}i$ eta $w_2 = -\sqrt{-a}i$, $a < 0$ bada.

$b \neq 0$ bada, $w^2 = z$ bete behar denez,

$$x^2 - y^2 + 2xyi = a + bi$$

eta parte errealak eta parte irudikariak berdinduz,

$$\begin{cases} x^2 - y^2 = a \\ 2xy = b. \end{cases}$$

$b \neq 0$ denez, $x \neq 0$ eta $y \neq 0$ dira eta bigarren ekuaziotik $y = \dfrac{b}{2x}$ da. Lehen ekuazioan ordezkatuz,

$$x^2 - y^2 = x^2 - \frac{b^2}{4x^2} = a \iff 4x^4 - 4ax^2 - b^2 = 0.$$

Ekuazio honen soluzioak honako hauek dira:

$$x^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} = \frac{a \pm \sqrt{a^2 + b^2}}{2}.$$

Negatiboarena baztertu egin behar dugu!

$x \in \mathbb{R}$ denez, $x^2 > 0$ da, beraz $x = \sqrt{\dfrac{a + \sqrt{a^2 + b^2}}{2}}$ edo $x = -\sqrt{\dfrac{a + \sqrt{a^2 + b^2}}{2}}$ eta z-ren bi erro karratuak honako hauek dira:

$$w_1 = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + \frac{b\,i}{2\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}} = \frac{\sqrt{a + \sqrt{a^2 + b^2}}}{\sqrt{2}} + \frac{b}{\sqrt{2}\sqrt{a + \sqrt{a^2 + b^2}}}i$$

$$w_2 = -w_1 = -\frac{\sqrt{a + \sqrt{a^2 + b^2}}}{\sqrt{2}} - \frac{b}{\sqrt{2}\sqrt{a + \sqrt{a^2 + b^2}}}i$$

Adibidea. Kalkulatuko ditugu $z = 3 + 4i$ zenbaki konplexuaren erro karratuak. $w = x + yi$ z-ren erro karratua bada, orduan

$$x^2 - y^2 + 2xyi = 3 + 4i \implies \begin{cases} x^2 - y^2 = 3 \\ 2xy = 4 \end{cases}$$

Bigarren ekuaziotik, $y = 2/x$ eta lehen ekuazioan ordezkatuz,

$$x^2 - \frac{4}{x^2} = 3 \iff x^4 - 3x^2 - 4 = 0 \iff x^2 = \frac{3 + \sqrt{9 + 16}}{2} = 4.$$

Orduan $x = 2$ edo $x = -2$ izan daiteke eta bilatzen ditugun erroak honako hauek dira:

$$w_1 = 2 + i, \quad w_2 = -2 - i.$$

8.4 Zenbaki konplexuen ordena

$\mathbb{R}$-ko ordena, $<$ edo $>$, osoa da eta eragetekin bateragarria, hau da, $a, b, c \in \mathbb{R}$ badira,

(i) $a < b$ edo $b < a$ edo $a = b$;

(ii) $a < b$ bada, $a + c < b + c$; → $b=0$ eta $c=-a$ hartuz, $a<0$ bada, $-a>0$

(iii) $a < b$ bada eta $c > 0$ orduan $ac < bc$.

Ezinezkoa da propietate horiek beteko dituen ordena bat izatea $\mathbb{C}$-n. Ordena hori balego, $i > 0$ edo $-i > 0$ izango genuke. Horietako edozeinek, (iii) erabiliz, $-1 > 0$ ematen du eta hemendik $(-1)^2 = 1 > 0$. Baina biak batera ezin dira gertatu.

8.5 Plano konplexuko distantzia

Definizioa. Izan bitez $z, w \in \mathbb{C}$. z eta w zenbakien arteko distantzia honela definitzen da:

$$d(z, w) = |z - w|.$$

Proposizioa 8.5 (Distantziaren propietateak)**.** *Izan bitez $z_1, z_2, z_3 \in \mathbb{C}$.*

(i) $d(z_1, z_2) \geq 0$, $z_1, z_2 \in \mathbb{C}$ guztietarako eta $d(z_1, z_2) = 0$ baldin eta soilik baldin $z_1 = z_2$.

(ii) $d(z_1, z_2) = d(z_2, z_1)$.

(iii) $d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$.

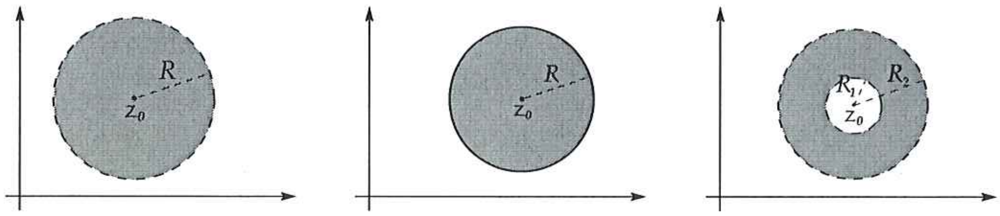
Froga. Lehen bi atalak zuzenean ondorioztatzen dira moduluaren propietateetatik. (iii) moduluaren desberdintza triangeluarraren ondorioa da. □

Definizioa. Izan bitez $z_0 \in \mathbb{C}$, $R, R_2 > 0$ eta $R_1 \geq 0$.

(i) $B_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$ z_0 puntuan zentratuta dagoen eta R erradioa duen *bola edo disko irekia* da.

(ii) $\overline{B_R(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq R\}$ z_0 puntuan zentratuta dagoen eta R erradioa duen *bola edo disko itxia* da.

(iii) $A(z_0, R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ z_0 puntuan zentratuta dagoen eta R_1 eta R_2 erradioak dituen *eraztuna* da.



$B_R(z_0)$ $\overline{B_R(z_0)}$ $A(z_0, R_1, R_2)$

Definizioa. Izan bedi $\Omega \subset \mathbb{C}$.

(i) Ω *irekia* dela diogu baldin eta Ω-ko edozein puntutarako, han zentratuta dagoen bola ireki bat existitzen bada, Ω-ren parte dena, hau da,

$$\forall z \in \Omega \quad \exists R > 0 \quad : \quad B_R(z) \subset \Omega.$$

(ii) Ω *itxia* dela diogu baldin eta $\mathbb{C} - \Omega$ irekia bada.

(iii) Ω-ren *muga* $\partial\Omega$ ikurraren bidez adieraziko dugu eta hurrengo multzoa da

$$\partial\Omega = \{z \in \mathbb{C} : \forall R > 0, \quad B_R(z) \cap \Omega \neq \emptyset, \ B_R(z) \cap (\mathbb{C} - \Omega) \neq \emptyset\}.$$

(iv) $K \subset \mathbb{C}$ *trinkoa* da itxia eta bornatua bada.

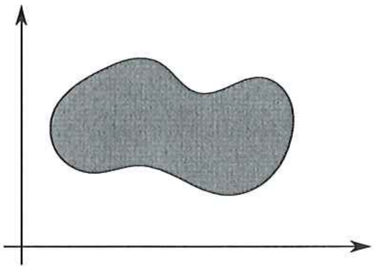
Definizioa. Izan bedi $\Omega \subset \mathbb{C}$.

(i) Esaten dugu Ω *konexua* dela $C, D \subset \mathbb{C}$ existitzen ez badira, C eta D irekiak, disjuntuak eta

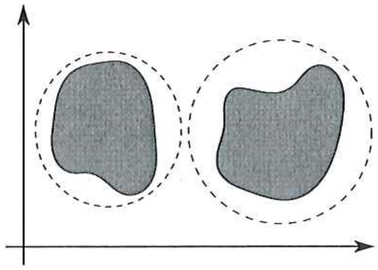
$$\Omega \cap C \neq \emptyset, \quad \Omega \cap D \neq \emptyset, \quad \Omega \subset C \cup D.$$

(ii) Ω *arkuz konexua* da baldin eta Ω-ko edozein bi puntu emanda, biak batzen dituen lerro poligonal bat existitzen bada. Ω-ren parte dena.

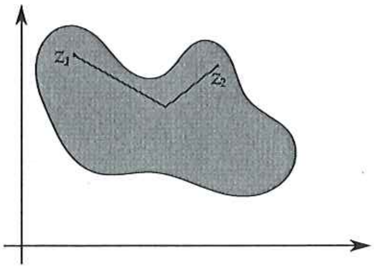
Gogoratu $\mathbb{C}$-ko n puntu ordenatu emanda, $n \geq 2$, bakoitzetik hurrengora doan zuzenkia eginez lortzen den planoko kurbari deitzen diogula lerro poligonala.



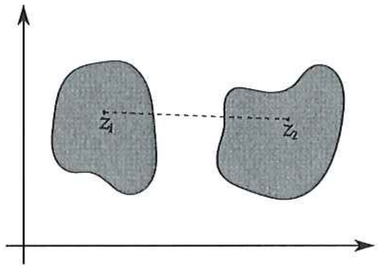
konexua



ez konexua



arkuz konexua



ez arkuz konexua

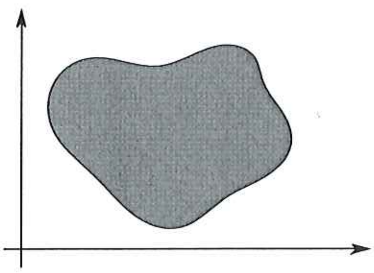
(iii) Ω eremua dela diogu baldin eta irekia eta konexua bada.

Proposizioa 8.6. *Izan bedi* $\Omega \subset \mathbb{C}$.

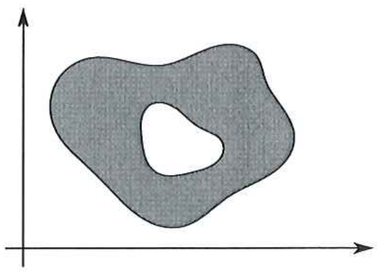
(i) Ω *arkuz konexua bada, konexua da.*

(ii) Ω *irekia izanez gero, arkuz konexua da baldin eta soilik baldin konexua bada.*

Definizioa. Esaten dugu $\Omega \subset \mathbb{C}$ eremu *sinpleki konexua* dela baldin eta kurba sinple itxi guztien barruko aldea multzoko parte bada, hau da, zulorik ez badauka.



sinpleki konexua



ez sinpleki konexua

ANALISI BEKTORIALA ETA KONPLEXUA

8. Gaia: ZENBAKI KONPLEXUAK

Ariketak

1. Idatz itzazu zenbaki hauek forma binomikoan:

   (i) $(2+i)+i(7-2i)$ — *Em.:* $4+8i$.

   (ii) $(2+i)(6+3i)$ — *Em.:* $9+12i$.

   (iii) $(-1-i)(5+i)$ — *Em.:* $-4-6i$.

   (iv) $\dfrac{3+i}{7-i}$ — *Em.:* $\dfrac{2+i}{5}$.

   (v) $\dfrac{-3+2i}{5+i}$ — *Em.:* $\dfrac{-1+i}{2}$.

   (vi) $i(1-i\sqrt{3})(\sqrt{3}+i)$ — *Em.:* $2+2\sqrt{3}i$.

   (vii) $\dfrac{1}{i}+\dfrac{1}{1+i}$ — *Em.:* $\dfrac{1-3i}{2}$.

   (viii) $-2e^{-i\pi/3}$ — *Em.:* $-1+i\sqrt{3}$.

   (ix) i^5+i^{16} — *Em.:* $1+i$.

   (x) $|3+4i|$ — *Em.:* 5.

2. $z=x+iy$ baldin bada, aurki itzazu honako zenbaki hauen parte erreala eta parte irudikaria.

   (i) $w=z^4$ — *Em.:* $\operatorname{Re} w = x^4-6x^2y^2+y^4$, $\operatorname{Im} w = 4x^3y-4xy^3$.

   (ii) $w=\dfrac{1}{z}$ — *Em.:* $\operatorname{Re} w = \dfrac{x}{x^2+y^2}$, $\operatorname{Im} w = \dfrac{-y}{x^2+y^2}$.

   (iii) $w=\dfrac{z-1}{z+1}$ — *Em.:* $\operatorname{Re} w = \dfrac{x^2+y^2-1}{x^2+y^2+2x+1}$, $\operatorname{Im} w = \dfrac{2y}{x^2+y^2+2x+1}$.

   (iv) $w=\dfrac{1}{z^2}$ — *Em.:* $\operatorname{Re} w = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, $\operatorname{Im} w = \dfrac{-2xy}{(x^2+y^2)^2}$.

3. Idatz itzazu jarraian ematen diren zenbakiak forma polarrean:

   (i) $-2+2\sqrt{3}i$ — *Em.:* $4e^{\frac{2\pi}{3}i}$.

   (ii) $|3+4i|$ — *Em.:* $5e^{0i}$.

   (iii) i^{503} — *Em.:* $e^{-\frac{\pi}{2}i}$.

   (iv) $-2e^{-i\pi/3}$ — *Em.:* $2e^{\frac{2\pi}{3}i}$.

   (v) $\dfrac{1+i}{1-i}$ — *Em.:* $e^{\frac{\pi}{2}i}$.

   (vi) $\dfrac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha-i\sin\alpha}$, $\alpha\in\mathbf{R}$ — *Em.:* $e^{\alpha i}$.

4. Kalkula itzazu zenbaki konplexu hauen modulua eta argumentu nagusia:

(i) $z = \cos\alpha - i\sin\alpha,\ \ \pi < \alpha < \dfrac{3\pi}{2}$ izanik *Em.:* $|z| = 1$, $\arg(z) = 2\pi - \alpha$.

(ii) $z = e^{it} + 1$, $t \in (-\pi, \pi)$ izanik *Em.:* $|z| = 2\cos\dfrac{t}{2}$, $\arg(z) = \dfrac{t}{2}$.

5. $z = re^{i\theta}$ eta $w = Re^{i\varphi}$ baldin badira, egiazta ezazu honako berdintza hau:

$$\mathrm{Re}\left(\frac{w+z}{w-z}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}.$$

6. Izan bitez $z, w \in \mathbf{C}$. Froga ezazu berdintza hauek betetzen direla:

(i) $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.

(ii) $|z\overline{w} + 1|^2 + |z-w|^2 = (1+|z|^2)(1+|w|^2)$.

(iii) $|z\overline{w} - 1|^2 - |z-w|^2 = (|z|^2 - 1)(|w|^2 - 1)$.

7. Kalkula itzazu batura hauek:

(i) $\sin x + \sin 2x + \cdots + \sin nx$ *Em.:* $\dfrac{\sin\dfrac{nx}{2}\sin\dfrac{(n+1)x}{2}}{\sin\dfrac{x}{2}}$.

(ii) $\cos x + \cos 2x + \cdots + \cos nx$ *Em.:* $\dfrac{\sin\dfrac{nx}{2}\cos\dfrac{(n+1)x}{2}}{\sin\dfrac{x}{2}}$.

(iii) $\sin x + \sin 3x + \cdots + \sin(2n-1)x$ *Em.:* $\dfrac{\sin^2(nx)}{\sin x}$.

(iv) $\cos x + \cos 3x + \cdots + \cos(2n-1)x$ *Em.:* $\dfrac{\sin(2nx)}{2\sin x}$.

Argibidea: Kontsidera itzazu e^{ix} arrazoiduneko batura geometrikoak.

8. De Moivre-ren formula erabiliz, idatz itzazu $\cos 5\alpha$ eta $\sin 5\alpha$, $\cos\alpha$ eta $\sin\alpha$ balioen menpe.

Em.: $\cos(5\alpha) = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$,
$\sin(5\alpha) = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$.

9. Kalkula itzazu berretura hauek, emaitza forma binomikoan eta polarrean emanez:

(i) $(1+i)^{12}$ *Em.:* $-64 = 64e^{\pi i}$.

(ii) $(1+i\sqrt{3})^{-10}$ *Em.:* $\dfrac{-1+i\sqrt{3}}{2048} = 2^{-10}e^{\frac{2\pi}{3}i}$.

(iii) $\left(\dfrac{1-i}{1+i}\right)^8$ *Em.:* $1 = e^{0i}$.

(iv) $\left(\dfrac{1+i\sqrt{3}}{1-i}\right)^{40}$ *Em.:* $-2^{19}(1+i\sqrt{3}) = 2^{20}e^{-\frac{2\pi}{3}i}$.

10. Kalkula itzazu erro hauek, emaitzak forma polarrean eta binomikoan emanez:

(i) $\sqrt[4]{-1}$ *Em.:* $e^{\frac{\pi}{4}i}=\frac{1+i}{\sqrt{2}}$, $e^{\frac{3\pi}{4}i}=\frac{-1+i}{\sqrt{2}}$, $e^{\frac{5\pi}{4}i}=\dfrac{-1-i}{\sqrt{2}}$, $e^{\frac{7\pi}{4}i}=\frac{1-i}{\sqrt{2}}$.

(ii) $\sqrt{1-i}$ *Em.:* $\sqrt[4]{2}e^{-\frac{\pi}{8}i}=\sqrt[4]{2}\Big(\frac{\sqrt{2+\sqrt{2}}}{2}-i\frac{\sqrt{2-\sqrt{2}}}{2}\Big)$,
$\sqrt[4]{2}e^{\frac{7\pi}{8}i}=\sqrt[4]{2}\Big(-\frac{\sqrt{2+\sqrt{2}}}{2}+i\frac{\sqrt{2-\sqrt{2}}}{2}\Big)$.

(iii) $\sqrt[3]{i}$ *Em.:* $e^{\frac{\pi}{6}i}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $e^{\frac{5\pi}{6}i}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $e^{\frac{3\pi}{2}i}=-i$.

(iv) $\sqrt[4]{-i}$ *Em.:* $e^{-\frac{\pi}{8}i}=\frac{\sqrt{2+\sqrt{2}}}{2}-i\frac{\sqrt{2-\sqrt{2}}}{2}$, $e^{\frac{3\pi}{8}i}=\frac{\sqrt{2-\sqrt{2}}}{2}+i\frac{\sqrt{2+\sqrt{2}}}{2}$,
$e^{\frac{7\pi}{8}i}=-\frac{\sqrt{2+\sqrt{2}}}{2}+i\frac{\sqrt{2-\sqrt{2}}}{2}$, $e^{\frac{11\pi}{8}i}=-\frac{\sqrt{2-\sqrt{2}}}{2}-i\frac{\sqrt{2+\sqrt{2}}}{2}$.

(v) $\sqrt[4]{64}$ *Em.:* $2\sqrt{2}e^{0i}=2\sqrt{2}$, $2\sqrt{2}e^{\frac{\pi}{2}i}=2\sqrt{2}i$,
$2\sqrt{2}e^{\pi i}=-2\sqrt{2}$, $2\sqrt{2}e^{\frac{3\pi}{2}i}=-2\sqrt{2}i$.

(vi) $\sqrt[3]{-1+i}$ *Em.:* $\sqrt[6]{2}e^{\frac{\pi}{4}i}=\sqrt[6]{2}\frac{1+i}{\sqrt{2}}$, $\sqrt[6]{2}e^{\frac{11\pi}{12}i}=\frac{\sqrt[6]{2}}{\sqrt{2}}\Big(-\frac{1+\sqrt{3}}{2}+\frac{\sqrt{3}-1}{2}i\Big)$,
$\sqrt[6]{2}e^{\frac{19\pi}{12}i}=\frac{\sqrt[6]{2}}{\sqrt{2}}\Big(\frac{-1+\sqrt{3}}{2}-\frac{\sqrt{3}+1}{2}i\Big)$.

(vii) $\sqrt{2-2\sqrt{3}i}$ *Em.:* $2e^{-\frac{\pi}{6}i}=\sqrt{3}-i$, $2e^{\frac{5\pi}{6}i}=-\sqrt{3}+i$.

(viii) $\sqrt[4]{\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}}$ *Em.:* $e^{\frac{\pi}{6}i}=\frac{\sqrt{3}+i}{2}$, $e^{\frac{2\pi}{3}i}=\frac{-1+i\sqrt{3}}{2}$, $e^{\frac{7\pi}{6}i}=-\frac{\sqrt{3}+i}{2}$,
$e^{\frac{5\pi}{3}i}=\frac{1-i\sqrt{3}}{2}$.

11. Egiazta ezazu $2+i$ zenbakia $z^3=2+11i$ ekuazioaren soluzio bat dela eta aurki itzazu beste bi soluzioak.

Em.: $-\dfrac{2+\sqrt{3}}{2}+\dfrac{2\sqrt{3}-1}{2}i$, $\dfrac{-2+\sqrt{3}}{2}-\dfrac{2\sqrt{3}+1}{2}i$.

12. Kalkula itzazu honako ekuazio hauen emaitzak:

(i) $z^2=i$ *Em.:* $z=\pm\dfrac{1+i}{\sqrt{2}}$.

(ii) $z^2=-i$ *Em.:* $z=\pm\dfrac{1-i}{\sqrt{2}}$.

(iii) $z^2=1+i$ *Em.:* $z=\pm\Big(\sqrt{\dfrac{1+\sqrt{2}}{2}}+\dfrac{i}{\sqrt{2}\sqrt{1+\sqrt{2}}}\Big)$.

(iv) $z^2=\dfrac{1-i\sqrt{3}}{2}$ *Em.:* $z=\pm\Big(\dfrac{\sqrt{3}}{2}-i\dfrac{1}{2}\Big)$.

13. Froga ezazu koefiziente errealetako bigarren mailako ekuazioen ebazpen-metodoa erabil daitekela ere $az^2+bz+c=0$ ekuazioaren soluzioak topatzeko, $a,b,c\in\mathbf{C}$ direnean.

14. Aurki itzazu ekuazio hauen soluzio konplexu guztiak:

(i) $z^2+iz+2=0$ *Em.:* $i, -2i$.

(ii) $z^4-2z^2+4=0$ *Em.:* $\pm\frac{\sqrt{3}\pm i}{\sqrt{2}}$.

(iii) $z^6 + 2z^3 + 2 = 0$ *Em.:* $\frac{1\pm i}{\sqrt[3]{2}}$, $\frac{(1-\sqrt{3})\pm(1+\sqrt{3})i}{2\sqrt[3]{2}}$, $\frac{(1+\sqrt{3})\pm(\sqrt{3}+1)i}{2\sqrt[3]{2}}$.

(iv) $z^6 + 7z^3 - 8 = 0$ *Em.:* $1, \frac{-1+i\sqrt{3}}{2}, -\frac{1+i\sqrt{3}}{2}, 1+i\sqrt{3}, -2, 1-i\sqrt{3}$.

(v) $z^3 + 3z^2 + 3z + 3 = 0$ *Em.:* $\frac{1-\sqrt[3]{4}+i\sqrt{3}}{\sqrt[3]{4}}$, $-1-\sqrt[3]{2}$, $\frac{1-\sqrt[3]{4}-i\sqrt{3}}{\sqrt[3]{4}}$.

(vi) $z^4 - 4z^3 + 6z^2 - 4z - 15 = 0$ *Em.:* $z = 3,\ 1+2i,\ -1, 1-2i$.

1. ARIKETA

i) $(2+i)+i(7-2i)$

$$2+i+7i-2i^2 = 2+i+7i+2 = \underline{4+8i}$$

ii) $(2+i)\cdot(6+3i)$

$$2\cdot 6+6i+6i+3i^2 = 12-3+6i+6i = \underline{9+12i}$$

iii) $(-1-i)(5+i)$

$$-5-i-5i-i^2 = \underline{-4-6i}$$

iv) $\dfrac{3+i}{7-i}$

$$\frac{3+i}{7-i}\cdot\frac{7+i}{7+i} = \frac{21+3i+7i+i^2}{49+7i-7i-i^2} = \frac{20+10i}{50} = \underline{\frac{2+i}{5}}$$

v) $\dfrac{-3+2i}{5+i}$

$$\frac{-3+2i}{5+i}\cdot\frac{5-i}{5-i} = \frac{-15+3i+10i-2i^2}{25-5i+5i-i^2} = \frac{-13+13i}{26} = \underline{\frac{-1+i}{2}}$$

vi) $i(1-i\sqrt{3})(\sqrt{3}+i)$

$$(\sqrt{3}+i)(\sqrt{3}+i) = 3+2\sqrt{3}\,i+i^2 = \underline{2+2\sqrt{3}\,i}$$

vii) $\dfrac{1}{i}+\dfrac{1}{1+i}$

$$\frac{1}{i}\cdot\frac{-i}{-i}+\frac{1}{1+i}\cdot\frac{1-i}{1-i} = \frac{-i}{-i^2}+\frac{1-i}{1-i^2} = -i+\frac{1-i}{2} = \underline{\frac{1-3i}{2}}$$

viii) $-2e^{-i\pi/3}$

$$-2\left(\cos(-\pi/3)+i\sin(-\pi/3)\right) = -2\left(1/2-\frac{\sqrt{3}}{2}i\right) = \underline{-1+\sqrt{3}\,i}$$

ix) i^5+i^{16}

$$\left(e^{i\frac{\pi}{2}}\right)^5+\left(e^{i\frac{\pi}{2}}\right)^{16} = e^{i\frac{5}{2}\pi}+e^{i8\pi} =$$

$$= \cos\left(\tfrac{5}{2}\pi\right)+i\sin\left(\tfrac{5}{2}\pi\right)+\cos(8\pi)+i\sin(8\pi) =$$

$= 0 + i + 1 + 0i = \underline{1+i}$

x) $|3+4i|$

$\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = \underline{5}$

2. ARIKETA

i) $\omega = z^4$

$\omega = (x+iy)^4 = [(x+iy)^2]^2 = (x^2+2xyi+y^2i^2)^2 =$

$= (x^2-y^2+2xyi)^2 = (x^2-y^2)^2 + 2\cdot 2xyi(x^2-y^2) + 4x^2y^2i^2 =$

$= x^4-2x^2y^2+y^4-4x^2y^2+(4x^3y-4xy^3)i =$

$= x^4-6x^2y^2+y^4+(4x^3y-4xy^3)i$

$\boxed{Re(\omega) = x^4-6x^2y^2+y^4 \quad ; \quad Im(\omega) = 4x^3y-4xy^3}$

ii) $\omega = \frac{1}{z}$

$\omega = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2-y^2i^2} = \frac{x-iy}{x^2+y^2}$

$\boxed{Re(\omega) = \frac{x}{x^2+y^2} \quad ; \quad Im(\omega) = \frac{-y}{x^2+y^2}}$

iii) $\omega = \frac{z-1}{z+1}$

$\omega = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy} =$

$= \frac{(x-1)(x+1)-(x-1)iy+(x+1)iy-y^2i^2}{(x+1)^2-y^2i^2} =$

$= \frac{x^2-1+(x+1-x+1)iy+y^2}{x^2+2x+1+y^2} = \frac{x^2+y^2-1+2yi}{x^2+y^2+2x+1}$

$$\boxed{Re(w) = \frac{x^2+y^2-1}{x^2+y^2+2x+1} \quad ; \quad Im(w) = \frac{2y}{x^2+y^2+2x+1}}$$

iv) $w = \frac{1}{z^2}$

$$w = \frac{1}{(x+iy)^2} = \frac{1}{x^2+2xyi+y^2i^2} = \frac{1}{x^2-y^2+2xyi} =$$

$$= \frac{1}{x^2-y^2+2xyi} \cdot \frac{x^2-y^2-2xyi}{x^2-y^2-2xyi} = \frac{x^2-y^2-2xyi}{(x^2-y^2)^2-4x^2y^2i^2} =$$

$$= \frac{x^2-y^2-2xyi}{x^4-2x^2y^2+y^4+4x^2y^2} = \frac{x^2-y^2-2xyi}{(x^2+y^2)^2}$$

$$\boxed{Re(w) = \frac{x^2-y^2}{(x^2+y^2)^2} \quad ; \quad Im(w) = \frac{-2xy}{(x^2+y^2)^2}}$$

3. ARIKETA

i) $-2+2\sqrt{3}\,i$

$$z = \sqrt{(-2)^2+(2\sqrt{3})^2} \cdot e^{i \cdot \arctan \frac{2\sqrt{3}}{-2}} = \sqrt{4+12}\; e^{i.\arctan(-\sqrt{3})} =$$

$$= \sqrt{16} \cdot e^{i \cdot \frac{2}{3}\pi} \longrightarrow \boxed{z = 4e^{\frac{2\pi}{3}i}}$$

ii) $|3+4i|$

$$\sqrt{3^2+4^2} = \sqrt{25} = 5 \longrightarrow \boxed{z = 5 \cdot e^{0i}}$$

iii) i^{503}

$$\left(e^{\frac{\pi}{2}i}\right)^{503} = e^{\frac{503}{2}\pi i} \longrightarrow \boxed{z = e^{-\frac{\pi}{2}i}}$$

iv) $-2e^{-\frac{\pi}{3}i}$

$$-2\left(\cos(-\pi/3) + i\sin(-\pi/3)\right) = -2\left(1/2 - \sqrt{3}/2\, i\right) =$$

$$= -1+\sqrt{3}\,i = \sqrt{1^2+(\sqrt{3})^2} \cdot e^{i\arctan(\sqrt{3})} \longrightarrow \boxed{z = 2e^{\frac{2\pi}{3}i}}$$

v) $\frac{1+i}{1-i}$

$$\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}=\frac{1+2i+i^2}{1^2-i^2}=\frac{2i}{2}=i\longrightarrow \boxed{z=e^{\frac{\pi}{2}i}}$$

vi) $\frac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha-i\sin\alpha}\;,\quad \alpha\in\mathbb{R}$

$$\frac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha-i\sin\alpha}\cdot\frac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha+i\sin\alpha}=\frac{(1+e^{\alpha i})^2}{(1+\cos\alpha)^2-i^2\sin^2\alpha}=$$

$$=\frac{1+2e^{i\alpha}+e^{2\alpha i}}{2+2\cos\alpha}=\frac{1+2\cos\alpha+2i\sin\alpha+\cos2\alpha+i\sin2\alpha}{2+2\cos\alpha}=$$

$$=\frac{1+2\cos\alpha+\cos^2\alpha-\sin^2\alpha+2\sin\alpha\cos\alpha\, i+2\sin\alpha\, i}{2+2\cos\alpha}=$$

$$=\frac{(1+\cos\alpha)^2-\sin^2\alpha+2\sin\alpha(\cos\alpha+1)i}{2+2\cos\alpha}$$

$$|z|=\frac{\sqrt{[(1+\cos\alpha)^2-\sin^2\alpha]^2+4\sin^2\alpha(\cos\alpha+1)^2}}{2+2\cos\alpha}=$$

$$=\frac{\sqrt{(1+\cos\alpha)^4+2\sin^2\alpha(1+\cos\alpha)^2+\sin^4\alpha}}{2+2\cos\alpha}=$$

$$=\frac{(1+\cos\alpha)^2+\sin^2\alpha}{2+2\cos\alpha}=\frac{2+2\cos\alpha}{2+2\cos\alpha}=1$$

$$\tan(\arg z)=\frac{2\sin\alpha(\cos\alpha+1)}{(1+\cos^2\alpha)^2-\sin^2\alpha}=\frac{2\sin\alpha\cos\alpha+2\sin\alpha}{1+2\cos^2\alpha+\cos^4\alpha-\sin^2\alpha}=$$

$$=\frac{2\sin\cos\alpha+2\sin\alpha}{3\cos^2\alpha+\cos^4\alpha}=\frac{2\sin\alpha(1+\cos\alpha)}{\cos^2\alpha(3+\cos^2\alpha)}=$$

$$=\tan\alpha\cdot\frac{2+2\cos\alpha}{3+\cos^2\alpha}=\tan\alpha\cdot\frac{2+2\cos\alpha}{3+\frac{1}{1+\tan^2\alpha}}=$$

$$= \tan x \cdot (1+\tan^2\alpha) \cdot \frac{2+2\cos\alpha}{4+3\tan^2\alpha} = \dots = \tan\alpha$$

$$\longrightarrow \boxed{z = e^{\alpha i}}$$

4. ARIKETA

i) $z = \cos\alpha - i\sin\alpha$, $\quad \pi < \alpha < \frac{3\pi}{2}$ izanik.

$$|z| = \sqrt{(\cos\alpha)^2 + (-\sin\alpha)^2} = 1.$$

$$\tan(\arg z) = y/x = \frac{-\sin\alpha}{\cos\alpha} = -\tan\alpha$$

$$-\tan(\arg z) = \tan\alpha \longrightarrow \boxed{\arg z = 2\pi - \alpha}$$

ii) $z = e^{it} + 1$, $\quad t \in (-\pi, \pi)$ izanik.

$$z = 1 + \cos t + i\sin t$$

$$|z| = \sqrt{(1+\cos t)^2 + (\sin t)^2} = \sqrt{2 + 2\cos t} = \sqrt{4\cos^2(t/2)}$$

$$= 2\cos(t/2)$$

$$\tan(\arg z) = \frac{\sin t}{1+\cos t} = \frac{2\sin t/2 \cos t/2}{2\cos^2(t/2)} = \tan t/2 \longrightarrow$$

$$\longrightarrow \boxed{\arg z = t/2}$$

5. ARIKETA

$z = re^{i\theta}$ eta $\omega = Re^{i\varphi}$

$$\cancel{\frac{\omega+z}{\omega-z} = \frac{(\omega+z)^2}{\omega^2-z^2} = \frac{\omega^2+2\omega z+z^2}{\omega^2-z^2} = \frac{R^2e^{2i\varphi}+2Rre^{i(\varphi+\theta)}+r^2e^{2i\theta}}{R^2e^{2i\varphi}-r^2e^{2i\theta}}} =$$

$$= \frac{e^i}{e^i} \cdot \frac{R^2 e^{2\varphi} + 2Rr e^{\varphi+\theta} + r^2 e^{2\theta}}{R^2 e^{2\varphi} - r^2 e^{2\theta}} = \frac{(Re^{\varphi} + re^{\theta})^2 e^i}{(Re^{\varphi} + re^{\theta})(Re^{\varphi} - re^{\theta}) e^i} =$$

$$= \frac{(Re^{\varphi} + re^{\theta}) e^i}{(Re^{\varphi} - re^{\theta}) e^i}$$

Forma binomikoan: $= r\cos\theta + r\sin\theta\, i$

$= R\cos\varphi + R\sin\varphi\, i$

$$\frac{\omega+z}{\omega-z} = \frac{(\omega+z)^2}{\omega^2 - z^2} = \frac{\omega^2 + 2\omega z + z^2}{\omega^2 - z^2} =$$

$$= \frac{r^2\cos^2\theta - r^2\sin^2\theta + 2r^2\cos\theta\sin\theta i + R^2\cos^2\varphi - R^2\sin^2\varphi + 2R^2\cos\varphi\sin\varphi i + 2\omega z}{r^2\cos^2\theta - r^2\sin^2\theta + 2r^2\cos\theta\sin\theta i - R^2\cos^2\varphi + R^2\sin^2\varphi - 2R^2\sin\varphi\cos\varphi i}$$

$$= \frac{r^2\cos 2\theta + r^2\sin 2\theta i + R^2\cos 2\varphi + R^2\sin 2\varphi i + 2\omega z}{r^2\cos 2\theta + r^2\sin 2\theta\, i - R^2\cos 2\varphi - R^2\sin 2\varphi\, i} =$$

$$= \frac{r^2(\cos 2\theta + \sin 2\theta\, i) + R^2(\cos 2\varphi + \sin 2\varphi\, i) + 2rR\cos\varphi\cos\theta - 2rR\sin\theta\sin\varphi + I\omega}{r^2(\cos 2\theta + \sin 2\theta\, i) - R^2(\cos 2\varphi + \sin 2\varphi\, i)}$$

$$Re\left(\frac{\omega+z}{\omega-z}\right) = \frac{r^2\cos 2\theta + R^2\cos 2\varphi + 2rR(\cos\varphi\cos\theta - \sin\theta\sin\varphi)}{r^2\cos 2\theta - R^2\cos 2\varphi} = \ldots$$

$$\ldots = \frac{r^2 - R^2}{R^2 - 2Rr\cos(\varphi - \theta) + r^2}$$

9. ARIKETA

i) $(1+i)^{12} = (\sqrt{2} \cdot e^{i \cdot \frac{\pi}{4}})^{12} = 64 \cdot e^{3\pi i} = 64 e^{\pi i} = -64$

ii) $(1+i\sqrt{3})^{-10} = (2 \cdot e^{i \cdot \frac{\pi}{3}})^{-10} = 2^{-10} e^{-\frac{10}{3}\pi i} = 2^{-10} e^{\frac{2\pi}{3} i} =$

$$= \frac{-1 + \sqrt{3}\, i}{2048}$$

iii) $\left(\frac{1-i}{1+i}\right)^8 = \left(\frac{(1-i)^2}{2}\right)^8 = \frac{1}{2^8}(1-i)^{16} = \frac{1}{2^8}\left(\sqrt{2}\, e^{-\frac{\pi}{4}i}\right)^{16} =$

$= \frac{1}{2^8}\cdot 2^8 e^{-4\pi i} = e^{0i} = 1$

iv) $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40} = \left(\frac{2\cdot e^{i\frac{\pi}{3}}}{\sqrt{2}\, e^{-\frac{\pi}{4}i}}\right)^{40} = \left(\sqrt{2}\, e^{\frac{7}{12}\pi i}\right)^{40} = 2^{20} e^{\frac{280}{12}\pi i} =$

$= 2^{20} e^{-\frac{2}{3}\pi i} = -2^{19}(1+\sqrt{3}\, i)$

10. ARIKETA

i) $\sqrt[4]{-1} = (-1\cdot e^{\pi i})^{1/4} = \sqrt[4]{|-1|}\cdot e^{i\left(\frac{\pi}{4}+\frac{2\pi k}{4}\right)}$, $k\in[0,3]$

$\omega_0 = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$; $\omega_2 = e^{i\cdot\frac{5\pi}{4}} = \frac{-1-i}{\sqrt{2}}$

$\omega_1 = e^{i\cdot\frac{3\pi}{4}} = \frac{-1+i}{\sqrt{2}}$; $\omega_3 = e^{i\cdot\frac{7\pi}{4}} = \frac{1-i}{\sqrt{2}}$

ii) $\sqrt{1-i} = \left(\sqrt{2}\cdot e^{-\frac{\pi}{4}i}\right)^{1/2} = \sqrt{\sqrt{2}}\cdot e^{i\left(-\frac{\pi}{8}+\frac{2\pi k}{2}\right)}$, $k=0,1$

$\omega_0 = \sqrt[4]{2}\, e^{-\frac{\pi}{8}i} = \sqrt[4]{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2} - i\,\frac{\sqrt{2-\sqrt{2}}}{2}\right)$

$\omega_1 = \sqrt[4]{2}\, e^{\frac{7\pi}{8}i} = \sqrt[4]{2}\left(-\frac{\sqrt{2+\sqrt{2}}}{2} + i\,\frac{\sqrt{2-\sqrt{2}}}{2}\right)$

iii) $\sqrt[3]{i} = (1\cdot e^{\frac{\pi}{2}i})^{1/3} = e^{i\left(\frac{\pi}{6}+\frac{2\pi k}{3}\right)}$, $k\in[0,2]$

$\omega_0 = e^{\frac{\pi}{6}i} = \frac{\sqrt{3}}{2}+\frac{i}{2}$; $\omega_1 = e^{\frac{5}{6}\pi i} = \frac{-\sqrt{3}+i}{2}$

$\omega_2 = e^{\frac{3\pi}{2}i} = -i$

iv) $\sqrt[4]{-i} = (e^{-\frac{\pi}{2}i})^{1/4} = e^{i(-\frac{\pi}{8}+\frac{2\pi k}{4})}$, $k \in [0,3]$

$$\left[\begin{array}{l} \omega_0 = e^{-\frac{\pi}{8}i} = \dfrac{\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}\,i}{2} \; ; \; \omega_1 = e^{\frac{3\pi}{8}i} = \dfrac{\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}\,i}{2} \\ \omega_2 = e^{\frac{7\pi}{8}i} = \dfrac{-\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}\,i}{2} \; ; \; \omega_3 = e^{\frac{11\pi}{8}i} = \dfrac{-\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}}\,i}{2} \end{array}\right]$$

v) $\sqrt[3]{-1+i} = (\sqrt{2}\,e^{\frac{3\pi}{4}i})^{1/3} = \sqrt[6]{2}\,e^{i(\frac{\pi}{4}+\frac{2\pi k}{3})}$, $k \in [0,2]$

$$\left[\begin{array}{l} \omega_0 = \sqrt[6]{2}\cdot e^{\frac{\pi}{4}i} = \dfrac{\sqrt[6]{2}}{\sqrt{2}}(1+i) \; ; \; \omega_1 = \sqrt[6]{2}\,e^{\frac{11\pi}{12}i} = \dfrac{\sqrt[6]{2}}{\sqrt{2}}\left(\dfrac{-1-\sqrt{3}+(\sqrt{3}-1)i}{2}\right) \\ \omega_2 = \sqrt[6]{2}\,e^{\frac{19}{12}\pi i} = \dfrac{\sqrt[6]{2}}{\sqrt{2}}\left(\dfrac{-1+\sqrt{3}+(-1-\sqrt{3})i}{2}\right) \end{array}\right]$$

vi) $\sqrt[4]{64} = (64e^{0i})^{1/4} = 2^{7/4}\,e^{i\cdot\frac{2\pi k}{4}}$, $k \in [0,3]$

$$\left[\begin{array}{l} \omega_0 = 2\sqrt{2}e^{0i} = 2\sqrt{2} \; ; \; \omega_1 = 2\sqrt{2}\,e^{i\frac{\pi}{2}} = 2\sqrt{2}\,i \\ \omega_2 = 2\sqrt{2}\,e^{\pi i} = -2\sqrt{2} \; ; \; \omega_4 = 2\sqrt{2}\,e^{\frac{3\pi}{2}i} = -2\sqrt{2}\,i \end{array}\right]$$

vii) $\sqrt{2-2\sqrt{3}i} = (4e^{-\frac{\pi}{3}i})^{1/2} = 2e^{i(-\frac{\pi}{6}+\frac{2\pi k}{2})}$, $k = 0, 1$

$$\left[\omega_0 = 2e^{-\frac{\pi}{6}i} = \sqrt{3}-i \; ; \; \omega_1 = 2e^{\frac{5\pi}{6}i} = -\sqrt{3}+i\right]$$

viii) $\sqrt[4]{\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}} = \left(\dfrac{2e^{i\cdot\frac{\pi}{3}}}{2e^{i\cdot -\frac{\pi}{3}}}\right)^{1/4} = (e^{\frac{2\pi}{3}i})^{1/4} = e^{i(\frac{\pi}{6}+\frac{2\pi k}{4})}$, $k \in [0,3]$

$$\left[\begin{array}{l} \omega_0 = e^{\frac{\pi}{6}i} = \dfrac{\sqrt{3}+i}{2} \; ; \; \omega_1 = e^{\frac{2\pi}{3}i} = \dfrac{-1+\sqrt{3}\,i}{2} \\ \omega_2 = e^{\frac{7\pi}{6}i} = \dfrac{-\sqrt{3}-i}{2} \; ; \; \omega_3 = e^{\frac{5\pi}{3}i} = \dfrac{1-\sqrt{3}\,i}{2} \end{array}\right]$$

11. ARIKETA

$z^3 = 2+11i$ ekuazioaren soluzio dela egiaztatatu $2+i$ zenbakia. Zein dira beste biak?

$$(2+i)^3 = (2+i)(2+i)(2+i) = (4+4i-1)(2+i) = (3+4i)(2+i) =$$

$$= 6+3i+8i-4 = 2+11i \longrightarrow \text{Ekuazioa betetzen du.}$$

Beste biak topa ditzagun:

$$z^3 = 2+11i = (2+i)^3 \longleftrightarrow \left(\frac{z}{2+i}\right)^3 = 1$$

$\omega \equiv \frac{z}{2+i}$ eginez, $\omega^3 = 1$ dugu.

$$\omega = \sqrt[3]{1} = e^{i\cdot\frac{2\pi k}{3}}$$

$$\omega_0 = e^{i\cdot 0} = 1 \longrightarrow z_0 = 2+i.$$

$$\omega_1 = e^{\frac{2\pi}{3}i} = \frac{-1+\sqrt{3}i}{2} \longrightarrow z_1 = (2+i)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) =$$

$$= -1+\sqrt{3}i-\frac{1}{2}i-\frac{\sqrt{3}}{2} \longrightarrow \boxed{z_1 = -\frac{2+\sqrt{3}}{2}+\frac{2\sqrt{3}-1}{2}i}$$

$$\omega_2 = e^{\frac{4\pi}{3}i} = \frac{-1-\sqrt{3}i}{2} \longrightarrow z_2 = (2+i)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) =$$

$$= -1-\sqrt{3}i-\frac{1}{2}i+\frac{\sqrt{3}}{2} \longrightarrow \boxed{z_2 = \frac{\sqrt{3}-2}{2}-\frac{2\sqrt{3}+1}{2}i}$$

12. ARIKETA

i) $z^2 = i$

$$z = \sqrt{i} = \left(e^{\frac{\pi}{2}i}\right)^{1/2} = e^{i\left(\frac{\pi}{4}+\pi k\right)}, \quad k = 0, 1$$

$$z_1 = e^{\frac{\pi}{4}i} \longrightarrow \boxed{z_1 = \frac{1+i}{\sqrt{2}}}$$

$$z_2 = e^{\frac{5\pi}{4}i} \longrightarrow \boxed{z_2 = -\frac{1+i}{\sqrt{2}}}$$

ii) $z^2 = -i$

$$z = \sqrt{-i} = \left(e^{-\frac{\pi}{2}i}\right)^{1/2} = e^{i\left(-\frac{\pi}{4} + \pi k\right)} \quad , k = 0, 1$$

$$z_1 = e^{-\frac{\pi}{4}i} \longrightarrow \boxed{z_1 = \frac{+i}{\sqrt{2}}}$$

$$z_2 = e^{\frac{3\pi}{4}i} \longrightarrow \boxed{z_2 = \frac{-1+i}{\sqrt{2}}}$$

iii) $z^2 = 1 + i$

$$z = (1+i)^{1/2} = \left(\sqrt{2}\, e^{\frac{\pi}{4}i}\right)^{1/2} = \sqrt[4]{2}\; e^{i\left(\frac{\pi}{8} + \pi k\right)} \quad , k = 0, 1$$

$$\boxed{z_1 = \sqrt[4]{2}\, e^{\frac{\pi}{8}i} \quad ; \quad z_2 = \sqrt[4]{2}\, e^{\frac{9}{8}\pi}}$$

iv) $z^2 = \dfrac{1 - i\sqrt{3}}{2}$

$$z = \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{2}i} = \left(e^{-\frac{\pi}{3}i}\right)^{1/2} = e^{i\left(-\frac{\pi}{6} + k\pi\right)} \quad , k = 0, 1$$

$$z_1 = e^{-\frac{\pi}{6}i} \longrightarrow \boxed{z_1 = \frac{\sqrt{3} - i}{2}}$$

$$z_2 = e^{\frac{5\pi}{6}i} \longrightarrow \boxed{z_2 = \frac{-\sqrt{3} + i}{2}}$$