
NO SEAS
RATA
ДЛЯ СЧАСТЬЯ НАРОДОВ!

1. GAIA: MUTURRAK

1. DERIBATU PARTZIALAK

$f: U \subset \mathbb{R}^1 \to \mathbb{R}$ deribagarria $\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h\cdot 1) - f(x)}{h}$

DEFINIZIOA

$f: U \subset \mathbb{R}^n \to \mathbb{R}$ n aldagaiko funtzio erreala eta

$\bar{x} = (x_1, \ldots, x_n) \in U$ puntu bat

$$f_{x_j}(\bar{x}) = \frac{\partial f}{\partial x_j}(\bar{x}) = \lim_{h \to 0} \frac{f(\bar{x} + h(\bar{e}_j)) - f(\bar{x})}{h} \quad \text{non}$$

$\bar{e}_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)$ $\quad \frac{\partial f}{\partial x_j}(\bar{x}) \equiv$ LEHEN ORDENAKO DE. PAR x_j-rekiko

DEFINIZIOA

$\bar{f}: U \subset \mathbb{R}^n \to \mathbb{R}^m$ n aldagaiko funtzio bektoriala

$\bar{f}(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$

$\bar{f}$ DIFERENTZIAGARRIA da $\bar{x}_0 \in U$ puntuan:

1) $\bar{f}$-ren deribatu partzialak $\bar{x}_0$-n existitzen diren

2) $$\lim_{\bar{x} \to \bar{x}_0} \frac{\| \bar{f}(\bar{x}) - \bar{f}(\bar{x}_0) - Df(x_0)(x - x_0) \|}{\| \bar{x} - \bar{x}_0 \|} = 0$$

TEOREMA 1.1:

$f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ eta f-ren lehen ordenako deribatu partzialak EXISTITZEN badira eta JARRAIAK badira $\bar{x} \in U$-n

$\Rightarrow \bar{f}$ DIFERENTZIAGARRIA da $\bar{x}$-n

DEFINIZIOA

$f: U \in \mathbb{R}^n \to \mathbb{R}$ $\wedge$ $\frac{\partial f}{\partial x}$ $\forall i, \ldots, n$ EXISTITZEN badira eta JARRAITUAK badira.

$\Rightarrow \bar{f}$ C^1 KLASEKOA

TEOREMA 1.2: KATEAREN ERREGELA

$\bar{f}: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\bar{g}: V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^p$ non $\bar{f} \circ \bar{g}$ ondo definituta dagoen

- $\bar{g}$ diferentziagarria $\bar{x}_0$-n
- $\bar{f}$ diferentziagarria $\bar{g}(x_0)$-n

$$\Rightarrow \underset{m\times p}{D(\bar{f} \circ \bar{g})(\bar{x}_0)} = \underset{m\times n}{Df(\bar{g}(\bar{x}_0))} \cdot \underset{n\times p}{D\bar{g}(\bar{x}_0)}$$

2. GOI ORDENAKO DERIBATUAK

DEF: $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ f C^2 KLASEKOA baldin:

bigarren ordeneko deribatuak $\exists$ ∧ jarraituak

BIGARREN ORDENAKO DERIBATU PARTZIALAK:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \quad \text{non } i, j = 1, \dots, n$$

$i \neq j \Rightarrow$ Deribatu partzial gurutzatuak

TEOREMA 1.3:

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ C^2 KLASEKOA

$$\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \forall i, j = 1, \dots, n$$ [ordenak ez du importa]

OHARRA: C^m izanik berdin gertatzen da

3. TAYLORREN TEOREMA

Gogoratu $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = f(x_0) + f'(x_0)\cdot(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{k)}}{k!}(x - x_0)^k + R_k(x, x_0)$$

$$\text{non } \lim_{x \to x_0} \frac{R_k(x, x_0)}{(x - x_0)^k} = 0$$

PLANO UKITZAILEA

⇒ TEOREMA 1.4: LEHEN ORDENAKO TAYLORREN TMA.

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $\bar{x}_0 \in U$, f dif. ∧ $\bar{h} = (h_1, \dots, h_n) = \bar{x} - x_0$

$$\Rightarrow f(\bar{x}) = f(\bar{x}_0 + h) = f(\bar{x}_0) + \sum_{i=1}^{n} h_i \cdot \frac{\partial f}{\partial i}(\bar{x}_0) + R_1(\bar{h}, \bar{x}_0) =$$

$$= f(\bar{x}_0) + Df(\bar{x}_0)\cdot h + R_1(\bar{h}, \bar{x}_0) \quad \text{non } \lim_{\bar{h} \to \bar{0}} \frac{R_1(\bar{h}, \bar{x}_0)}{\|\bar{h}\|} = 0$$

TEOREMA 1.5: BIGARREN ORDENAKO TAYLORREN TEOREMA

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad C^2$ KLASEKOA

$$f(\bar{x}_0+\bar{h}) = f(\bar{x}_0) + \sum_{i=1}^{n} h_i \cdot \frac{\partial f}{\partial x_i}(\bar{x}_0) + \frac{1}{2!}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}_0) + R_2(\bar{h}_j, \bar{x}_0)$$

non $\displaystyle\lim_{h \to 0} \frac{R_2(\bar{x}, \bar{x}_0)}{\|\bar{h}\|^2} = 0$

OHARRA: Tayloren 2. hurbilketa

$$f(x,y) \sim f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) +$$
$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + \frac{1}{2!}\cdot\frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2 +$$
$$+ \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\cdot(x-x_0)\cdot(y-y_0) + R_2$$

4. MUTUR LOKALAK

DEFINIZIOA

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R} \wedge \bar{x}_0 \in U$ MUTUR LOKALAK:

i) x_0 f-ren MINIMO LOKALA da $\exists V$ x_0-ren ingurune bat non $f(\bar{x}) \geq f(\bar{x}_0)$ $\forall x \in V$

ii) x_0 f-ren MAXIMO LOKALA da $\exists V$ x_0-ren ingurune bat non $f(\bar{x}) \leq f(\bar{x}_0)$ $\forall x \in V$

TEOREMA 1.6:

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $\bar{x}_0$ puntuan diferentziagarria eta $\bar{x}_0$ f-ren mutur lokala bada

$\Rightarrow \nabla f(\bar{x}_0) = \bar{0}$ $\quad \left[\frac{\partial f}{\partial x_i}(\bar{x}_0) = 0 \quad \forall i = 1, \dots, n\right]$

DEFINIZIOA:

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $\bar{x}_0 \in U$ f-ren PUNTU KRITIKOA

1) f x_0-n ez-diferentziagarria

2) f diferentziagarria $\bar{x}_0$ eta $\nabla f(\bar{x}_0) = \bar{0}$

DEFINIZIOA

$\bar{x}_0 \in U$ f-ren PUNTU KRITIKOA $\wedge$ $\neg$MUTUR LOKALA

$\Rightarrow \bar{x}_0$ ZELADURA PUNTUA

MATRIZE HESSIARRA $\Rightarrow \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\bar{x}_0) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\bar{x}_0) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\bar{x}_0) \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\bar{x}_0) & & \dots & \frac{\partial^2 f}{\partial x_n^2}(\bar{x}_0) \end{pmatrix}$

TEOREMA 1.11: $f: U \subset \mathbb{R}^n \to \mathbb{R}$ C^2 klasekoa, $(x_0, y_0) \in U$

eta $D = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}$ = MAT. HESSIARRAREN DETERM.

1) $\bar{x}_0 = (x_0, y_0)$ MINIMO LOK:

- $\bar{x}_0$-n PUNTU KRITIKOA
- $\frac{\partial^2 f}{\partial x^2}(\bar{x}_0) > 0$
- $D(\bar{x}_0) > 0$

2) $\bar{x}_0 = (x_0, y_0)$ MAXIMO LOK:

- $\bar{x}_0$-n PUNTU KRITIKOA
- $\frac{\partial^2 f}{\partial x^2}(\bar{x}_0) < 0$
- $D(\bar{x}_0) > 0$

3) $\bar{x}_0 = (x_0, y_0)$ ZELADURA PUNT.

- $\bar{x}_0$ PUNTU KRITIKOA
- $D(\bar{x}_0) < 0$

* 3 DIMENTSIOTAN [HESSIARRAN]

min $\Rightarrow |1\times 1| > 0, |2\times 2| > 0, |3\times 3| > 0$

max $\Rightarrow |1\times 1| < 0, |2\times 2| > 0, |3\times 3| < 0$

zel $\Rightarrow |1\times 1| \neq 0, |2\times 2| \neq 0, |3\times 3| \neq 0$

5. MUTUR BALDINTZATUAK

$g(\bar{x}) = C$ EKUAZIO BALDINTZATUA izango dela esango dugu

TEOREMA 1.12: Izan bitez $f, g: U \subset \mathbb{R}^n \to \mathbb{R}$ C^1 k.

$S = \{(x_1, \dots, x_n) \in U : g_i(\bar{x}_0) \neq \bar{0}$ den $\forall i = 1, \dots, m$

eta $\bar{x}_0 \in S$ non $\nabla g_i(\bar{x}_0) \neq \bar{0}$ den $\forall i = 1, \dots, m$

Izan bedi $h(\lambda_1, \dots, \lambda_m, x_1, \dots, x_n)$ = LAGRANGEN FUNTZIOA

$= f(x_1, \dots, x_n) - \lambda_1 \cdot (g_1(x_1, \dots, x_n) - C_1) - \dots + \lambda_m (g_m(x_1, \dots, x_n) - C_m$

$\Rightarrow$ i) $f_{|S}$ funtzioak f (S-ra murriztuta) maximo eta minimo lokala du $\bar{x}_0$-n $\Rightarrow \exists \lambda_i \in \mathbb{R}$ non $\nabla h(\bar{x}_0) = \bar{0}$

ii) $f_{|S}$ funtzioak $\bar{x}_0$ puntuan maximo edo minimo lokala badu $\Rightarrow \nabla f(\bar{x}_0) \perp S$ x_0-n

DEFINIZIOA: Izan bitez $f, g : U \subset \mathbb{R}^{2=n} \rightarrow \mathbb{R}$ C^2

$S = \{(x,y) : g(x,y) = c\}$ $\wedge$ $\bar{x}_0 \in S$: $\nabla g(\bar{x}_0) \neq \bar{0}$

Demagun $\exists \lambda \in \mathbb{R}$ non $\boxed{h(\lambda, x, y) = f(x,y) - \lambda(g(x,y) - c)}$

izanik $\nabla h(\bar{x}_0) = \bar{0}$

HESSIAR MUGATUA $\Rightarrow$ $|\bar{H}| = \begin{vmatrix} 0 & -\frac{\partial g}{\partial x} & -\frac{\partial g}{\partial y} \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial y \partial x} \\ -\frac{\partial g}{\partial y} & \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial y^2} \end{vmatrix}$

TEOREMA 1.13:

Aurreko definizioaren $(n=2)$ kasuan notazioa eta baldintzak kontuan hartuz,

i) $|\bar{H}| > 0 \Rightarrow f|_S$-ak $\bar{x}_0$ puntuan MAXIMO LOKALA

ii) $|\bar{H}| < 0 \Rightarrow f|_S$-ak $\bar{x}_0$ puntuan MINIMO LOKALA

$|\bar{H}| = 0$?

[Minorrak aztertu *]

6. MUTUR ABSOLUTUAK

DEFINIZIOA: Izan bitez $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ $\wedge$ $\bar{x}_0 \in A$

(A ez da derrigorrez irekia)

i) f-k $\bar{x}_0$-n MAXIMO ABSOLUTUA duela esaten da:

$f(\bar{x}) \leq f(\bar{x}_0)$ $\forall \bar{x} \in A$ bada

ii) f-k $\bar{x}_0$-n MINIMO ABSOLUTUA duela esaten da

$f(\bar{x}) \geq f(\bar{x}_0)$ $\forall \bar{x} \in A$ bada

TEOREMA 1.14:

Izan bedi $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ jarraitua, D eremu trinkoa (itxia $\wedge$ bornatua) izanik.

Paulak

$\Rightarrow$ f-k maximo $\wedge$ minimo absolutuak lortzen ditu D-n

3D.

$$\boxed{h(\lambda, \mu, x, y, z) = f(x,y,z) - \lambda(g_1(x,y,z) - c_1) - \mu(g_2(x,y,z) - c_2)}$$

2. GAIA: FUNTZIO INPLIZITUAK

$F(x,y)=0$ izanik $\Rightarrow$ lortzea $y=f(x)$ non $F(x,f(x))=0$

TEOREMA 2.1: FUNTZIO INPLIZITUEN TMA-ren KASU PARTIKULARRA

$y=f(x) \Rightarrow$ ESPLIZITUA
$F(x,y)=0 \Rightarrow$ INPLIZITUA

$F: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ C^1 KLASEKOA

Idatz dezagun $(\bar{x}, z) \in \mathbb{R}^{n+1}$ non $\bar{x}=(x_1,\dots,x_n) \in \mathbb{R}^n$ $z \in \mathbb{R}$. Izan bedi $(\bar{x}_0, z_0) \in \mathbb{R}^{n+1}$ non

$$\Rightarrow \begin{cases} 1) \ F(\bar{x}_0, z_0)=0 \\ 2) \ \dfrac{\partial F}{\partial z}(\bar{x}_0, \bar{z}_0) \neq 0 \end{cases}$$

$\Rightarrow$ $\exists$ dira $U \in \mathbb{R}^n$ $\bar{x}_0$-ren ingurune bat, $V \subset \mathbb{R}$ z_0-ren ingurune bat eta $g: U \to V$ funtzio BAKAR bat non $F(\bar{x}, \underbrace{g(\bar{x})}_{z})=0$

Gainera
- g diferentziagarria
- g-ren deribatu partzialak jarraituak

$$\frac{\partial g}{\partial x_i} = \frac{-\dfrac{\partial F}{\partial x_i}}{\dfrac{\partial F}{\partial z}} \quad \forall i=1,\dots,n$$

OHARRA:

$z=g(x)$ funtzioa existitzen dela jakinda eta diferentziagarria dela frogatuta, g-ren deribatu partzialak kalkulatzeko DIFERENTZIAZIO INPLIZITUA erabili.

TEOREMA 2.2: FUNTZIO INPLIZITUEN TEOREMA OROKORRA

Kontsidera dezagun honako sistema

$$* \begin{cases} F_1(x_1,\dots,x_n,z_1,\dots,z_m)=0 \\ \vdots \\ F_m(x_1,\dots,x_n,z_1,\dots,z_m)=0 \end{cases} \quad \text{non } F_i \ C^1 \text{ den } \forall i=1,\dots,m$$

$$\Delta = \begin{vmatrix} \dfrac{\partial F_1}{\partial z_1} & \dots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \\ \dfrac{\partial F_m}{\partial z_1} & \dots & \dfrac{\partial F_m}{\partial z_m} \end{vmatrix}$$

Izan bitez $\bar{x}_0 \in \mathbb{R}^n$, $\bar{z}_0 \in \mathbb{R}^m$ eta demagun

$F_i(\bar{x}_0, \bar{z}_0) = 0$ dela $\forall_{i=1,\dots,m}$ $\wedge$ $\Delta(\bar{x}_0, \bar{z}_0) \neq 0$

$\Rightarrow (\bar{x}_0, \bar{z}_0)$ puntuko ingurune batean $\exists h_1, \dots, h_m$

C^1 klasekoak non $z_i = h_1(x_1, \dots, x_n)$

* Sistemaren soluzioak dira; h_i funtzioak bakarrak dira $(\bar{x}_0, \bar{z}_0)$ puntuaren ingurune batean eta beraien deribatu partzialak kalkulatzeko deribatu inplizitua erabili daiteke.

2.2 ALDERANTZIZKO FUNTZIOAREN TEOREMA

Funtzio inplizituaren kasu partikularra

$f: \mathbb{R} \to \mathbb{R}$

$y = f(x) \Rightarrow \exists g$ funtzioa (f-ren alderantzizkoa)

$x = g(y)$, non $y = f(g(y))$

$F(x, y) = y - f(x) = 0$

TEOREMA 2.3: ALDERANTZIZKO FUNTZIOAREN TEOREMA

Izan bitez,

- $U \subset \mathbb{R}^n$ irekia
- $f: U \subset \mathbb{R}^n \to \mathbb{R}$ C^1 klasekoa
- $\bar{x}_0 \in U$
- $\bar{f} = (f_1, \dots, f_n)$

Baldin eta $J(\bar{f})(\bar{x}_0) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}_0) & \cdots & \frac{\partial f_n}{\partial x_n}(\bar{x}_0) \\ \\ \frac{\partial f_n}{\partial x_1}(\bar{x}_0) & \cdots & \frac{\partial f_n}{\partial x_n}(\bar{x}_0) \end{vmatrix} \neq 0$

$\Rightarrow \left\{ \begin{array}{l} y_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ y_n = f_n(x_1, \dots, x_n) \end{array} \right.$ funtzioa ebatzi daiteke

$\bar{x} = \bar{g}(\bar{y})$ funtzioaren bidez, $\bar{x}$ $\bar{x}_0$-tik hurbil eta $\bar{y}$ $f(\bar{x}_0)$-tik hurbil daudenean. Gainera, soluzioa bakarra da eta $\bar{g} = (g_1, \dots, g_n)$ C^1 klasekoa.

$$\Rightarrow \begin{cases} x_1 = g_1(y_1, \dots, y_n) \\ \vdots \\ x_n = g_n(y_1, \dots, y_n) \end{cases}$$

3. GAIA: INTEGRAL BIKOITZA

1. INTEGRAL BIKOITZA ERREKTANGELU BATEN GAINEAN

DEFINIZIOA

Izan bitez $a, b, c, d \in \mathbb{R}$, $D = [a,b] \times [c,d]$ eta $f: D \to \mathbb{R}$ bornatua eta kontsidera ditzagun $a = x_0 < x_1 < \dots < x_n = b$ eta $c = y_0 < y_1 < \dots < y_n = d$

Izan bedi $P_{ij} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $i = 1, \dots, n$ eta $j = 1, \dots, n$ bakoitzerako baldin eta existitu

- $\exists \lim\limits_{n \to \infty} \sum\limits_{i,j=1}^{n} f(P_{ij}) \cdot (x_i - x_{i-1}) \cdot (y_j - y_{j-1}) = L$
- L finitua

$\Rightarrow$ f integragarria D-n

$$\iint_D f(x,y)\, dA = \iint_D f(x,y)\, dx\, dy = L$$

TEOREMA 3.1:

$f: D = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ jarraitua $\Rightarrow$ f integragarria.

TEOREMA 3.2:

Izan bedi $f: D = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ bornatua eta demagun f-ren etenguneak funtzio jarraituen bildura FINITU batean kokatzen direla $\Rightarrow$ f integragarria

CAVALIERIREN PRINTZIPIOA BOLUMENERAKO

$f: [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ jarraitua eta $f(x,y) > 0$ $\forall (x,y) \in D$. f-ren azpian geratzen den bolumena:

A) Gorputza

3. GAIA: INTEGRAL BIKOITZA

1. INTEGRAL BIKOITZA ERREKTANGELU BATEN GAINEAN

DEFINIZIOA

Izan bitez $a, b, c, d \in \mathbb{R}$, $D = [a,b] \times [c,d]$
eta $f: D \to \mathbb{R}$ bornatua eta kontsidera ditzagun
$a = x_0 < x_1 < \dots < x_n = b$ eta $c = y_0 < y_1 < \dots < y_n = d$ partiketak

Izan bedi $P_{ij} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $\begin{matrix} i = 1, \dots, n \\ j = 1, \dots, n \end{matrix}$

Baldin eta existitzen bada eta finitua bada

$$\lim_{n \to \infty} \sum_{i,j=1}^{n} f(P_{ij}) \cdot (x_i - x_{i-1}) \cdot (y_j - y_{j-1}) = L$$

$\Rightarrow$ f integragarria D-n

$$\iint_D f(x,y)\, dA = \iint_D f(x,y)\, dx\, dy = L$$

TEOREMA 3.1:

$f: D = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ jarraitua $\Rightarrow$ f integragarria

TEOREMA 3.2:

Izan bedi $f: D = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ bornatua eta demagun f-ren etenguneak funtzio jarraituen bildura FINITU batean kokatzen direla

$\Rightarrow$ f integragarria D eremuan.

CAVALIERIREN PRINTZIPIOA BOLUMENA KALKULATZEKO

$f: [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ jarraitua $\wedge$ $f(x,y) \geq 0$ $\forall (x,y) \in D$

f-ren azpian geratzen den bolumena:

1) Gorputza $x = x_0$ planoarekin ebakitzean $z = f(x_0, y)$ aldagai bateko funtzioaren grafikoaren azpian geratzen den azalera. $A(x)$ deitu eta x a-tik b-ra mugitu:

$$B=\int_a^b A(x)\,dx=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

2) Berdin $y=y_0$ planoarekin eginez, $z=f(x,y_0)$ aldagai bakarreko funtzioaren grafikoaren azpian geratzen den azalera $\hat{A}(y)$ deituz eta y c-tik d-ra mugituz

$$B=\int_c^d A(x)\,dy=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

DEFINIZIOA:

Izan bedi $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ bornatua

f-ren INTEGRAL ITERATUAK dira:

$$\int_a^b\int_c^d f(x,y)\,dy\,dx \quad\wedge\quad \int_c^d\int_a^b f(x,y)\,dx\,dy$$

TEOREMA 3.3: FUBINIREN TEOREMA

$f:D=[a,b]\times[c,d]\to\mathbb{R}$ bornatua eta bere etenguneen multzoa funtzio jarraituen grafikoan bildura finitua.

i) $\int_c^d f(x,y)\,dy$ $\exists$ bada $\forall x\in[a,b]$ $\Rightarrow$

$\Rightarrow \exists \int_a^b\int_c^d f(x,y)\,dy\,dx \quad\wedge\quad \iint_D f\,dA=\int_a^b\int_c^d f\,dy\,dx$

ii) $\int_a^b f(x,y)\,dx$ $\exists$ bada $\forall x\in[c,d]$ $\Rightarrow$

$\Rightarrow \exists \int_c^d\int_a^b f(x,y)\,dx\,dy \quad\wedge\quad \iint_D f\cdot dA=\int_c^d\int_a^b f\,dx\,dy$

$$\overset{i)\wedge ii)}{\Longrightarrow} \iint_D f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

OHARRA:

$$\iint_D f(x)g(y)\,dy\,dx=\int_a^b\int_c^d f(x)g(y)\,dy\,dx=\left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

3.2. INTEGRAL BIKOITZA EREMU OROKORRAGOETAN

EREMU ELEMENTALAK

$a, b, c, d \in \mathbb{R}$ $\quad a < b \wedge c < d$

i) 1. MOTAKO EREMUA:

Izan bitez $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ jarraituak

eta $\phi_1(x) \leq \phi_2(x) \quad \forall x \in [a,b]$

$\Rightarrow D = \{(x,y) \in \mathbb{R}^2 : x \in [a,b], \phi_1(x) \leq y \leq \phi_2(x)\}$

ii) 2. MOTAKO EREMUA:

Izan bitez $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ jarraituak

eta $\psi_1(y) \leq \psi_2(y) \quad \forall y \in [c,d]$

$\Rightarrow D = \{(x,y) \in \mathbb{R}^2 : y \in [c,d], \psi_1(y) \leq x \leq \psi_2(y)\}$

iii) 3. MOTAKO EREMUA:

$D \subset \mathbb{R}^2$ $1 \wedge 2$ motakoa bada

PROPOSIZIOA 3.4: INTEGRAL BIKOITZAREN PROPIETATEAK

Izan bitez $D \subset \mathbb{R}^2$ eremu elementala

$f, g : D \to \mathbb{R}$ integragarriak D-n

i) $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ integragarria da eta

$$\iint_D \alpha f + \beta g \, dA = \alpha \iint_D f \, dA + \beta \iint_D g \, dA$$

ii) $f(x,y) \geq g(x,y) \quad \forall (x,y) \in D$

$$\iint_D f \, dA \geq \iint_D g \, dA$$

iii) $D_i \subset \mathbb{R}^2$ eremu elementalak $\forall i = 1, \dots, m$

$D_i \cap D_j = \phi \quad \wedge \quad D = \bigcup_{i=1}^{m} D_i$

$$\Rightarrow \iint_D f \, dA = \sum_{i=1}^{m} \iint_{D_i} f \, dA$$

iv) $|f|$ integragarria $\Rightarrow \left|\iint_D f\,dA\right| \leq \iint_D |f|\,dA$

TEOREMA 3.5: BATAZ BESTEKO BALIOAREN TEOREMA

Izan bitez $D\subset\mathbb{R}^2$ eremu elementala eta $f: D\to\mathbb{R}$ jarraitua

$\Rightarrow \exists\, \bar{x}_0 \in D$ non $\iint_D f\,dA = f(x_0,y_0)\cdot A(D)$

PROPOSIZIOA 3.6: INTEGRAL BIKOITZA ETA SIMETRIA

Izan bitez D OX (OY) ardatzarekiko simetrikoa den 1. motako (2. motako) eremu elementala eta D^+ $y\geq 0$ $(x\geq 0)$ planoerdian agertzen den D-ren zatia.

1) f y (x) aldagaiaren BIKOITIA $f(x,-y)=f(x,y)$

$\Rightarrow \iint_D f\,dA = 2\iint_{D^+} f\,dA$

2) f y (x) aldagaiaren BAKOITIA $f(x,-y)=-f(x,y)$

$\Rightarrow \iint_D f\,dA = 0$

3.3. ALDAGAI-ALDAKETA INTEGRAL BIKOITZETAN

$$T = g: [a,b]\in\mathbb{R} \longrightarrow [g(a), g(b)]\subset\mathbb{R}$$
$$t \longrightarrow g(t) = x$$

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(t))g'(t)\,dt$$

$\iint_D f(x,y)\,dA$ integralean aldagai-aldaketa aplikatzeko:

$$T: D^*\subset\mathbb{R}^2 \longrightarrow D\subset\mathbb{R}^2$$
$$(u,v)\longrightarrow T(u,v) = (x(u,v), y(u,v)) \qquad (x,y)\in D$$

D emanik, D^* ateratzeko $T^{-1}(D)$ egin behar ($\exists T^{-1}$ behar)

DEFINIZIOA:

Izan bedi $T: D^*\subset\mathbb{R}^2\longrightarrow\mathbb{R}^2$ C^1 klasekoa non $T(u,v) = (x(u,v), y(u,v))$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} \Rightarrow \text{T-ren DETERMINANTE JACOBIARRA}$$

$J\neq 0 \Rightarrow \exists T$

TEOREMA 3.7: ALDAGAI-ALDAKETA INTEGRAL BIKOITZEAN

Izan bitez D a D^* planoko bi eremu elemental eta $T: D^* \to D$ C^1 klasekoak TRANSFORMAZIO INJEKTIBOA $T(D^*) = D$ izanik. Orduan f integragarria D-n

$$\iint_D f(x,y)\, dA = \iint_{D^*} f(x(u,v), y(u,v)) \cdot |J|\, du\, dv$$

DEFINIZIOA:

$(x,y) \in \mathbb{R}^2$ (x,y) puntuko KOORDENATU POLARRAK (ρ, θ)

$$T(\rho,\theta) = (x_0 + \rho\cos\theta,\ y_0 + \rho\sin\theta) \qquad \begin{matrix} \rho \in [0,+\infty) \\ \theta \in [0, 2\pi) \end{matrix}$$

$$J = \frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = [\dots] = \rho \Rightarrow \underline{J = \rho}$$

3.4. INTEGRAL BIKOITZEN APLIKAZIOAK

i) $A(D) = \iint_D 1\, dx\, dy \Rightarrow$ AZALERA

ii) $z = f(x,y)$ a $z = g(x,y)$ gainazalen arteko BOLUMENA $D \subset \mathbb{R}^2$-ren gainean $\Rightarrow$

$$B = \iint_D [f(x,y) - g(x,y)]\, dx\, dy$$

DENTSITATEA

iii) D-ren MASA $\Rightarrow$ $m(D) = \iint_D \rho(x,y)\, dx\, dy$

iv) D-ren MASA ZENTRUA $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\iint_D x\,\rho(x,y)\,dx\,dy}{m(D)} \quad ; \quad \bar{y} = \frac{\iint_D y\,\rho(x,y)\,dx\,dy}{m(D)}$$

v) $D \subset \mathbb{R}^2$ eremu elementala eta $f: D \to \mathbb{R}$ jarraitua. f-ren BATAZBESTEKO BALIOA D-n

$$[f]_m = \frac{\iint_D f\, dx\, dy}{A(D)}$$

4. INTEGRAL HIRUKOITZA

4.1. INTEGRAL HIRUKOITZA PARALELEPIPEDO BATEN GAINEAN

DEFINIZIOA

$B = [a,b]\times[c,d]\times[e,g] \in \mathbb{R}^3$ paralelepipedoa

$$\Rightarrow \iiint_B f \cdot dV = \iiint_B f(x,y,z)\,dx\,dy\,dz$$

TEOREMA 4.1:

Izan bitez $B=[a,b]\times[c,d]\times[e,g] \subset \mathbb{R}^3$ paralelepipedoa eta $f: B \to \mathbb{R}$. f jarraitua bada eta f-ren etenguneen multzoa bi aldagaiko funtzio jarraituen bildura FINITUA bada

$\Rightarrow$ f integragarria da B eremuan

TEOREMA 4.2:

Izan bitez $B = [a,b]\times[c,d]\times[e,g] \subset \mathbb{R}^3$ eta

$f: B \to \mathbb{R}$ integragarria

$\Rightarrow$ Edozein integral iteratu existitzen bada, hirukoitzaren berdina da:

$$\iiint_B f\,dV = \int_a^b\int_c^d\int_e^g f\,dz\,dy\,dx = \int_a^b\int_e^g\int_c^d f\,dy\,dz\,dx = \int_e^g\int_c^d\int_a^b f\,dx\,dy\,dz \Rightarrow \text{INTEGRAL ITERATUAK}$$

4.2. INTEGRAL HIRUKOITZA ESKUALDE OROKORRAGOETAN

DEFINIZIOA: ESKUALDE ELEMENTALAK

Izan $W \subset \mathbb{R}^2$ bornatua

i) W I MOTATAKO E.E.: $\exists D \subset \mathbb{R}^2$ E.E (EREMU) eta $\gamma_1, \gamma_2: D \to \mathbb{R}$ jarraitua non $W = \{(x,y,z) \subset \mathbb{R}^3 : (x,y) \in D,\ \gamma_1(x,y) \leq z \leq \gamma_2(x,y)\}$

ii) W II MOTAKO E.E.: $\exists D \subset \mathbb{R}^2$ E.E $\wedge$ $\gamma_1, \gamma_2: D \to \mathbb{R}$ jarraitua non $W = \{(x,y,z) \in \mathbb{R}^2 : (y,z) \in D,\ \gamma_1(y,z) \leq x \leq \gamma_2(y,z)\}$

iii) W III MOTAKO E.E: $\exists D \in \mathbb{R}^2$ E.E $\wedge$ $\gamma_1, \gamma_2 : D \to \mathbb{R}$ jarraitua non $W = \{(x,y,z) \in \mathbb{R}^2 : (x,z) \in D,\ \gamma_1(x,z) \leq y \leq \gamma_2(x,z)\}$

iv) W IV MOTAKO E.E da I, II eta III motakoa bada.

PROPOSIZIOA 4.3: INTEGRAL HIRUKOITZAREN OINARRIZKO PROP.

Izan $W \subset \mathbb{R}^3$ E.E $f, g : W \to \mathbb{R}$ W-n integragarriak

i) $\iiint_W f + g \, dV = \iiint_W f \, dV + \iiint_W g \, dV$

ii) $\iiint_W \lambda f \, dV = \lambda \iiint_W f \, dV$

iii) $f \leq g \quad \forall (x,y,z) \in W \Rightarrow \iiint_W f \, dV \leq \iiint_W g \, dV$

iv) $W = \bigcup_{i=1}^{m} W_i$ non W_i E.E $\forall i = 1, \dots, m \; \wedge \; W_i \cap W_j = \emptyset \quad \forall i \neq j$

$$\Rightarrow \iiint_W f \, dV = \sum_{i=1}^{m} \iiint_{W_i} f \, dV$$

v) $|f|$ integragarria W-n $\Rightarrow \iiint_W f \, dV \leq \iiint_W |f| \, dV$

TEOREMA 4.4: BATAZBESTEKO BALIOAREN TEOREMA

Izan bitez $W \subset \mathbb{R}^3$ E.E finituen bildura $\wedge$ $f : W \to \mathbb{R}$ J

$\Rightarrow \exists (x_0, y_0, z_0) \in W$ non $\iiint_W f \, dV = f(x_0, y_0, z_0) \cdot \iiint_W 1 \, dV$ → BOLUMENA

4.3. ALDAGAI-ALDAKETA INTEGRAL HIRUKOITZETAN

DEFINIZIOA: $T : \mathbb{R}^3 \to \mathbb{R}^3$ C^1 klasekoa

Transformazioa $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$

T-ren DETERMINANTE JACOBIARRA hau da

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

TEOREMA 4.5: ALD-ALD INTEGRAL HIRUKOITZETAN

Izan $W \wedge W^* \subset \mathbb{R}^3$ E.E $T : W^* \to W$ C^1 motako T injektiboa eta $T : W \to \mathbb{R}$ integragarria

BALIO ABS

$$\iiint_W f(x,y,z) \, dV = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot |J| \, du\,dv\,dw$$

DEFINIZIOA:

$(x,y,z) \in \mathbb{R}^3$ puntu bat emanik (ρ, θ, z)

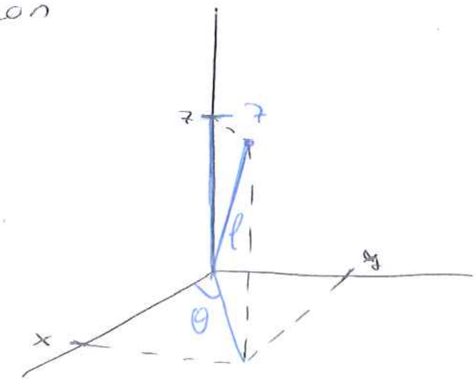
KOORDENATU ZILINDRIKOAK dira non

$T(\rho,\theta,z) = (x_0 + \rho\cos\theta,\ y_0 + \rho\sin\theta,\ z_0 + z)$

$(x_0, y_0, z_0) \Rightarrow$ ZENTRUA

$\theta \in [0, 2\pi] \ \wedge\ \rho \geq 0$

$\boxed{J = \rho}$



DEFINIZIOA:

$(x,y,z) \in \mathbb{R}^3$ puntu bat emanik (ρ, θ, φ)

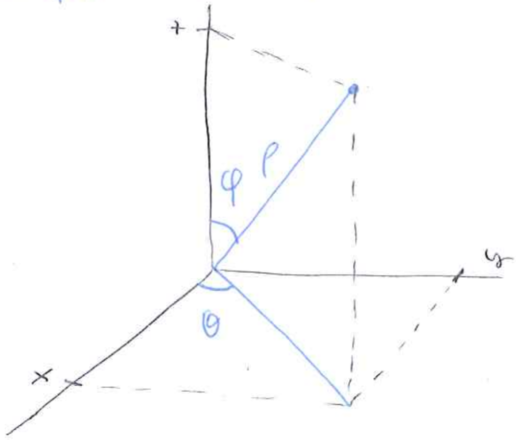
KOORDENATU ESFERIKOAK dira non

$T(\rho,\theta,\varphi) = (x_0 + \rho\cos\theta\sin\varphi,\ y_0 + \rho\sin\theta\sin\varphi,\ z_0 + \rho\cos\varphi)$

$(x_0, y_0, z_0) \Rightarrow$ ZENTRUA

$\theta \in [0, 2\pi] \ \wedge\ \varphi \in [0, \pi]$

$\boxed{J = -\rho^2 \sin\varphi}$



4.4. INTEGRAL HIRUKOITZAREN APLIKAZIOAK

$W \subset \mathbb{R}^3$

1) $B(W) = \iiint_W 1\, dx\, dy\, dz \longrightarrow$ W-ren BOLUMENA

2) $m(W) = \iiint_W \rho(x,y,z)\, dx\, dy\, dz \longrightarrow$ W-ren MASA

3) $(\bar{x}, \bar{y}, \bar{z}) \Rightarrow$ W-ren MASA ZENTRUA

$$\boxed{\bar{x}} = \frac{\iiint_W \boxed{x}\, \rho(x,y,z)\, dx\, dy\, dz}{m(W)}$$

4) L $\mathbb{R}^3$-KO ZUZENA

$\delta(x,y,z) \to (x,y,z) \in W$ puntutik L-rako distantzia izanik. W-ren L-rekiko MOMENTUA:

$$I_L = \iiint_W (\delta(x,y,z))^2 \rho(x,y,z)\,dx\,dy\,dz$$

$$I_x = \iiint_W (y^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz$$

$\hookrightarrow$ INERTZIA MOMENTUAK (ARDATZEKIKO)

5) PLANOEKIKO INERTZIA MOMENTUAK [Ikusi 11.orr]

6) $W \subset \mathbb{R}^3$ eremu elementala $\wedge$ $f: W \to \mathbb{R}$ jarraitua

$$[f]_m = \frac{\iiint_W f\,dV}{B(W)} \Rightarrow$$ f-ren BATAZBESTEKO BALIOA W eremuan

5. LERRO INTEGRALAK

1. IBILBIDEAK: ARKU-LUZERA

DEFINIZIOA

$$\sigma : [a,b] \subset \mathbb{R} \to \mathbb{R}^n$$
$$t \to \sigma(t) = (x_1(t), \dots, x_n(t)) \to \mathbb{R}^n\text{-ko IBILBIDEA}$$

- σ-ren IRUDIA ($t \in [a,b]$ denean) kurba bat da
- $\sigma(a)$ $\wedge$ $\sigma(b)$ $\to$ σ-ren MUTURRAK

DEFINIZIOA

$\sigma : I \subset \mathbb{R} \to \mathbb{R}^n$ C^1 klasekoa

i) $\sigma'(t) = (x_1'(t), \dots, x_n'(t)) \to$ σ-ren ABIADURA BEKTOREA

ii) $\|\sigma'(t)\| = \sqrt{\sum_{i=1}^{n} (x_i'(t))^2} \Rightarrow$ σ-ren ABIAZTASUNA

DEFINIZIOA

$\sigma : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ C^1 klasekoa

$$\sigma(t) = (x(t), y(t), z(t))$$

$$\ell(\sigma) = \int_a^b \|\sigma'(t)\| dt \Rightarrow \sigma\text{-ren ARKU-LUZERA}$$

OHARRA: Zatika C^1 bada ere balio du

2. LEHEN ETA BIGARREN MAILAKO LERRO INTEGRALAK

$f : D \subset \mathbb{R}^n \to \mathbb{R}$ funtzio eskalarra

$\vec{F} : D \subset \mathbb{R}^n \to \mathbb{R}^m$ funtzio bektoriala

DEFINIZIOA

$\sigma : [a,b] \subset \mathbb{R} \to \mathbb{R}^3$ C^1 klaseko ibilbidea

$f : \mathbb{R}^3 \to \mathbb{R}$ funtzio skalar jarraitua

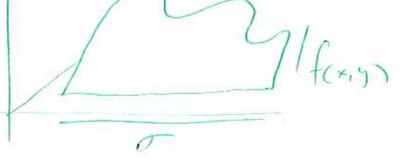
$$\int_\sigma f\, ds = \int_a^b f(\sigma(t)) \|\sigma'(t)\| dt \Rightarrow \text{LEHEN MAILAKO LERRO-INT.}$$

IBILBIDE INTEGRALA

ESANAHI GEOMETRIKOA

$f(x,y) \geq 0$

[$f(x,y)$ altuera]



$\int_\sigma f\, ds \equiv$ AZALERA
-hormarena-

PROP 5.1:

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ funtzio eskalarra

$\rho: \rho(\theta) \longrightarrow \sigma$ ibilbidearen ekuazio polarra

$$\theta \in [\theta_1, \theta_2] \Rightarrow \int_{\theta_1}^{\theta_2} f(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)\sqrt{\rho^2+(\rho')^2}\,d\theta$$

$\Rightarrow$ LEHEN MAILAKO LERRO INTEGRALA POLARRETAN

DEFINITZIOA

$\sigma: [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ C^1 klaseko ibilbidea

$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ funtzio bektorial jarraitua

$$\int_\sigma \vec{F}\,ds = \int_a^b \vec{F}(\sigma(t)) \overset{\text{ESKALARRA}}{\cdot} \sigma'(t)\,dt \Rightarrow \text{BIGARREN MAILAKO LERRO-INT}$$

OHARRAK:

1) Esanahi fisikoa: partikula $\sigma(a)$-tik $\sigma(b)$-ra mugitzean $\vec{F}$ indarrak horren gainean egiten duen lana

2) $\vec{F} = (F_1, F_2, F_3) \wedge \sigma(t) = (x(t), y(t), z(t))$

$$\sigma'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \Rightarrow \int_\sigma \vec{F}\,ds = \int_\sigma F_1\,dx + F_2\,dy + F_3\,dz$$

TEOREMA 5.2:

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ C^1 klasekoa $\wedge$ $\sigma: [a,b] \longrightarrow \mathbb{R}^3$ C^1 ibilbidea

$$\Rightarrow \int_\sigma \underline{\nabla f}\,ds = f(\sigma(b)) - f(\sigma(a))$$

$\vec{F}$ emanik $\exists f$ eskalarra non $\nabla f = \vec{F}$ orduan

f, $\vec{F}$-ren POTENTZIALA da.

3. BIRPARAMETRIZAZIOA

$\sigma \wedge \rho$ IBILBIDE desberdin $\wedge$ IRUDI berdinarekin

DEFINITZIOA

$h: [\alpha, \beta] \longrightarrow [a,b]$ C^1 funtzio bijektiboa

izan bitez $\sigma: [a,b] \longrightarrow \mathbb{R}^3 \wedge \rho: [\alpha,\beta] \longrightarrow \mathbb{R}^3$

non $\rho = \sigma \circ h \Rightarrow \rho$ σ-ren BIRPARAMETRIZAZIOA

DEFINIZIOA

$\sigma : [a,b] \longrightarrow \mathbb{R}^3$ ibilbidea

$h : [\alpha, \beta] \longrightarrow [a,b]$ C^1 funtzio bijektiboa

$\rho : [\alpha, \beta] \longrightarrow \mathbb{R}^3$ ibilbidean $\Rightarrow \rho = \sigma \circ h$

$\Rightarrow$

i) $\begin{cases} \rho(\alpha) = \sigma(a) \\ \rho(\beta) = \sigma(b) \end{cases} \Rightarrow \rho$-k ORIENTAZIOA MANTENDU

ii) $\begin{cases} \rho(\alpha) = \sigma(b) \\ \rho(\beta) = \sigma(a) \end{cases} \Rightarrow \rho$-k ORIENTAZIOA ALDATU

TEOREMA 5.3

$\sigma : [a,b] \longrightarrow \mathbb{R}^3$ C^1 ibilbidea

ρ, σ-ren birparametrizazioa

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ funtzio eskalar jarraitua

$$\Rightarrow \int_\sigma f\,ds = \int_\rho f\,ds$$

TEOREMA 5.4

$\sigma : [a,b] \longrightarrow \mathbb{R}^3$ C^1 ibilbidea

ρ, σ-ren birparametrizazioa

$\vec{F} = \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ funtzio bektorial jarraitua

$$\int_\sigma \vec{F}\,ds \underset{\text{Orientazioa mantendu}}{=} \int_\rho \vec{F}\,ds \quad \wedge \quad \int_\sigma \vec{F}\,ds \underset{\text{Orientazioa aldatu}}{=} -\int_\rho \vec{F}\,ds$$

4. LERRO-INTEGRALAK KURBA GEOMETRIKOETAN

DEFINIZIOA

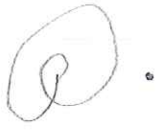
$\sigma : [a,b] \longrightarrow \mathbb{R}^3$ zatika C^1 $\wedge$ injektiboa

σ-ren IRUDIA Γ kurba sinple bat da

eta σ Γ-ren parametrizazio bat da.

- $\sigma(a), \sigma(b) \rightarrow \Gamma$-ren MUTURRAK
- Γ-k bi orientazio
- Γ orientazio batekin $\Rightarrow$ KURBA SINPLE NORABIDETUA

DEFINIZIOA



$$\left.\begin{array}{l} \sigma:[a,b]\to\mathbb{R}^3 \text{ zatika } C^1 \\ \sigma(a)=\sigma(b) \ \wedge\ \Gamma \ \sigma\text{-ren IRUDIA} \end{array}\right\} \Rightarrow \Gamma \text{ KURBA ITXIA}$$

$$\left.\begin{array}{l} \sigma:[a,b]\to\mathbb{R}^3 \text{ zatika } C^1 \\ \sigma(a)=\sigma(b) \text{ INJEKTIBOA} \\ \Gamma,\ \sigma\text{-ren IRUDIA} \end{array}\right\} \Rightarrow \Gamma \text{ KURBA SINPLE ITXIA}$$

DEFINIZIOA

Γ kurba sinple norabideztua σ orientazioa mantentzen duen Γ-ren parametrizazioa

$f:\mathbb{R}^3\to\mathbb{R} \ \wedge\ F:\mathbb{R}^3\to\mathbb{R}^3$ jarraituak

$$\Rightarrow \begin{cases} \int_\Gamma f\,ds = \int_\sigma f\,ds = \int_a^b f(\sigma(t))\,\|\sigma'(t)\|\,dt \\ \int_\Gamma \vec{F}\,ds = \int_\sigma \vec{F}\,ds = \int_a^b \vec{F}(\sigma(t))\cdot\sigma'(t)\,dt \end{cases}$$

TEOR 5.5:

Γ kurba sinple norabideztua Γ^- kontrako norantzan

$\vec{F}:\mathbb{R}^3\to\mathbb{R}^3$ jarraitua $\Rightarrow \int_\Gamma \vec{F}\cdot ds = -\int_{\Gamma^-}\vec{F}\,ds$

OHARRA: f jarraitua $\Rightarrow$ berdin

TEOREMA 5.6:

$\Gamma_i,\ i=1,\dots,m$ K.S.N. $\wedge$ Γ_i-ren bukaerako muturra Γ_{i+1}-en hasierakoa.

$\Rightarrow \Gamma = \Gamma_1+\Gamma_2+\dots+\Gamma_m \ \wedge\ \vec{F}:\mathbb{R}^3\to\mathbb{R}^3$ jarraitua

$$\Rightarrow \int_\Gamma \vec{F}\,ds = \sum_{i=1}^{m}\int_{\Gamma_i}\vec{F}\,ds$$

6. GAINAZAL INTEGRALAK

1. GAINAZAL PARAMETRIZATUAK, AZALERA

DEFINIZIOA

$\phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\qquad S = \phi(D)$ GAINAZALA

$(u,v) \longrightarrow \phi(u,v) = (x(u,v), y(u,v), z(u,v))$

↑

GAINAZAL PARAMETRIZATUA

· ϕ $C^1 \Rightarrow S$ *diferentziagarria* v C^1

· $\phi : D \longrightarrow \mathbb{R}^3$

$(u_0, v_0) \in D$ *puntu diferentziagarria* $\Rightarrow$

ABIADURA BEKTOREA

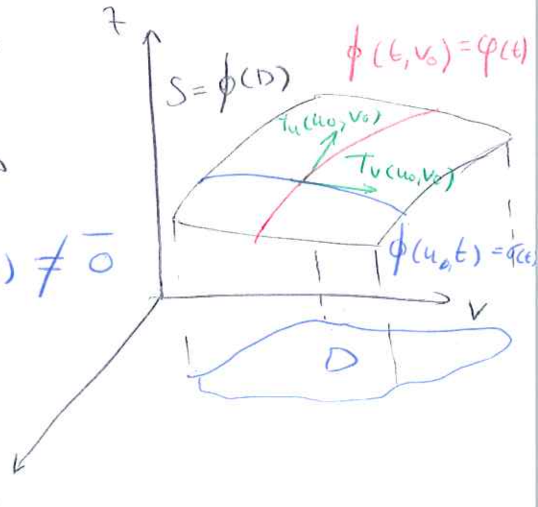
$$T_u(u_0,v_0) = \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right)$$

$$T_v(u_0,v_0) = \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right)$$

DEFINIZIOA:

$\phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ C^1 *gainazal parametrizatua*. $S = \phi(D)$ GAINAZALA

$\phi(u_0,v_0)$ *puntuan* LEUNA $\Longleftrightarrow T_u \times T_v(u_0,v_0) \neq \bar{0}$



DEFINIZIOA:

$\phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ C^1 *gain. par.*

$(u_0,v_0) \in D \wedge \phi(u_0,v_0)$ *puntuan gainazala leuna da*

$S = \phi(D)$*-ren* PLANO UKITZAILEA

$\phi(u_0,v_0) = (x_0, y_0, z_0)$ $\qquad \vec{n} = \dfrac{T_u \times T_v}{\|T_u \times T_v\|}$

$(x - x_0, y - y_0, z - z_0) \cdot \vec{N} = 0$ *non* $\vec{N} = T_u \times T_v(u_0,v_0)$

OHARRA: *S gainazala* $G(x,y,z)$ *funtzio baten maila gainazal bezala adierazi badaiteke*

[$\exists k \in \mathbb{R}$ *non S* $G = k$ *ekuazioak definitzen duen*]

$\Rightarrow \vec{n} = \dfrac{\nabla G}{\|\nabla G\|}$ ← UNITARIOA

- DEFINIZIOA

$\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ gainazal parametrizatu leuna D-n

$\wedge$ $S = \phi(D)$. S-ren AZALERA:

$$A(S) = \iint_D \| T_v \times T_u \| \, du \, dv$$ [INT. BIKOITZA] (Zatikako era)

OHARRA: Aurkitu badaiteke $g : \tilde{D} \subset \mathbb{R}^2 \to \mathbb{R}$

C^1 klasekoa non $g(\tilde{D}) = S$

$$\Rightarrow A(S) = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2} \, du \, dv$$

2. LEHEN ETA BIGARREN MAILAKO GAINAZAL-INTEGRALAK

DEFINIZIOA:

$\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ gainazal parametrizatu leuna

$\phi(u,v) = (x(u,v), y(u,v), z(u,v))$ non $S = \phi(D)$

f jarraitua $\Rightarrow$

$$\iint_S f \, dS = \iint_D f(\phi(u,v)) \, \| T_u \times T_v \| \, du \, dv$$ $\Rightarrow$ f-ren S GAINEKO LEHEN MAILAKO GAINAZAL INTEGRALA

OHARRAK:

1) $f(x,y,z) = 1 \Rightarrow \iint_S 1 \, dS = A(S)$

2) Aurkitu ahal badugu $g : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ C^1 klasekoa

non $g(D) = S$.

Proiekzioa

deribatu partzialak

[...] $$\iint_S f(x,y,z) \, dS = \iint_{D} f(x, y, g(x,y)) \cdot \sqrt{1 + (g_x)^2 + (g_y)^2} \, dx \, dy$$

DEFINIZIOA

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ funtzio bektoriala jarraitua

$\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ gainazal parametrizatu leuna

$$\iint_\phi \vec{F} \, dS = \iint_D \vec{F}(\phi(u,v)) \cdot T_u \times T_v \, du \, dv$$ $\Rightarrow$ $\vec{F}$-ren ϕ-ren GAINEKO 2. MAILAKO GAINAZAL INTEGRALA

$$\vec{F} = \iint_S \vec{F} \, dS = \iint_S -F_1 g_x - F_2 g_y + F_3 \, dx \, dy$$

DEFINIZIOA

S GAINAZAL NORABIDETUA da, bi bektitu, positiboa (kanpokoa) eta negatiboa (barrukoa)

DEFINIZIOA

$\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$

S gainazal parametrizatu bat izanik

ϕ bere parametrizazioa

$\vec{n}$ bek. normal unitarioa konponentz

$$\vec{n} = \pm \frac{T_u \times T_v}{\|T_u \times T_v\|}$$

$\oplus \Rightarrow \phi$ orientazioa mantendu || $\ominus \Rightarrow \phi$ orientazioa aldatu

TEOREMA 6.1:

S gainazal norabidetua $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ funtzio bektorial jarraitua ϕ_1 eta ϕ_2 gain. para. izanik

$$\Rightarrow \begin{cases} \phi_1 \wedge \phi_2 \text{ orientazioa mantendu} \Rightarrow \iint_{\phi_1} \vec{F} ds = \iint_{\phi_2} \vec{F} ds \\ \phi_1 \wedge \phi_2 \text{ orientazioa aldatu} \Rightarrow \iint_{\phi_1} \vec{F} ds = -\iint_{\phi_2} \vec{F} ds \end{cases}$$

DEFINIZIOA

- S gainazal norabidetua
- ϕ orientazioa mantendu $\Rightarrow$ $\iint_S \vec{F} ds = \iint_\phi \vec{F} ds$
- $\vec{F}$ funtzio jarraitua

$\vec{F}$-ren FLUXUA S-ren gainean

TEOREMA 6.2:

- S gainazal norabidetua
- $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^2$ jarraitua $\Rightarrow$ $\iint_S \vec{F} ds = \iint \vec{F} \cdot \vec{n} \, dS$

non $\vec{n}$ S-ren BEK. NOR. UNIT.

OHARRAK:

1) S gainazala era honetan definitu ahal bada

$\exists g : D \subset \mathbb{R}^2 \to \mathbb{R}$ non $g(D) = S$

$(x,y) \to g(x,y)$

$$\phi(x,y) = (x, y, g(x,y)) \qquad \vec{n} = \frac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$$

$$\vec{F} = (F_1, F_2, F_3) \Rightarrow \iint_S \vec{F} dS \stackrel{!}{=} \iint_S (-F_1 g_x - F_2 g_y + F_3) \, dx \, dy$$

2) $\vec{F}(F_1, F_2, F_3)$ jomaitua

$$\iint \vec{F}\, ds = \iint F_1\, dy\, dz + F_2\, dx\, dz + F_3\, dx\, dy$$

Jakin beharreko Kurba edo gainazalak

$\mathbb{R}^2$ Planoan

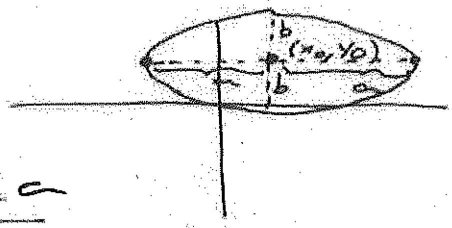
+ $ax+by+c=0 \qquad (a,b,c\in\mathbb{R})$
  zuzena

+ $ay+bx^2+cx+d=0 \qquad (a,b,c,d\in\mathbb{R})$
  parabola

+ $ax+by^2+cy+d=0 \qquad (a,b,c,d\in\mathbb{R})$
  parabola

+ $(x-x_0)^2+(y-y_0)^2=R^2 \qquad (x_0,y_0,R\in\mathbb{R})$
  (x_0,y_0) zentruko eta R erradioko
  zirkunferentzia

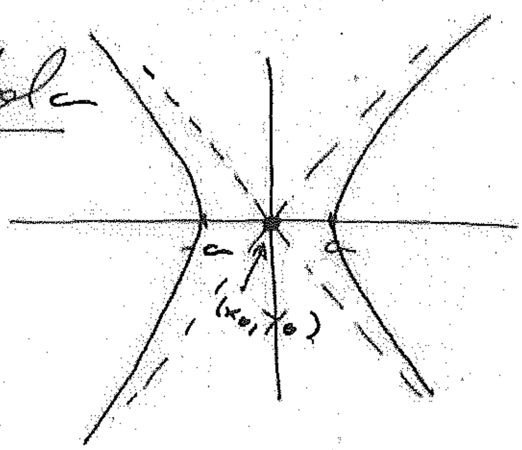
+ $$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$



  (x_0,y_0) zentruko elipsea
  $(x_0,y_0,a,b\in\mathbb{R})$

+ $$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$ hiperbola

  $(x_0,y_0,a,b\in\mathbb{R})$



LEMNISKATA

2

$\boxed{\mathbb{R}^3}$ espazioan

+ $ax+by+cz+d=0$ $\quad (a,b,c,d \in \mathbb{R})$

  Planoa

+ $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$ $\quad (x_0,y_0,z_0,R \in \mathbb{R})$

  (x_0,y_0,z_0) zentruko eta R erradioko esfera

+ $$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$

  (x_0,y_0,z_0) zentruko elipsoidea

  $(a,b,c,x_0,y_0,z_0 \in \mathbb{R})$

+ Paraboloideak
  + Zirkularrak

    $$(x-x_0)^2+(y-y_0)^2+c=k\cdot z$$
    $$(y-y_0)^2+(z-z_0)^2+c=k\cdot x$$
    $$(x-x_0)^2+(z-z_0)^2+c=ky$$

    $(x_0,y_0,z_0,c,k \in \mathbb{R})$

  + Eliptikoak

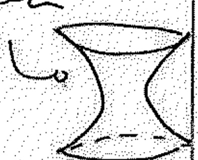
    $$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+d=kz$$
    $$\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}+d=k\cdot z$$
    $$\frac{(x-x_0)^2}{a^2}+\frac{(z-z_0)^2}{c^2}+d=kz$$

    $(x_0,y_0,z_0,a,b,c,d,k \in \mathbb{R})$

+ Hiperboloidea

  $$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=1$$

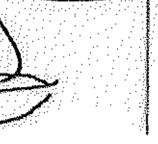
  Orri batekoa



  $$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=-1$$

  $(a,b,c,x_0,y_0,z_0 \in \mathbb{R})$

  Bi orritakoa



$\mathbb{R}^3$ Espazioan

+ Konoak

+ Zirkularra

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{a^2}-\frac{(z-z_0)^2}{c^2}=0$$

$\longrightarrow (a, b, c, x_0, y_0, z_0 \in \mathbb{R})$

+ Eliptikoa

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=0$$

+ Zilindroak

+ Zirkularrak

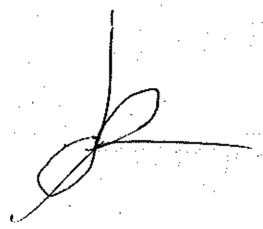
- $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{a^2}=1 \Longleftrightarrow (x-x_0)^2+(y-y_0)^2=a^2$
- $(y-y_0)^2+(z-z_0)^2=a^2 \qquad (x_0, y_0, z_0, a \in \mathbb{R})$
- $(x-x_0)^2+(z-z_0)^2=a^2$

+ Eliptikoak

- $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1 \qquad (x_0, y_0, z_0, a, b, c \in \mathbb{R})$
- $\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$
- $\frac{(x-x_0)^2}{a^2}+\frac{(z-z_0)^2}{c^2}=1$

LEMNISKATA

$$(x^2+y^2)^2 = a^2(x^2-y^2) \rightarrow$$



7. Analisi Bektorialeko Teoremak

7.1. Eragile bektorialak

DEF: Nabla eragilea

$$\bar{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

DEF: Gradientea

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \qquad f: \mathbb{R}^3 \to \mathbb{R} \quad C^1$$

DEF:

$$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3 \quad C^1 \qquad \vec{F} = (F_1, F_2, F_3)$$

Dibergentzia: $\operatorname{div}\vec{F} = \bar{\nabla}\cdot\vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

Errotazionala: $\operatorname{rot}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

[Propietateak]

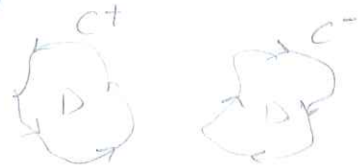
Teorema 7.1:

$$f: \mathbb{R}^3 \to \mathbb{R} \quad C^2 \Rightarrow \operatorname{rot}(\bar{\nabla} f) = \vec{0}$$

Teorema 7.2:

$$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3 \quad C^2 \Rightarrow \operatorname{div}(\operatorname{rot}\vec{F}) = 0$$

7.2. Greenen teorema



Teorema 7.5: Greenen teorema

7. fatika ere

$D \subset \mathbb{R}^3$ 3. motako eremu elementala C kurba muga eta $P, Q: D \subset \mathbb{R}^2 \to \mathbb{R}$ C^1 klaseko funtzioak

2. mailako lerro int. 5. gaia $\leftarrow$ $$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$ $\to$ integral bikoitza 3. gaia

Teorema 7.6: Planoko eremu baten azalera

$D \subset \mathbb{R}^2$ 3. motako eremu eta ∂D bere muga ori

$$A(D) = \frac{1}{2}\int_{\partial D} x\,dy$$

$$\int_\sigma f\,ds = \int_a^b f(\sigma(t))\cdot\|\sigma'(t)\|\,dt$$

$$\int_\sigma \vec{F}\,ds = \int_a^b \vec{F}(\sigma(t))\cdot\sigma'(t)\,dt$$

Teorema 7.7: Greenen ;

$D \subset \mathbb{R}^2$ greenen hipotesia

$$\int_\sigma \vec{F}\,ds = \int_\sigma F_1\,dx + F_2\,dy + F_3\,dz$$

2. mailako lerro int. 5. gaia $\leftarrow$ $$\int_{\partial D} \vec{F}\,ds = \iint_D \operatorname{ro}$$

$$\int_\sigma \nabla f\,ds = f(\sigma(b)) - f(\sigma(a))$$

Teorema 7.8: Dibergentziaren teorema planoan

$D \subset \mathbb{R}^2$ Greenen hipotesiak, ∂D bere muga:

- $\vec{n}$ ∂D-ren bektore normal unitarioa kanporantz
- $\sigma(t) = (x(t), y(t))$ orientazioa mantendu ∂D-ren parametr.

$$\Rightarrow \vec{n} = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} \Rightarrow \int_{\partial D} \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA$$

7.3. Stokesen teorema

Teorema 7.9: Stokesen

$D \subset \mathbb{R}^2$ Greenen hipotesi

$g: D \subset \mathbb{R}^2 \to \mathbb{R}$ C^1 ...

1. mailako gainazal int, 6. gaia $\Leftarrow$ $\Rightarrow \iint_S \operatorname{rot} \vec{F} ds = \int_{\partial S} \vec{F}$

$$\iint_S f\, ds = \iint_D f(\phi(u,v)) \cdot \| T_u \times T_v \|\, du\, dv$$

$$\iint_S f\, ds = \iint_D f(x, y, g(x,y)) \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy$$

$$\iint_\phi \vec{F} ds = \iint_D \vec{F}(\phi(u,v)) \cdot T_u \times T_v\, du\, dv$$

$\to$ FLUXUA $\iint_S \vec{F} ds = \iint \vec{F} \cdot \vec{n}\, dS$

$$\iint_\phi \vec{F} ds = \iint -F_1 g_x - F_2 g_y + F_3\, dx\, dy$$

Teorema 7.10: Stokese

$\phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ injek

$S = \phi(D)$ gainazal parametrizatu norabidetua

2. mailako gainazal-int, 6. gaia $\Leftarrow$ $\Rightarrow \iint_S \operatorname{rot} \vec{F} ds = \int_{\partial S} \vec{F} dS \to$ 2. mailako lerro-int, 5. gaia

7.4 Eremu kontserbakorrak $\to$ ZIRKULAZIOA

Teorema 7.11:

$\vec{F}$ C^1 funtzio bektoriala $\mathbb{R}^2$ - puntu kop. finitu eremuan definitua

Baliokideak:

i) $\int_C \vec{F} ds = 0$ $\forall$ kurba sinple itxietarako

ii) $\int_{C_1} \vec{F} ds = \int_{C_2} \vec{F} ds$ C_1, C_2 kurba sinple norabidetuak eta mutur berdinekin

iii) $\exists f$ non $\nabla f = \vec{F}$ f, $\vec{F}$-ren potentziala

iv) $\operatorname{rot} \vec{F} = \vec{0}$

DEF: $\vec{F}$ kontserbakorra

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ C^1 $\mathbb{R}^3$ - puntu kop. finitu

$\int_C \vec{F} ds = 0$ $\forall C$ kurba sinple itxi

DEF: $\vec{F}$ IRROTAZIONALA

$$\text{rot}\,\vec{F} = \vec{0}$$

OHARRA:

1) $\vec{F}$ IRROTAZIONALA $\Longleftrightarrow$ $\vec{F}$ KONTSERBAKORRA

2) $\vec{F} = (P, Q)$ C^1 $\quad \vec{k} \to (0,0,1)$

$$\vec{F} = (P, Q, 0) \Rightarrow \text{rot}\,\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot \vec{k}$$

KOROLARIOA 7.12:

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^2$ C^1 $\wedge$ $\vec{F} = (P, Q)$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \exists f: \mathbb{R}^2 \to \mathbb{R} \; C^1 \text{ non } \nabla f = \vec{F}$$

TEOREMA 7.13:

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ C^1 $\wedge$ $\text{div}\,\vec{F} = 0$

$$\Rightarrow \exists \vec{G}: \mathbb{R}^3 \to \mathbb{R}^3 \; C^1 \text{ non rot}\,\vec{G} = \vec{F}$$

Nola?

$$\vec{G} = (G_1, G_2, G_3) \quad \begin{cases} G_1(x,y,z) = \int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt \\ G_2(x,y,z) = -\int_0^z F_1(x,y,t)\,dt \\ G_3(x,y,z) = 0 \end{cases}$$

7.5. GAUSSEN TEOREMA

Dibergentziaren teorema: $\int_{\partial D} \vec{F} \cdot \vec{n}\, ds = \iint_D \text{div}\,\vec{F}\, dA$

TEOREMA: 7.14 GAUSSEN DIBERGENTZIAREN TEOREMA

$\Omega \subset \mathbb{R}^3$ IV motako (edo 3 motakoak) eremu elementala, $\vec{F}: \Omega \to \mathbb{R}^3$ C^1,

$\partial\Omega$ Ω-ren gainazal itxi norabidetua orientazio positiboarekin

2. MAILAKO GAINAZAL-INT 5. GAIA

$$\Rightarrow \iint_{\partial\Omega} \vec{F} ds = \iint_{\partial\Omega} \vec{F} \cdot \vec{n}\, ds = \iiint_{\Omega} \text{div}\,\vec{F}\, dV \to$$

INTEGRAL HIRUKOITZA 4. GAIA

$\iint_{\partial\Omega} \vec{F} ds$ = FLUXUA

$\iint_{\partial\Omega} \vec{F} \cdot \vec{n}\, ds$: 1. MAILAKO GAINAZAL-INT 5. GAIA

OHARRA:

1) $\text{div}\,\vec{F} = 0 \Longleftrightarrow \iint_S \vec{F} ds$

2) $\text{div}\,\vec{F} > 0 \Rightarrow$ FLUXUA ATERA

3) $\text{div}\,\vec{F} = 0 \Rightarrow$ Kanpoantz $\wedge$ barruantz fluxu kant. berdina

ANALISI BEKTORIALA ETA KONPLEXUA

1. MUTURRAK

1.1. DERIBATU PARTZIALAK

Gogoratu $f: U \subset \mathbb{R}^1 \longrightarrow \mathbb{R}$ deribagarria

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h\cdot 1) - f(x)}{h}$$

DEFINIZIOA

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ n aldagaiko funtzio erreala eta

$\bar{x} = (x_1, \dots, x_n) \in U$ puntu bat

$$f_{x_j}(\bar{x}) = \frac{\partial f}{\partial x_j}(\bar{x}) = \lim_{h \to 0} \frac{f(\bar{x} + h(\bar{e}_j) - f(\bar{x})}{h}$$ non

$$\bar{e}_j = (0, \dots, 0, \overset{\to\, j.\ \text{GAIAN}}{1}, 0, \dots, 0)$$

$\frac{\partial f}{\partial x_j}(\bar{x}) \longrightarrow$ f-ren lehen ordenako deribatu partziala x_j aldagaiarekiko $\bar{x}$ puntuan.

OHARRA! Praktikan, $\frac{\partial f}{\partial x}$ kalkulatzeko, x_j ez diren aldagaiak konstanteak direla suposatzen da.

ADIBIDEA:

$$f(x,y) = x^2y + y^3$$

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = 3x^2 + x^2$$

DEFINIZIOA

$\bar{f}: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ n aldagaiko funtzio bektoriala

$$\bar{f}(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

$\bar{f}$ DIFERENTZIAGARRIA da $\bar{x}_0 \in U$ puntuan:

1) $\bar{f}$-ren deribatu partzialak $\bar{x}_0$-n existitzen badira

2) $$\lim_{\bar{x} \to \bar{x}_0} \frac{\|\bar{f}(\bar{x}) - \bar{f}(\bar{x}_0) - Df(x_0)\cdot(\bar{x} - \bar{x}_0)\|}{\|\bar{x} - \bar{x}_0\|} = 0$$

non $$D\bar{f}(\bar{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\bar{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\bar{x}_0) \end{pmatrix}_{m\times n}$$

OHARRA: $m = 1$ bada,

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ eta

$$Df(\bar{x}_0) = \nabla f_{(\bar{x}_0)} = \left(\frac{\partial f}{\partial x_1}(\bar{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\bar{x}_0)\right)$$

TEOREMA 1.1:

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ eta f-ren lehen ordenako deribatu partzialak existitzen badute eta jarraiak badira $\bar{x} \in U$-n $\Rightarrow \bar{f}$ DIFERENTZIAGARRIA da $\bar{x}$ puntuan

DEFINIZIOA:

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ eta $\frac{\partial f}{\partial x_i}$ $\forall i = 1, \ldots, n$ existitzen dira eta eta jarraituak badira, f C^1 KLASEKOA dela esaten da.

TEOREMA 1.2: KATEAREN ERREGELA

$\bar{f}: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\bar{g}: V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^p$

non $\bar{f} \circ \bar{g}$ ondo definituta dagoen

- $\bar{g}$ diferentziagarria $\bar{x}_0$ puntuan
- $\bar{f}$ diferentziagarria $\bar{g}(\bar{x}_0)$ puntuan

$$\Rightarrow \underbrace{D(\bar{f} \circ \bar{g})(\bar{x}_0)}_{m\times p} = \underbrace{Df(\bar{g}(\bar{x}_0))}_{m\times n} \cdot \underbrace{D\bar{g}(\bar{x}_0)}_{n\times p}$$

ADIBIDEA

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\bar{g}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad \bar{g}(x,y) = (\underbrace{2x - y}_{u(x,y)}, \underbrace{x^2 + y^2}_{v(x,y)})$

KALKULATU $D(\bar{f} \circ \bar{g})(\bar{x}_0)$

$$h(x,y) = (f\circ \bar{g})\cdot(x,y) = f(\bar{g}(x,y)) = f(u(x,y), v(x,y))$$

$$h: \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

KATEAREN ERREGELA

$$Dh(\bar{x}_0) = D(f\circ \bar{g})(\bar{x}_0) = Df(\bar{g}(\bar{x}_0))\cdot D\bar{g}(\bar{x}_0)$$

$$\begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}}_{1\times 2} \cdot \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}}_{2\times 2} =$$

$$= \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 2x & 2y \end{pmatrix} = 2\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v} \quad -\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}$$

$$\Rightarrow \quad \frac{\partial h}{\partial x} = 2\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v}$$

$$\frac{\partial h}{\partial y} = -\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}$$

- BESTE ERA BAT

$$h(x,y) = f(u(x,y), v(x,y))$$

DERIBATU

$$h \to f \begin{cases} u \begin{cases} x \\ y \end{cases} \\ v \begin{cases} x \\ y \end{cases} \end{cases}$$

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\cdot \underbrace{\frac{\partial u}{\partial x}}_{"2} + \frac{\partial f}{\partial v}\cdot \underbrace{\frac{\partial v}{\partial x}}_{"2x}$$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial u}\cdot \underbrace{\frac{\partial u}{\partial y}}_{"-1} + \frac{\partial f}{\partial v}\cdot \underbrace{\frac{\partial v}{\partial y}}_{"2y}$$

1.2. GOI ORDENAKO DERIBATUAK

DEFINIZIOA:

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

f C^2 KLASEKOA da bigarren ordenako deribatu partzial denak existitzen badira eta jarraituak badira.

- $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad C^2$ KLASEKOA

f-ren BIGARREN ORDENAKO DERIBATU PARTZIALAK

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \quad \text{non} \quad i, j = 1, \ldots, n$$

- $i = j \longrightarrow \dfrac{\partial^2 f}{\partial x_i^2}$
- $i \neq j \longrightarrow \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ eta $\dfrac{\partial^2 f}{\partial x_j \partial x_i} \Rightarrow$ DERIBATU PARTZIAL GURUTZATUAK

ADIBIDEA:

$\boxed{n=2}$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad f(x,y)$ Bigarren ordenako deribatuak

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

ADIBIDEA

$$f(x,y) = xy + (x+2y)^2$$

$\left(\begin{array}{l} n = 2 \\ C^2 \text{ KLAS} \end{array} \right)$ → Polinomioa, ∞ aldiz deriba.

LEHEN ORDENAKO DERIBATU PARTZIALAK

$$\frac{\partial f}{\partial x} = y + 2\cdot(x+2y) = y + 2x + 4y = 5y + 2x$$

$$\frac{\partial f}{\partial y} = x + 2\cdot(x+2y)\cdot 2 = x + 4x + 8y = 5x + 8y$$

Bigarren ordenako deribatuak

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\cdot\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\cdot(2x+5y) = 2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\cdot\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(5x+8y) = 8$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(5x+8y) = 5$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x+5y) = 5$$

TEOREMA 1.3:

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad C^2$ KLASEKOA

$$\Longrightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \forall i,j = 1,\dots,n$$

[ORDENAK EZ DU INPORTA]

OHARRAK

1) f C^m KLASEKOA da $(m \in \mathbb{N})$ m ordenako deribatu partzial denak existitzen badira eta jarraiak badira.

2) f C^m bada, deribatu partzialak egitekoan ordenak ez du inporta

3) Orokorrean, f k aldiz deribatzea
k_1 aldiz x_1-ekiko
...
k_n aldiz x_n-rekiko

$$\Rightarrow \frac{\partial^k f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

ADIBIDEA

$$f(x,y,z) = e^z + x e^{-y}$$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\wedge$ f C^3 KLASEKOA [Esponentziala polinomikoa]

$\frac{\partial^3 f}{\partial x \partial y^2}$ KALKULATU. Ordenak inporta ez duenez 2 era.

$$\bullet \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(-x e^{-y}\right)\right) =$$

$$= \frac{\partial}{\partial y}\left(-e^{-y}\right) = e^{-y}$$

$$\bullet \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(e^{-y}\right)\right) =$$

$$= \frac{\partial}{\partial y}\left(-e^{-y}\right) = e^{-y}$$

1.3. TAYLORREN TEOREMA

Gogoratu $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}\cdot(x-x_0)^2 + \dots + \frac{f^{k)}}{k!}(x-x_0)^k +$$

$$+ R_k(x, x_0) \quad \text{non} \quad \lim_{x \to x_0} \frac{R_k(x, x_0)}{(x-x_0)^k} = 0$$

TEOREMA 1.4: LEHEN ORDENAKO TAYLORREN TEOREMA

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $\bar{x}_0 \in U$ f dif. eta

$\bar{h} = (h_1, \dots, h_n) = \bar{x} - x_0$ bada

$$f(\bar{x}) = f(\bar{x}_0 + h) = f(\bar{x}_0) + \sum_{i=1}^{n} h \cdot \frac{\partial f}{\partial i}(\bar{x}_0) + R_1(\bar{h}, \bar{x}_0) =$$

$$= f(\bar{x}_0) + Df(\bar{x}_0) \underset{\substack{\downarrow \\ \text{ESKALARRA}}}{\cdot} h + R_1(\bar{h}, \bar{x}_0) \quad \text{non} \quad \lim_{\bar{h} \to \bar{0}} \frac{R_1(\bar{h}, \bar{x}_0)}{\|\bar{h}\|} = 0$$

OHARRA:

$f: \mathbb{R}^2 \to \mathbb{R}$ dif

f-ren lehen ordenako Taylorren hurbilketa

$\bar{x}_0 = (x_0, y_0)$ puntuaren ingurune batean

$$f(x,y) \sim f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)$$

TEOREMA 1.5: BIGARREN ORDENAKO TAYLORREN TEOREMA

$f: U \subset \mathbb{R}^n \to \mathbb{R}$ C^2 KLASEKOA

$$f(\bar{x}_0 + \bar{h}) = f(\bar{x}_0) + \sum_{i=1}^{n} h_i \cdot \frac{\partial f}{\partial x_i}(\bar{x}_0) + \frac{1}{2!}\sum_{i,j=1}^{n} h_i h_j \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(\bar{h}, \bar{x}_0)$$

$$\text{non} \quad \lim_{\bar{h} \to \bar{0}} \frac{R_2(\bar{x}, \bar{x}_0)}{\|\bar{h}\|^2} = 0$$

OHARRA

f-ren bigarren ordenako Taylorren hurbilketa (x_0,y_0)-n

$$f(x,y) \sim f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$
$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\cdot(x-x_0)^2 + \frac{1}{2!}\cdot\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\cdot(y-y_0)^2$$
$$+ \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\cdot(x-x_0)\cdot(y-y_0)$$

ADIBIDEA

$f(x,y) = \sin(xy)$ $\quad (1, \frac{\pi}{2})$ PUNTUAN

Bilatu Taylorren lehen eta bigarren ordeneko hurbilketa

$$\frac{\partial f}{\partial x} = \cos(xy)\cdot y \quad \wedge \quad \frac{\partial f}{\partial y} = \cos(xy)\cdot x$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(xy)y^2 \quad \wedge \quad \frac{\partial^2 f}{\partial y^2} = -\sin(xy)x^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin(xy)\cdot y^2 + \cos(xy)$$

PUNTUA ORDEZKATU

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial^2 f}{\partial x^2} = \frac{-\pi^2}{4}, \quad \frac{\partial^2 f}{\partial y^2} = -1, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{-\pi}{2}$$

- LEHEN ORDENAKOA

$$f(x,y) \sim \overbrace{f(1,\tfrac{\pi}{2})}^{1} + \cancel{\frac{\partial f}{\partial x}(1,\pi/2)(x-1)} + \cancel{\frac{\partial f}{\partial y}(1,\tfrac{\pi}{2})\cdot(y-\tfrac{\pi}{2})} = \boxed{1}$$

- BIGARREN ORDENAKOA

$$f(x,y) \simeq \overbrace{f(1,\pi/2)}^{1} + \cancel{\frac{\partial f}{\partial x}(1,\pi/2)(x-1)} + \cancel{\frac{\partial f}{\partial y}(1,\pi/2)(y-\pi/2)}$$

$$+ \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}\overbrace{(1,\pi/2)}^{-\pi^2/4}\cdot(x-1)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\overbrace{(1,\pi/2)}^{-1}(y-\pi/2)^2$$

$$+ \frac{\partial^2 f}{\partial x \partial y}\overset{-\pi/4}{\overbrace{(1,\pi/2)}^{1}}\cdot(x-1)(y-\pi/2)$$

4. MUTUR LOKALAK

DEFINIZIOA

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ eta $\bar{x}_0 \in U$ MUTUR LOKALAK:

i) x_0 f-ren MINIMO LOKALA da existitzen bada V x_0-ren ingurune bat non $f(\bar{x}) \geq f(\bar{x}_0)$ $\forall x \in V$

ii) x_0 f-ren MAXIMO LOKALA da existitzen bada V x_0-ren ingurune bat non $f(\bar{x}) \leq f(\bar{x}_0)$ $\forall x \in V$

TEOREMA 1.6:

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $\bar{x}_0$ puntuan diferentziagarria eta $\bar{x}_0$ f-ren mutur lokala bada,

$$\Rightarrow \nabla f(\bar{x}_0) = \bar{0} \qquad \left[\frac{\partial f}{\partial x_i}(\bar{x}_0) = 0 \quad \forall i = 1, \dots, n\right]$$

DEFINIZIOA

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $\bar{x}_0 \in U$ f-ren PUNTU KRITIKOA:

1) f ez bada x_0 puntuan diferentziagarria

2) f diferentziagarria $\bar{x}_0$-n eta $\nabla f(\bar{x}_0) = \bar{0}$

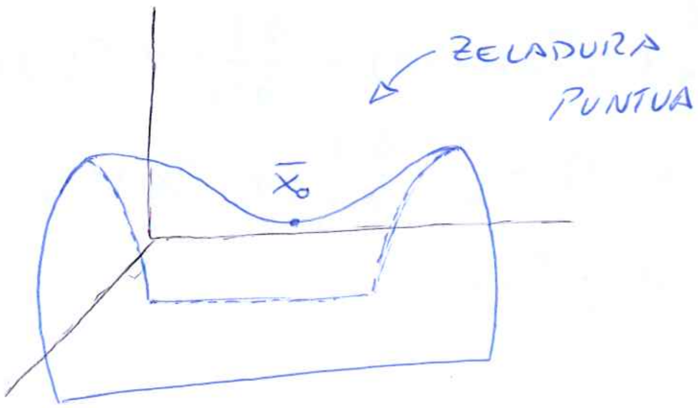
OHARRA

1) $\bar{x}_0$-n mutur lokala $\Rightarrow$ $\bar{x}_0$-n puntu kritikoa

2) $\bar{x}_0$-n puntu kritikoa $\not\Rightarrow$ $\bar{x}_0$-n mutur lokala

DEFINIZIOA

$\bar{x}_0 \in U$ f-ren puntu kritikoa bada eta ez de mutur lokala $\Rightarrow$ x_0 f-ren ZELADURA PUNTUA



DEFINIZIOA

$f: U \in \mathbb{R}^n \longrightarrow \mathbb{R}$ eta 2. ordeneko deribatu partzialak existitzen dira

$$Hf(x_0)(h_1,\dots,h_n) = \frac{1}{2}(h_1,\dots,h_n)\cdot \underbrace{\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\bar{x}_0) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\bar{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\bar{x}_0) \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\bar{x}_0) & & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\bar{x}_0) \end{pmatrix}} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

$\Rightarrow$ f-ren MATRIZE HESSIARRA

OHARRA

$n=2 \Rightarrow$ MATRIZE HESSIARRA: $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

TEOREMA 4.11

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ C^2 klasekoa, $(x_0, y_0) \in U$ eta

$D = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}$ = MATRIZE HESSIARRAREN DETERMINANT

1) $\bar{x}_0 = (x_0, y_0)$ MINIMO LOKALA:

- $\bar{x}_0$ PUNTU KRITIKOA
- $\frac{\partial^2 f}{\partial x^2}(\bar{x}_0) > 0$
- $D(\bar{x}_0) > 0$

2) $\bar{x}_0 = (x_0, y_0)$ MAXIMO LOKALA

- $\bar{x}_0$ PUNTU KRITIKOA
- $\frac{\partial^2 f}{\partial x^2}(\bar{x}_0) < 0$
- $D(\bar{x}_0) > 0$

3) $\bar{x}_0 = (x_0, y_0)$ ZELADURA PUNTUA

- $\bar{x}_0$ PUNTU KRITIKOA
- $D(\bar{x}_0) < 0$

OHARRA: 3 DIMENTSIO

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

1) MINIMO LOKALA

$|1\times1| > 0$, $|2\times2| > 0$, $|3\times3| > 0$

2) MAXIMO LOKALA

$|1\times1| < 0$, $|2\times2| > 0$ $|3\times3| < 0$

3) ZELADURA PUNTUA

$|1\times1| \neq 0$, $|2\times2| \neq 0$ $|3\times3| \neq 0$

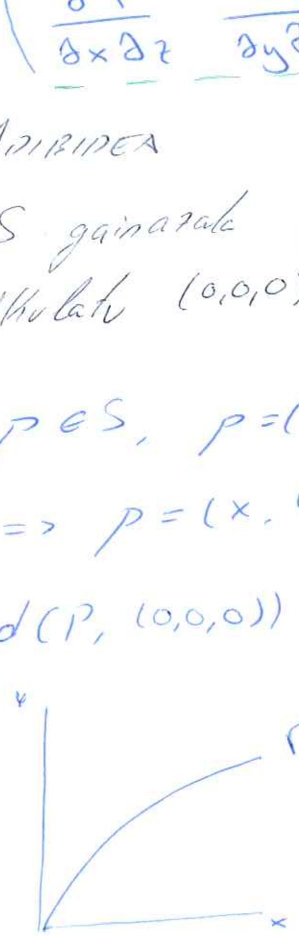
ARIKETA

• S gainazala $g(x,y) = \frac{1}{xy}$ funtzioaren grafikoa $\mathbb{R}^3$-n
Kalkulatu $(0,0,0)$ puntutik hurbilen dauden S-en puntuak
distantzia minimoa

$p \in S$, $p = (x, y, z) \in S \Rightarrow z = \frac{1}{xy}$

$\Rightarrow p = (x, y, \frac{1}{xy})$

$d(P, (0,0,0)) = \sqrt{(x-0)^2 + (y-0)^2 + (\frac{1}{xy} - 0)^2}$



$\sqrt{x}$-ren minimoa = x-en minimoa
d-ren minimoa aztertu beharrean
$f(x,y) = x^2 + y^2 + \frac{1}{x^2y^2}$ funtzioaren
minimoa aztertuko dugu

$\nabla f = \bar{0} \Rightarrow \frac{\partial f}{\partial x} = 2x - \frac{2}{x^3y^2} = \frac{2x^4y^2 - 2}{x^3y^2} = 0$

$\frac{\partial f}{\partial y} = 2y - \frac{2}{x^2y^3} = \frac{2x^2y^4 - 2}{x^2y^3} = 0$

$2x^4y^2-2=0 \Rightarrow x^4y^2=1 \Rightarrow y^2=\frac{1}{x^4}$ *

$2x^2y^4-2=0 \Rightarrow x^2y^4=1 \Rightarrow x^2\cdot\frac{1}{x^8}=1$

$\frac{1}{x^6}=1 \Rightarrow x^6=1 \Rightarrow x=\pm 1$

$x=1 \xrightarrow{*} y^2=1 \Rightarrow y=\pm 1$

$x=-1 \xrightarrow{*} y^2=1 \Rightarrow y=\pm 1$

PUNTU KRITIKOAK

$\overset{1}{(1,1)}, \overset{2}{(1,-1)}, \overset{3}{(-1,1)}, \overset{4}{(-1,-1)}$

$\frac{\partial^2 f}{\partial x^2}=2+\frac{6}{x^4y^2}>0$

$\frac{\partial^2 f}{\partial x\partial y}=\frac{4}{x^3y^3}$ $\quad \forall P_i : i=1,4$

$\frac{\partial^2 f}{\partial y^2}=2+\frac{6}{x^2y^4}>0$

$D(1,1)=(2+6)(2+6)-4^2=48>0$

$D(1,-1)=8\cdot 8-(-4)^2=48>0$

$D(-1,1)=48>0$

$D(-1,-1)=48>0$

$\xRightarrow{\text{TEOR 11}}$ P_1, P_2, P_3 ∧ P_4 MINIMO LOKALAK DIRA

$d(P_i,(0,0,0))=\sqrt{3} \quad \forall i=1,2,3,4$

ADIBIDEA

$f(x,y)=x^5y+xy^5+xy$ AZTERTU PUNTU KRITIKOAK

- f C^∞ klasekoa $\mathbb{R}^2$ osoan ⇒ ez daude puntu kritiko ez diferentziagarriak
- $\nabla f=\bar{0} \Rightarrow (0,0)$

⇒ $(0,0)$ puntu kritiko bakarra

$D(0,0)=-1<0 \xRightarrow{\text{TEOR 11}} (0,0)$ ZELADURA PUNTUA

ADIBIDEA

$f(x,y,z)=x^2+y^2+z^2+xy$ MUTUR LOKALAK

- f polinomioa denez, C^∞ klasekoa da $\mathbb{R}^3$-n

⇒ ez dago puntu kritiko ez-dif.

• $\nabla f = \bar{0} \Rightarrow \frac{\partial f}{\partial x} = 2x + y = 0$

$$\frac{\partial f}{\partial y} = 2y + x = 0 \qquad \Rightarrow (0,0,0)$$

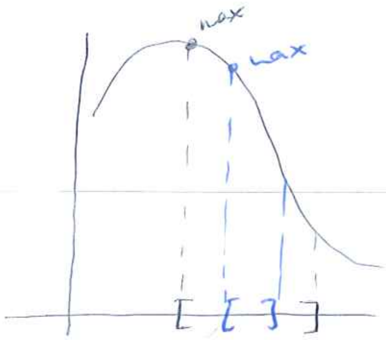
$$\frac{\partial f}{\partial z} = 2z = 0$$

$$H(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & & \\ \frac{\partial^2 f}{\partial x y} & & \\ & & \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$|1\times 1| = 2 > 0 \qquad |2\times 2| = 2\cdot 2 - 1 > 0 \qquad |3\times 3| = 2^3 - 2 > 0$

$\Rightarrow (0,0,0)$ MINIMO LOKALA

1.5. MUTUR BALDINTZATUAK



$g(\bar{x}) = c$ ekuazio baldintza izango dela esango dugo.

TEOREMA 12:

Izan bitez $f, g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ C^1 klasekoak.

$S = \{(x_1, \dots, x_n) \in U : g_i(x_1, \dots, x_n) = \overset{\in\mathbb{R}}{c_i},\ \forall i = 1, \dots, m\}$

eta $\bar{x}_0 \in S$ non $\nabla g_i(\bar{x}_0) \neq \bar{0}$ den $\forall i = 1, \dots, m$

Izan bedi $h(\lambda_1, \dots, \lambda_m, x_1, \dots, x_n) =$ LAGRANGEREN FUNTZIOA

$= f(x_1, \dots, x_n) - \lambda_1(g_1(x_1, \dots, x_n) - c_1) - \dots - \lambda_m(g_m(x_1, \dots, x_n) - c_m)$

↳ LAGRANGEREN biderkatzaileak

$\Rightarrow$ i) $f|_S$ funtzioak f (S-ra murriztuta) maximo eta minimo lokala du $\bar{x}_0$-n $\Rightarrow \exists \lambda_i \in \mathbb{R}$ ($\forall i = 1, \dots, m$) non $\nabla h(\bar{x}_0) = \bar{0}$

ii) $f|_S$ funtzioak $\bar{x}_0$ puntuan maximoa edo minimo lokala badu $\nabla f(\bar{x}_0) \perp S$ $\bar{x}_0$ puntuan

DEFINIZIOA:

Izan bitez $f, g : U \subset \mathbb{R}^{2=n} \to \mathbb{R}$ C^2 klasekoak $S = \{(x,y) : g(x,y) = c\}$ eta $\bar{x}_0 \in S$ non $\nabla g(\bar{x}_0) \neq \bar{0}$

Demagun $\exists \lambda \in \mathbb{R}$ non $h(\lambda, x, y) = f(x,y) - \lambda(g(x,y) - c)$ izanik $\nabla h(\bar{x}_0) = \bar{0}$

Kasu honetan, HESSIAR NUGATUA honako hau da:

$$|\bar{H}| = \begin{vmatrix} 0 & -\frac{\partial g}{\partial x} & -\frac{\partial g}{\partial y} \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial y \partial x} \\ -\frac{\partial g}{\partial y} & \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial y^2} \end{vmatrix}$$

$n > 2$

$$|\bar{H}| = \begin{vmatrix} 0 & -\frac{\partial g}{\partial x_1} & \cdots & -\frac{\partial g}{\partial x_n} \\ -\frac{\partial g}{\partial x_1} & \frac{\partial^2 h}{\partial x_1^2} & \cdots & \frac{\partial^2 h}{\partial x_n \partial x_1} \\ \vdots & & \ddots & \\ -\frac{\partial g}{\partial x_n} & \frac{\partial^2 h}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

TEOREMA 13:

Aurreko definizioaren ($n = 2$) kasuan notazioa eta baldintzak kontuan hartuz,

i) $|\bar{H}| > 0 \Rightarrow f_{|S}$-ak $\bar{x}_0$ puntuan maximo lokala du

ii) $|\bar{H}| < 0 \Rightarrow f_{|S}$-ak $\bar{x}_0$ puntuan minimo lokala du

OHARRA:

$|\bar{H}| = 0$ bada, ezin dugu ezer ziurtatu, beste zerbait erabili behar da.

OHARRA:

$|\bar{H}|$-ren minoreak aztertuz jakin dezakegu zein motatako puntuak ditugun.

i) $|3\times 3|<0$, $|4\times 4|<0$, ... $\Rightarrow \bar{x}_0$ MINIMO LOKALAK

ii) $|3\times 3|>0$, $|4\times 4|<0$, (alternatzen) $\Rightarrow \bar{x}_0$ MAXIMO LOKALA

iii) $|\text{Denak}| \neq 0 \wedge \bar{x}_0$ ez da max ez min

$\Rightarrow \bar{x}_0$ $f_{|S}$-ren ZELADURA PUNTUA

ADIBIDEA:

$f(x,y) = x^2 - y^2$-ren muturrak $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ en gainean

- f C^1 da $\mathbb{R}^2$-n [polinomioa]
- $g(x,y) = x^2+y^2 = 1$ C^2 klasekoa

Definitzen dugu:

$$h(\lambda, x, y) = f(x,y) - \lambda(g(x,y) - c) = x^2 - y^2 - \lambda(x^2+y^2-1)$$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} (1)\ \dfrac{\partial h}{\partial \lambda} = -(x^2+y^2-1) = 0 \\ (2)\ \dfrac{\partial h}{\partial x} = 2x - 2\lambda x = 0 \Rightarrow 2x(1-\lambda) = 0 < \begin{matrix} x=0 \\ \vee \\ \lambda = 1 \end{matrix} \\ (3)\ \dfrac{\partial h}{\partial y} = -2y - 2\lambda y = 0 \end{cases}$$

$$x = 0 \overset{(1)}{\Rightarrow} y^2 - 1 = 0 \Rightarrow y = \pm 1 \ *$$

$$\lambda = 1 \overset{(3)}{\Rightarrow} -2y - 2y = 0 \Rightarrow y = 0 \overset{(1)}{\Rightarrow} x = \pm 1 < \begin{matrix} (1,0) \\ (-1,0) \end{matrix}$$

$*$ $(0,1) \wedge (0,-1)$

$\boxed{n=2}$

$$|\bar{H}| = \begin{vmatrix} 0 & -\dfrac{\partial g}{\partial x} & -\dfrac{\partial g}{\partial y} \\ -\dfrac{\partial g}{\partial x} & \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ -\dfrac{\partial g}{\partial y} & \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 0 & -2x & -2y \\ -2x & 2(1-\lambda) & 0 \\ -2y & 0 & -2(1+\lambda) \end{vmatrix}$$

$|\bar{H}|_{(0,\pm 1)} = -16 < 0 \Rightarrow (0,\pm 1)$ minimo lokala $f_{|S}$-rena

$|\bar{H}|(\pm 1, 0) = 16 > 0 \Rightarrow (\pm 1, 0)$ maximo lokala $f_{|S}$-rena

ADIBIDEA:

Teorema 12 dio $\nabla g_i(\bar{x}_0) \neq \bar{0} \quad \forall i = 1, \dots, m$ hipotesi berala horrben du atera zaizkigun emaitzak lortzeko. Baina, zer gertatzen da $\nabla g_i(\bar{x}_0) = \bar{0}$ betetzen duten puntuetan?

Gure kasuan $m = 1 \rightarrow g(x,y) = x^2 + y^2$

$\nabla g = \bar{0}$? $\Rightarrow \nabla g = (2x, 2y) = \bar{0} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix} \Rightarrow (0,0) \notin S$

$\Rightarrow (0,0)$ puntuan ez dugu aztertu behar

ADIBIDEAK: eGeleko 24 ∧ 26 orrialdeetan

1.6. MUTUR ABSOLUTUAK

DEFINIZIOA

Izan bitez $f: A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ eta $\bar{x}_0 \in A$

(A ez da derrigorrez irekia)

i) f-k $\bar{x}_0$ puntuan MAXIMO ABSOLUTUA duela esaten da

$f(\bar{x}) \leq f(\bar{x}_0) \quad \forall \bar{x} \in A$ bada.

ii) f-k $\bar{x}_0$ puntuan MINIMO ABSOLUTUA duela esaten da $f(\bar{x}) \geq f(\bar{x}_0) \quad \forall \bar{x} \in A$ bada.

TEOREMA 14:

Izan bedi $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ jarraitua, D eremu trinko bat (itxia eta bornatua) izanik

$\Rightarrow$ f-k bere maximoa eta minimo absolutuak lortzen dituela D-n.

PAUSUAK:

1) Aztertu muturrak D-ren barnean $(\mathring{D})$

2) Aztertu muturrak D-ren mugan (∂D)

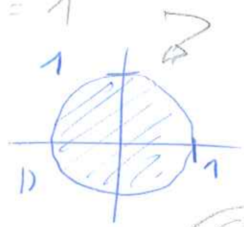
3) Ebaluatu puntu denak eta konparatu

ADIBIDEA:

$(x-0)^2+(y-0)^2=1$

$f(x,y) = x^2+y^2-x-y+1$

$D = \{(x,y) : x^2+y^2 \leq 1\}$ $\longrightarrow$



- f polinomioa $\Rightarrow$ f jarraitua D-n
- D trinkoa

$\overset{\text{TEOR 14}}{\Longrightarrow}$ f-k D-n MAX $\wedge$ min ABS. ditu

1) D-ren barneko muturrak:

Ez dute betetzen baldintzarik (ekuaziorik)

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 2x-1=0 \\ \dfrac{\partial f}{\partial y} = 2y-1=0 \end{cases} \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \in D$$

Konprobatu $\Rightarrow \left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2 \leq 1 \Rightarrow$ Bai

2) D-ren mugako muturrak:

"$g(x,y) = x^2+y^2=1$" betetzen dute

$$h(\lambda,x,y) = f(x,y) - \lambda(g(x,y)-c) =$$
$$= x^2+y^2-y+1-\lambda\cdot(x^2+y^2-1)$$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial h}{\partial x} = 2x-1-2\lambda x=0 \\ \dfrac{\partial h}{\partial y} = 2y-1-2\lambda y=0 \\ \dfrac{\partial h}{\partial \lambda} = -x^2-y^2+1=0 \end{cases} \Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \wedge \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Kontuan izan: $\nabla g = \bar{0} \Rightarrow \nabla g = (2x, 2y) = 0$
$\downarrow$
$(0,0)$

Baina $(0,0) \notin \partial D$

3) Ebaluatu

$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ MIN. ABS. f-rena D-n

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2-\sqrt{2}$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2+\sqrt{2} \Rightarrow \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ MAX. ABS. f-rena D-n

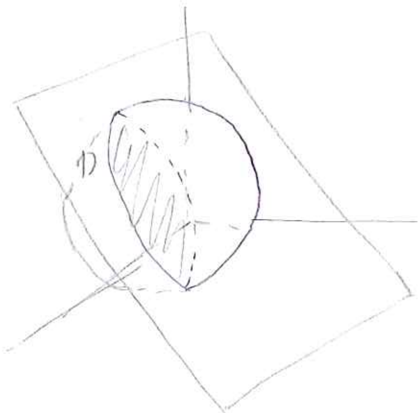
ADIBIDEA

$$f(x,y,z) = x^2+y^2+z$$

$(x-0)^2+(y-0)^2+(z-0)^2=1^2$

$(0,0,0)$ ZENTRUKO $r=1$ ESFERA

$$D = \{(x,y,z) \in \mathbb{R}^3 : \underbrace{z = x}_{\text{PLANOA}} \wedge x^2+y^2+z^2 \leq 1$$



f-ren MUTUR ABSOLUTUAK D-n

1) D-ren barnean

MUTUR BALDINTZATUEN PROBLEMA

$$z = x \longrightarrow g(x,y,z) = x - z = 0^{c}$$

LAGRANGE

$$h(\lambda,x,y,z) = f - \lambda(g-c) = x^2+y^2+z - \lambda\cdot(x-z-0)$$

$$\nabla h = \bar{0} \qquad \frac{\partial h}{\partial \lambda} = 0 \quad \frac{\partial h}{\partial x} = 0 \quad \frac{\partial h}{\partial y} = 0 \quad \frac{\partial h}{\partial z} = 0$$

$$\Rightarrow P_1 = \left(-\frac{1}{2}, 0, -\frac{1}{2}\right)$$

$$\begin{matrix} x^2+y^2+z^2 \leq 1 \\ x = z \end{matrix} \xRightarrow{\text{Konprobatu}} -\frac{1}{2} = -\frac{1}{2} \checkmark \quad \left(-\frac{1}{2}\right)^2+0^2+\left(\frac{-1}{2}\right)^2 \leq 1 \checkmark$$

$$\nabla g = \bar{0} \Rightarrow \nabla g = (1,0,-1) \neq (0,0,0) \Rightarrow$$ Ez dugu punturik lortzen

2) D-ren mugan

MUTUR BALDINTZATUEN PROBLEMA

$$g_1(x,y,z) = x - z = 0$$

$$g_2(x,y,z) = x^2+y^2+z^2 = 1$$

$$h(\lambda,x,y,z) = f - (g_1-c_1) + \mu(g_2-c_2) =$$

$$= x^2+y^2+z^2 - \lambda(x-z-0) - \mu(x^2+y^2+z^2-1)$$

$$\nabla h = \bar{0} \Rightarrow \frac{\partial h}{\partial \lambda} = -z+x = 0 \qquad \frac{\partial h}{\partial \mu} = -x^2-y^2-z^2+1 = 0$$

$$\frac{\partial h}{\partial x} = 2x - \lambda - 2x\mu = 0 \qquad \frac{\partial h}{\partial y} = 2y - 2\mu y = 0 \qquad \frac{\partial h}{\partial z} = 1+\lambda-2\mu z = 0$$

$$\Rightarrow P_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \quad P_3 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \quad P_4 = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

$$P_5 = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

Gogoratu:

$\nabla g_1 = \bar{0} \implies \nabla g_1 = (1,0,-1) \neq \bar{0}$

$\nabla g_2 = \bar{0} \implies \nabla g_2 = (2x, 2y, 2z) = \bar{0}$

$\implies (0,0,0) \notin \partial D \implies 0+0+0 \neq 1 \implies$ Ez du balio

3) Ebaluatu

$f(P_1) = -\frac{1}{4}$ $\quad$ $f(P_3) = \frac{1-\sqrt{2}}{2}$

$f(P_2) = \frac{1+\sqrt{2}}{2}$ $\quad$ $f(P_4) = f(P_5) = \frac{5}{4}$

$P_4 \wedge P_5$ MAXIMO ABSOLUTUAK

P_1 MINIMO ABSOLUTUAK

ADIBIDEA

Zuek $\to$ ebatzi 31. orrialdean

ANALISI BEKTORIALA ETA KONPLEXUA

1. Gaia: MUTURRAK

Ariketak

1. Funtzio hauetarako, lehen eta bigarren ordenako deribatu partzialak kalkulatu:

(i) $f(x,y) = x^4 + y^4 - 4x^2y^2$ (ii) $f(x,y) = xy + \dfrac{x}{y}$

(iii) $f(x,y) = \ln(x + y^2)$ (iv) $f(x,y) = \sin x \sin^2 y$

(v) $f(x,y,z) = e^{x^2+y^2+z^2}$ (vi) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

2. Eskatzen diren goi-ordenako deribatuak kalkula itzazu:

(i) $\dfrac{\partial^3 f}{\partial x^2 \partial y}$, $f(x,y) = x\ln(xy)$ funtzioa bada. *Em.:* 0

(ii) $\dfrac{\partial^6 f}{\partial x^3 \partial y^3}$, $f(x,y) = x^3 \sin y + y^3 \sin x$ funtzioa bada. *Em.:* $-6(\cos x + \cos y)$

(iii) $\dfrac{\partial^4 f}{\partial x \partial y \partial z^2}$, $f(x,y,z) = e^{xyz}$ funtzioa bada. *Em.:* $e^{xyz}(4xy + 5x^2y^2z + x^3y^3z^2)$

(iv) $\dfrac{\partial^6 f}{\partial z \partial y \partial x^2 \partial y \partial z}$, $f(x,y,z) = x^2yz + xy^2z + xyz^2$ bada. *Em.:*0

3. Izan bedi $f(x,y) = xu(x+y) + yv(x+y)$, $u, v\colon \mathbf{R} \to \mathbf{R}$ C^∞ klaseko funtzioak izanik. Froga ezazu honako berdintza hau:

$$\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0.$$

4. Izan bedi $f(x,y) = u\left(\dfrac{y}{x}\right) + xv\left(\dfrac{y}{x}\right)$, $u, v\colon \mathbf{R} \to \mathbf{R}$ C^∞ klaseko funtzioak izanik. Kalkulatu:

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}.$$

Em.: 0.

5. Funtzio hauen bigarren mailako Taylorren formula eman, adierazten diren puntuetan:

(i) $f(x,y) = e^x \sin y$, $(0,0)$ puntuan.

(ii) $f(x,y) = (1+x)^m(1+y)^n$, $(0,0)$ puntuan, $m, n \in \mathbf{N}$ izanik.

(iii) $f(x,y) = e^{x+y}$, $(1,-1)$ puntuan.

(iv) $f(x,y) = \dfrac{1}{1+x^2+y^2}$, $(0,0)$ puntuan.

Em.: (i) $y + xy + R_2$; (ii) $1 + mx + ny + \dfrac{m(m-1)}{2}x^2 + mnxy + \dfrac{n(n-1)}{2}y^2 + R_2$; (iii) $1 + x + y + \dfrac{x^2}{2} + \dfrac{y^2}{2} + xy + R_2$; (iv) $1 - x^2 - y^2 + R_2$.

$\dfrac{\partial w}{\partial x} = -\dfrac{y}{x^2}$ $\dfrac{\partial w}{\partial y} = \dfrac{1}{x}$ $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{2y}{x^3}$

6. Honako funtzio hauen mutur lokalak aurkitu:

(i) $f(x,y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$ *Em.:* $f(1,1)$ minimo lokala

(ii) $f(x,y) = e^{2x}(x + y^2 + 2y)$ *Em.:* $f(1/2,-1)$ minimo lokala

(iii) $f(x,y) = x^3 + 6x^2 + 3y^2 - 12xy + 9x$ *Em.:* $f(3,6)$ minimo lokala

(iv) $f(x,y) = (x-y)(xy-1)$ *Em.:* Ez dago

(v) $f(x,y) = \ln(x^2 + y^2 + 1)$ *Em.:* $f(0,0)$ minimo lokala

7. Froga ezazu $f(x,y) = (1+e^y)\cos x - ye^y$ funtzioak infinitu puntutan lortzen dituela maximo lokalak, baina ez duela minimo lokalik.

8. Aurkitu $f(x,y,z) = x + \dfrac{y^2}{4x} + \dfrac{z^2}{y} + \dfrac{2}{z}$ funtzioaren mutur lokalak bere definizio-eremu osoan.

Em.: $f(1/2,1,1)$ minimo lokala eta $f(-1/2,-1,-1)$ maximo lokala.

9. Funtzio hauen mutur absolutuak kalkulatu, adierazten diren eremuetan:

(i) $f(x,y) = x^2 - y^2$, $x^2 + y^2 \leq 4$.

(ii) $f(x,y) = 2x + y^2$, $x^2 + y^2 \leq 2$, $y^2 - x \geq 0$.

(iii) $f(x,y) = x^2 - xy + y^2$, $|x| + |y| \leq 1$.

(iv) $f(x,y,z) = xyz$, $x^2 + y^2 + z^2 = 1$, $x + y + z = 0$.

(v) $f(x,y,z) = x + y + z$, $x^2 + y^2 \leq z \leq 1$.

(vi) $f(x,y,z) = x^2 + y^2 + z^2 + x + y + z$, $y + z = 1$, $x^2 + y^2 + z^2 \leq 1$.

10. Eskualde batean dauden bi ibaiek $y = x^2$, eta $x - y - 2 = 0$ ekuazioen forma dute. Bi ibai horiek lotuko dituen kanal zuzen bat egin nahi da. Aztertu zein puntutan jarri behar den kanal hori, ahal den laburrena izan dadin.

Em.: $P = (1/2, 1/4)$ eta $Q = (11/8, -5/8)$ puntuetan

11. Idatz ezazu 120 zenbakia hiru zenbakiren batura modura, binaka hartutako biderkaduren batura maximoa izan dadin.

Em.: $120 = 40 + 40 + 40$

12. Izan bedi $2z = 16 - x^2 - y^2$, $x + y = 4$ ekuazioen bidez definitutako C kurba. $x, y, z \geq 0$ betetzen duten puntuen artean, aurki itzazu jatorritik hurbilen eta urrunen dauden puntuak. Zeintzuk dira puntu horietatik jatorrira dauden distantziak?

Em.: $(2+\sqrt{3}, 2-\sqrt{3}, 1)$ eta $(2-\sqrt{3}, 2+\sqrt{3}, 1)$ hurbilen daudenak eta $(2,2,4)$ urrunen.

13. Aurki ezazu $x^2 + y^2 + z^2 = 54$ eta $2x + y - z = 2$ gainazalen ebakidura den kurbaren punturik altuena.

Em.: $(10/3, 5/3, 19/3)$

14. Kutxa errektangeluar bat lehen oktantean kokatzen da, erpin bat jatorrian eta hiru aurpegi auzokideak plano koordenatuetan dituela. Gainera, (x,y,z) erpina, non $x > 0$, $y > 0$, $z > 0$ diren, $2x^2 + y^2 + z = 1$ ekuazioko paraboloidean dago. Aurki itzazu erpin horren koordenatuak kutxaren bolumena maximoa izan dadin.

Em.: $\left(\dfrac{1}{2\sqrt{2}}, \dfrac{1}{2}, \dfrac{1}{2}\right)$.

1. GAIA: MUTURRAK

ARIKETAK

1. ARIKETA

i) $f(x,y) = x^4 + y^4 - 4x^2y^2$

$\frac{\partial f}{\partial x} = 4x^3 - 8y^2x$ $\qquad$ $\frac{\partial f}{\partial y} = 4y^3 - 8x^2y$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 8y^2$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = 12y^2 - 8x^2$

$\frac{\partial^2 f}{\partial \cancel{y} \partial \cancel{y}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -16y$ $\qquad$ $\frac{\partial^2 f}{\partial \cancel{x} \partial y} = -16x$

ii) $f(x,y) = xy + \frac{x}{y}$

$\frac{\partial f}{\partial x} = y + \frac{1}{y}$ $\qquad$ $\frac{\partial f}{\partial y} = x + \frac{-x}{y^2}$

$\frac{\partial^2 f}{\partial x^2} = 0$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{2}{y^3}$ $\qquad$ $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(0) = 0$

iii) $f(x,y) = \ln(x + y^2)$

$\frac{\partial f}{\partial x} = \frac{1}{x+y^2}$ $\qquad$ $\frac{\partial f}{\partial y} = \frac{1}{x+y^2} \cdot 2y$

$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(x+y^2)^2}$

$\frac{\partial^2 f}{\partial \cancel{y}^2} = \frac{\frac{-1}{(x+y^2)\cancel{2}} \cdot 2y 2y - 2 \cdot \frac{1}{\cancel{x+y^2}}}{(x+y^2)^{\cancel{2}}} = \frac{-2}{x+y^2} \cdot \left[\frac{2y^2}{x+y^2} + 1\right]$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{-1}{(x+y^2)^2} \cdot 2y$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial y} = \frac{-2y}{(x+y^2)^2}$

iv) $f(x,y) = \sin x \sin^2 y$

$$\frac{\partial f}{\partial x} = \cos x \sin^2 y \qquad \frac{\partial f}{\partial y} = \sin x \, 2 \sin y \cos y$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x \sin^2 y$$

$$\frac{\partial^2 f}{\partial y^2} = \sin x \cdot \left[2 \cos y \cos y + 2 \sin y(-\sin y)\right] =$$

$$= 2 \cdot \sin x \cdot \left[\cos^2 y - \sin^2 y\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = \cos x \, 2 \sin y \cos y$$

$\Rightarrow C^2$ KLASECKOA

$$\frac{\partial^2 f}{\partial x \partial y} = \cos x \, 2 \sin y \cos y$$

v) $f(x,y,z) = e^{x^2+y^2+z^2}$

$$\frac{\partial f}{\partial x} = 2x \cdot e^{x^2+y^2+z^2} \qquad \frac{\partial f}{\partial y} = 2y \cdot e^{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial z} = 2z \cdot e^{x^2+y^2+z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 4x^2 \cdot e^{x^2+y^2+z^2} \qquad \frac{\partial^2 f}{\partial z^2} = 4z^2 \cdot e^{x^2+y^2+z^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 4y^2 \cdot e^{x^2+y^2+z^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4xy \cdot e^{x^2+y^2+z^2} \qquad \frac{\partial^2 f}{\partial x \partial z} = 4xz \cdot e^{x^2+y^2+z^2}$$

$$\frac{\partial^2 f}{\partial y \partial z} = 4zy \cdot e^{x^2+y^2+z^2}$$

$\hookrightarrow C^2$ KLASECKOA

vi) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f}{\partial x\,\partial y} = \frac{yx}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial f}{\partial x\,\partial z} = \frac{zx}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f}{\partial y\,\partial z} = \frac{zy}{\sqrt{x^2+y^2+z^2}} \longrightarrow C^2 \text{ KLASEKOA}$$

2. ARIKETA

i) $f(x,y) = x\ln(xy)$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial}{\partial x^2}\left(\frac{\cancel{x}}{\cancel{x}y}\right) = 0$$

ii) $f(x,y) = x^3 \sin y + y^3 \sin x$

$$\frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y^2}\left(x^3\cos y + 3y^2 \sin x\right) = \frac{\partial}{\partial y}\left(-x^3 \sin y + 6y\sin x\right) =$$

$$= -x^3\cos y + 6\sin x$$

$$\frac{\partial^3}{\partial x^3}\left(\frac{\partial^3 f}{\partial y^3}\right) = \frac{\partial^2}{\partial x^2}\left(-3x^2\cos y + 6\cos x\right) =$$

$$= \frac{\partial}{\partial x}\left(-6x\cos y - 6\sin x\right) = -6\cdot(\cos y + \cos x)$$

$$\frac{\partial^6 f}{\partial x^3 \partial y^3} = \boxed{-6\cdot[\cos y + \cos x]}$$

iii) $f(x,y,z) = e^{xyz}$

$$\frac{\partial^4 f}{\partial x\,\partial y\,\partial z^2} = \frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial}{\partial z}\left(xy\cdot e^{xyz}\right)\right) = \frac{\partial^2}{\partial x\,\partial y}\left(x^2y^2\cdot e^{xyz}\right) =$$

$$= \frac{\partial}{\partial x}\cdot\left(2x^2y\cdot e^{xyz} + x^3y^2z\,e^{xyz}\right) =$$

$$= 4xy\cdot e^{xyz} + 2x^2y^2z\,e^{xyz} + 3x^2y^2z\,e^{xyz} + x^3y^3z^2\,e^{xyz} =$$

$$= e^{xyz}\cdot\left[4xy + 2x^2y^2z + x^3y^3z^2\right]$$

iv) $f(x,y,z) = x^2yz + xy^2z + xyz^2$

$$\frac{\partial^6 f}{\partial z \partial y \partial x^2 \partial y \partial z} = \frac{\partial^5}{\partial z \partial y \partial x^2}\left(\frac{\partial}{\partial y}\left(x^2y + xy^2 + 2xyz\right)\right) =$$

$$= \frac{\partial^3}{\partial z \partial y \partial x}\left(\frac{\partial}{\partial x}\left(x^2 + 2xy + 2xz\right)\right) =$$

$$= \frac{\partial^2}{\partial z \partial y}\left(\frac{\partial}{\partial x}\left(2x + 2y + 2z\right)\right) =$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}(2)\right) = 0$$

3. ARIKETA

$$f(x,y) = x\,u(\overbrace{x+y}^{w}) + y\,v(x+y) \quad : u,v : R \to R \Rightarrow C^\infty$$

$$? \quad \frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0 \qquad u \wedge w \longrightarrow w \begin{cases} x \\ y \end{cases}$$

$$\frac{\partial f}{\partial x} = 1 \cdot u(w) + x \cdot \frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} =$$

$$= u(w) + x \cdot \frac{\partial u}{\partial w} \cdot \underbrace{\frac{\partial w}{\partial x}}_{=1} + y \cdot \frac{\partial v}{\partial w} \cdot \underbrace{\frac{\partial w}{\partial x}}_{=1}$$

$$\frac{\partial f}{\partial x} = u(w) + x\frac{\partial u}{\partial w} + y\frac{\partial v}{\partial w} \qquad \underbrace{\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)}_{\frac{\partial^2 u}{\partial w^2}} \underbrace{\frac{\partial w}{\partial x}}_{1}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} + \frac{\partial u}{\partial w} + x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial w}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial w}\right) =$$

$$= \frac{\partial u}{\partial w} + \frac{\partial u}{\partial w} + x\frac{\partial^2 u}{\partial w^2} + y\frac{\partial}{\partial w}\left(\frac{\partial v}{\partial w}\right)\underbrace{\frac{\partial w}{\partial x}}_{=1} =$$

$$= \boxed{2\frac{\partial u}{\partial w} + x\frac{\partial^2 u}{\partial w^2} + y\frac{\partial^2 v}{\partial w^2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial u}{\partial y} + x\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial w}\right) + \frac{\partial v}{\partial w} + y\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial w}\right) =$$

$$= \frac{\partial u}{\partial w}\underbrace{\frac{\partial w}{\partial y}}_{=1} + x\frac{\partial^2 u}{\partial w^2}\underbrace{\frac{\partial w}{\partial y}}_{=1} + \frac{\partial v}{\partial w} + y\frac{\partial^2 v}{\partial w^2} \cdot \underbrace{\frac{\partial w}{\partial y}}_{=1} =$$

$$= \boxed{\frac{\partial u}{\partial w} + x\frac{\partial^2 u}{\partial w^2} + \frac{\partial v}{\partial w} + y\frac{\partial^2 v}{\partial w^2}}$$

$$\frac{\partial f}{\partial y} = x\frac{\partial u}{\partial y} + v(w) + y\frac{\partial v}{\partial y} = x\frac{\partial u}{\partial w}$$

$$= x\frac{\partial u}{\partial w}\frac{\partial w}{\partial y} + v(w) + y\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} = \boxed{x\frac{\partial u}{\partial w} + v(w) + y\frac{\partial v}{\partial w}}$$

$$\frac{\partial^2 f}{\partial y^2} = x\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial w}\right) + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial w} + y\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial w}\right) =$$

$$= x\frac{\partial^2 u}{\partial w^2}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial w} + y\frac{\partial^2 v}{\partial w^2}\frac{\partial w}{\partial y} =$$

$$\boxed{= x\frac{\partial^2 u}{\partial w^2} + 2\frac{\partial v}{\partial w} + y\frac{\partial^2 v}{\partial w^2}}$$

[ORDEZKATU ETA ZUZTATU] (Zuek)

5. ARIKETA

iii) $f(x,y) = e^{x+y}$, $(1,-1)$ puntuan

$$f(x,y) = f(1,-1) + \frac{\partial f}{\partial x}(1,-1)(x-1) + \frac{\partial f}{\partial y}(1,-1)\cdot(y+1) +$$

$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(1,-1)(x-1)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(1,-1)(y+1)^2 +$$

$$+ \frac{\partial^2 f}{\partial x\partial y}(1,-1)\cdot(x-1)(y+1) + R_2 =$$

$$\frac{\partial f}{\partial x} = e^{x+y} \xrightarrow{(1,-1)} 1 \qquad \frac{\partial f}{\partial y} = e^{x+y} \xrightarrow{(1,-1)} 1$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x+y} \xrightarrow{(1,-1)} 1 \qquad \frac{\partial^2 f}{\partial y^2} = e^{x+y} \xrightarrow{(1,-1)} 1$$

$$\frac{\partial^2 f}{\partial x\partial y} = e^{x+y} \xrightarrow{(1,-1)} 1$$

$$= 1 + (x-1) + (y+1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(y+1)^2 + (x-1)(y+1) + R_2$$

$$= \cancel{1} + \cancel{x} - \cancel{1} + y + \cancel{1} + \frac{1}{2}x^2 - \cancel{x} + \frac{1}{2} + \frac{1}{2}y^2 + \cancel{y} + \frac{1}{2} + xy + x - \cancel{y} - \cancel{1} + R_2$$

$$= y + \frac{1}{2}x^2 + 1 + \frac{1}{2}y^2 + xy + x + R_2$$

$$\boxed{f(x,y) = 1 + x + y + \frac{x^2}{2} + \frac{y^2}{2} + xy + R_2}$$

5

6. ARIKETA

iii) $f(x,y) = x^3 + 6x^2 + 3y^2 - 12xy + 9x$

$\hookrightarrow C^\infty$ $\mathbb{R}^2$ osoan [polinomioa]

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 3x^2 + 12x - 12y + 9 = 0 \\ \frac{\partial f}{\partial y} = 6y - 12x = 0 \Rightarrow \boxed{y = 2x} \end{cases}$$

$$= x^2 - 4x + 3 = 0 \Rightarrow \boxed{x = 3 \wedge 1}$$

$P_1 = (3,6) \wedge (1,2) = P_2$ PUNTU KRITIKOAK

$$\frac{\partial^2 f}{\partial x^2} = 6x + 12 \begin{cases} \xrightarrow{P_1} 30 \\ \xrightarrow{P_2} 6 \end{cases} \qquad Hf(3,6) = \begin{pmatrix} 30 & -12 \\ -12 & 6 \end{pmatrix}$$

$\wedge$

$$\frac{\partial^2 f}{\partial x \partial y} = 12 \xrightarrow{P_1, P_2} 12 \qquad Hf(1,2) = \begin{pmatrix} 6 & -12 \\ -12 & 6 \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial y^2} = 6 \xrightarrow{P_1, P_2} 6$$

$D = |Hf(3,6)| > 0$ TEOR 11

$\Rightarrow$ MINIMO LOKALA $(3,6)$-n

$\frac{\partial^2 f}{\partial x^2}(3,6) > 0$

$D = |Hf(1,2)| < 0 \Rightarrow$ ZELADURA PUNTUA $(1,2)$-n

7. ARIKETA

$f(x,y) = (1+e^y)\cos x - ye^y \longrightarrow \mathbb{R}^2$ osoa C^∞

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} P_1 = (2k\pi, 0) \\ P_2 = ((2k+1)\pi, -2) \end{cases} \text{ P.K.}$$

$$Hf(2k\pi, 0) = \begin{pmatrix} \overset{<0}{-2} & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{MAXIMO LOKALA}$$

$$Hf((2k+1)\pi, -2) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \Rightarrow \text{ZELADURA PUNTUA}$$

5. Ariketa

i) $f(x,y) = e^x \sin y$ $(0,0)$ puntuan

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) +$$

$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(0,0)y^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)xy + R_2 =$$

$$\frac{\partial f}{\partial x} = e^x \sin y \xrightarrow{(0,0)} 0 \qquad \frac{\partial^2 f}{\partial x^2} = e^x \sin y \xrightarrow{(0,0)} 0$$

$$\frac{\partial f}{\partial y} = e^x \cos y \xrightarrow{(0,0)} 1 \qquad \frac{\partial^2 f}{\partial y^2} = -e^x \sin y \xrightarrow{(0,0)} 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^x \cos y \xrightarrow{(0,0)} 1$$

$$= 0 + 0 + y + 0 + 0 + xy + R_2 \Rightarrow \boxed{f(x,y) = y + xy + R_2}$$

ii) $f(x,y) = (1+x)^m (1+y)^n$ $(0,0)$ puntuan : $m, n \in \mathbb{N}$

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y +$$

$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(0,0)y^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)xy + R_2 =$$

$$\frac{\partial f}{\partial x} = (1+y)^n \cdot m \cdot (1+x)^{m-1} \xrightarrow{(0,0)} m$$

$$\frac{\partial^2 f}{\partial x^2} = (1+y)^n \cdot m \cdot (m-1) \cdot (1+x)^{m-2} \xrightarrow{(0,0)} m \cdot (m-1)$$

$$\frac{\partial f}{\partial y} = (1+x)^m \cdot n \cdot (1+y)^{n-1} \xrightarrow{(0,0)} n$$

$$\frac{\partial^2 f}{\partial y^2} = (1+x)^m \cdot n \cdot (n-1) \cdot (1+y)^{n-2} \xrightarrow{(0,0)} n \cdot (n-1)$$

$$\frac{\partial^2 f}{\partial x \partial y} = m \cdot (1+x)^{m-1} \cdot n \cdot (1+y)^{n-1} \xrightarrow{(0,0)} m \cdot n$$

$$= 1 + mx + ny + \frac{1}{2} m\cdot(m-1)x^2 + \frac{1}{2}n(n-1)y^2 + mnxy + R_2$$

iv) $f(x,y) = \dfrac{1}{1+x^2+y^2}$ $(0,0)$ puntuan

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y +$$
$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(0,0)y^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)xy + R_2$$

$$\frac{\partial f}{\partial x} = \frac{-2x}{(1+x^2+y^2)^2} \xrightarrow{(0,0)} 0$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{(1+x^2+y^2)^2} \xrightarrow{(0,0)} 0$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-2\cdot(1+x^2+y^2)^2 + 2x\cdot 4x\cdot(1+x^2+y^2)}{(1+x^2+y^2)^2} \xrightarrow{(0,0)} -2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2\cdot(1+x^2+y^2)^2 + 2y\cdot 4y\cdot(1+x^2+y^2)}{(1+x^2+y^2)^2} \xrightarrow{(0,0)} -2$$

$$\frac{\partial^2 f}{\partial x \partial y} = + 2y\cdot(+4x)\cdot\frac{1}{(1+x^2+y^2)^3} \xrightarrow{(0,0)} 0$$

$$= 1 + \frac{1}{2}\cdot(-2)x^2 + \frac{1}{2}\cdot(-2)y^2 + R_2$$

$$\boxed{f(x,y) = 1 - x^2 - y^2 + R_2}$$

6. ARIKETA

ii) $f(x,y) = e^{2x}\cdot(x+y^2+2y)$

Esponentziala + polinomikoa $\Rightarrow$ $\mathbb{R}^2$ osoan C^∞

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 2e^{2x}\cdot(x+y^2+2y) + e^{2x} = 0 \\ \dfrac{\partial f}{\partial y} = e^{2x}\cdot(2y+2) = 0 \end{cases}$$

$$2e^{2x}\cdot(x+y^2+2y)+e^{2x}=0$$

$$e^{2x}\cdot(2y+2)=0 \Rightarrow 2y+2=0 \Rightarrow y=-1$$

$$2e^{2x}\cdot(x+1-2)+e^{2x}=0$$

$$2e^{2x}\cdot x-2e^{2x}+e^{2x}=0 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$$

$$P=\left(\frac{1}{2},-1\right)$$

$$\frac{\partial^2 f}{\partial x^2}=4e^{2x}\cdot(x+y^2+2y)+2e^{2x}+2e^{2x}=$$

$$=4e^{2x}\cdot(x+y^2+2y+1)\xrightarrow{(1/2,-1)}2e$$

$$\frac{\partial^2 f}{\partial y^2}=2e^{2x}\xrightarrow{(\frac{1}{2},-1)}2e$$

$$\frac{\partial^2 f}{\partial x\partial y}=2e^{2x}\cdot(2y+2)\xrightarrow{(\frac{1}{2},-1)}2e\cdot 0=0$$

C^n denez $\frac{\partial^2 f}{\partial y\partial x}=$

$$Hf\left(\frac{1}{2},-1\right)=\begin{pmatrix}2e & 0\\ 0 & 2e\end{pmatrix}$$

$$D\left|Hf\left(\frac{1}{2},-1\right)\right|=4e^2>0$$

$$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{2},-1\right)=2e>0$$

$\xRightarrow{\text{TEOR 11}}$ $\boxed{\left(\frac{1}{2},-1\right)\text{-n MINIMO LOKALA}}$

iv) $f(x,y)=(x-y)(xy-1)$

Polinomikoa $\Rightarrow$ $\mathbb{R}^2$ osoan C^∞ klasekoa

$$\nabla f=\bar{0} \Rightarrow \begin{cases}\frac{\partial f}{\partial x}=xy-1+y\cdot(x-y)=0\\ \frac{\partial f}{\partial y}=-xy+1+x\cdot(x-y)=0\end{cases}$$

$$xy-1+yx-y^2=0 \Rightarrow 2xy-y^2-1=0$$

$$-xy+1+x^2-xy=0 \Rightarrow \underline{-2xy+x^2+1=0}$$

$$x^2-y^2=0 \Rightarrow x^2=y^2$$

v) $f(x,y) = \ln(x^2+y^2+1)$

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{1}{x^2+y^2+1} \cdot 2x = 0 \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{1}{x^2+y^2+1} \cdot 2y = 0 \end{cases}$$

$$\frac{2x}{x^2+y^2+1} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

$$\frac{2y}{x^2+y^2+1} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$$

ez dago arazoik $y = x^2+y^2+1$ ez delako inoiz 0

$P = (0,0) \Rightarrow$ PUNTU KRITIKOA

$$\frac{\partial^2 f}{\partial x^2} = \frac{2\cdot(x^2+y^2+1) - 2x \cdot 2x}{(x^2+y^2+1)^2} \xrightarrow{(0,0)} 2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2\cdot(x^2+y^2+1) - 2y \cdot 2y}{(x^2+y^2+1)^2} \xrightarrow{(0,0)} 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x \frac{2y}{(x^2+y^2+1)^3} \xrightarrow{(0,0)} 0 \qquad \frac{\partial^2 f}{\partial y \partial x} =$$

$$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$|Hf(0,0)| = 4 > 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0$$

$\Rightarrow \boxed{P=(0,0) \text{ MINIMO LOKALA}}$

i) $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = y - \dfrac{1}{x^2} = 0 \Rightarrow y = \dfrac{1}{x^2} \\[2ex] \dfrac{\partial f}{\partial y} = x - \dfrac{1}{y^2} = 0 \Rightarrow x = \dfrac{1}{y^2} \end{cases}$$

10

$$x = \frac{1}{\left(\frac{1}{x^2}\right)^2} \Rightarrow x = x^{4} \Rightarrow x = 1$$

$$y = \frac{1}{x^2} = \frac{1}{1} = 1 \Rightarrow y = 1$$

$P = (1, 1)$ PUNTU KRITIKOA

$$\frac{\partial^2 f}{\partial x^2} = +2 \cdot \frac{1}{x^3} \xrightarrow{(1,1)} +2$$

$$\frac{\partial^2 f}{\partial y^2} = +2 \cdot \frac{1}{y^3} \xrightarrow{(1,1)} +2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1 \xrightarrow{(1,1)} 1 \qquad \frac{\partial^2 f}{\partial y \partial x} =$$

$$Hf(1,1) = \begin{pmatrix} +2 & 1 \\ 1 & +2 \end{pmatrix}$$

$$|Hf(1,1)| = 3 > 0$$

$$\frac{\partial^2 f}{\partial x^2} = +2 > 0 \Rightarrow \boxed{P = (1,1)\text{-n MINIMO LOKALA}}$$

4. ARIKETA

GAITZ

$$f(x,y) = u\left(\underbrace{\frac{y}{x}}_{w}\right) + x\,v\left(\underbrace{\frac{y}{x}}_{w}\right) \longrightarrow u(w) + x\,v(w) \quad : u, v : \mathbb{R} \to \mathbb{R} \quad C^\infty \text{ k.}$$

$$x^2 \cdot \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} \qquad u \wedge v \longrightarrow w <^{x}_{y}$$

$$\frac{\partial w}{\partial x} = -\frac{y}{x^2} \qquad \frac{\partial w}{\partial y} = \frac{1}{x} \qquad \frac{\partial^2 w}{\partial x^2} = 2\frac{y}{x^3} \qquad \frac{\partial^2 w}{\partial y^2} = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} + v(w) + x \cdot \frac{\partial v}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} + x \cdot \frac{\partial v}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{-1}{x^2} = \frac{\partial^2 w}{\partial y \partial x} \qquad [C^\infty]$$

$$\frac{\partial^2 f}{\partial x^2} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial w}\right)}_{\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)\cdot\frac{\partial w}{\partial x} \;\to\; \frac{\partial^2 u}{\partial w}} \cdot \underbrace{\frac{\partial w}{\partial x}}_{\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial x} \;\to\; \frac{\partial^2 w}{\partial x^2}} + \frac{\partial u}{\partial w}\frac{\partial^2 w}{\partial x^2} + \frac{\partial v}{\partial w}\cdot\frac{\partial w}{\partial x} + \underbrace{\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial w}\right]}_{\frac{\partial}{\partial w}\left(\frac{\partial v}{\partial w}\right) \;\to\; \frac{\partial^2 v}{\partial w}} \underbrace{\frac{\partial w}{\partial x}}_{\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial x} \;\to\; \frac{\partial^2 w}{\partial x^2}} + \frac{\partial v}{\partial w}\frac{\partial^2 w}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 u}{\partial w}\cdot\frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial w}\frac{\partial^2 w}{\partial x^2} + \frac{\partial v}{\partial w}\cdot\frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial w}\cdot\frac{\partial^2 w}{\partial x^2} + \frac{\partial v}{\partial w}\frac{\partial^2 w}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial w}\cdot\cancel{\frac{\partial^2 w}{\partial y^2}} + x\cdot\left[\frac{\partial^2 v}{\partial w}\cdot\cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}}\right] = 0$$

$$\frac{\partial^2 f}{\partial x\partial y} = \underbrace{\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial w}\right]}_{\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \;\to\; \frac{\partial^2 u}{\partial w^2}}\cdot\frac{\partial w}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial^2 w}{\partial y\partial x} + x\cdot\left[\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial w}\right)}_{\frac{\partial}{\partial w}\left(\frac{\partial v}{\partial w}\right)\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \;\to\; \frac{\partial^2 v}{\partial w^2}}\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x\partial y}\frac{\partial v}{\partial w}\right]$$

$$2xy\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 u}{\partial w^2}\cdot\frac{2y^2}{x^3} + \frac{\partial u}{\partial w}\frac{4y^2}{x^3} + \frac{\partial v}{\partial w}\cdot 2y + \frac{\partial^2 v}{\partial w^2}\cdot\frac{2y^2}{x^3} + \frac{\partial v}{\partial w}\cdot\frac{4y^2}{x^2}$$

$$x^2\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 u}{\partial w^2}\cdot\frac{-y^2}{x^3} + \frac{\partial u}{\partial w}\frac{-4y^2}{x^3} + \frac{\partial v}{\partial w}(-2y)^{??} + \frac{\partial^2 v}{\partial w^2}\cdot\frac{-y^3}{x^3} + \frac{\partial v}{\partial w}\frac{-3y^2}{x^2}$$

$$y^2\cdot\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial w^2}\cdot\frac{-y^2}{x^3} + \frac{\partial^2 v}{\partial w^2}\cdot\frac{-y^2}{x^2}$$

8. ARIKETA

$$f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$$

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 1 - \frac{y^2}{4x^2} = 0 \\ \frac{\partial f}{\partial y} = \frac{y}{2x} - \frac{z^2}{y^2} = 0 \\ \frac{\partial f}{\partial z} = \frac{2z}{y} - \frac{2}{z^2} = 0 \end{cases}$$

$$\frac{\cancel{2}z}{y} = \frac{\cancel{2}}{z^2} \Rightarrow z^3 = y$$

$$1 - \frac{y^2}{4x^2} = 0 \Rightarrow \frac{y^2}{4x^2} = 1 \Rightarrow y^2 = 4x^2 \Rightarrow y = \pm 2x$$

$$\frac{y}{2x} = \frac{z^2}{y^2} \Rightarrow y^3 = z^2 \cdot 2x \Rightarrow \begin{matrix} y^2 = z^2 \\ y = z^3 \end{matrix} \Rightarrow \begin{matrix} y = z = -1 \\ y = z = 1 \\ y = 0 = z \Rightarrow \text{EZIN} \end{matrix}$$

$P_1 = \left(\frac{1}{2}, 1, 1\right)$ $\quad P_2 = \left(-\frac{1}{2}, -1, -1\right)$ PUNTU KRITIKOAK

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2}{2x^3} \quad (1/2,1,1) \to 4 \quad (-1/2,-1,-1) \to -4$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{2x} + \frac{2z^2}{y^3} \quad (1/2,1,1) \to 3 \quad (-1/2,-1,-1) \to -3$$

$1+2$ $\quad -1-2$

$$\frac{\partial^2 f}{\partial z^2} = \frac{2}{y} + \frac{4}{z^3} \quad (1/2,1,1) \to 6 \quad (-1/2,-1,-1) \to -6$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{y}{2x^2} \quad (1/2,1,1) \to -2 \quad (-1/2,-1,-1) \to 2$$

$$\frac{\partial^2 f}{\partial z \partial x} = 0 = \frac{\partial^2 f}{\partial x \partial z}$$

$$\frac{\partial^2 f}{\partial y \partial z} = -\frac{2z}{y^2} \quad (1/2,1,1) \to -2 \quad (-1/2,-1,-1) \to 2$$

$$\begin{array}{ccc} 0 & 0 & 0 \\ \wedge & \wedge & \wedge \end{array}$$

$$D\left(\tfrac{1}{2},1,1\right) = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix}$$

$$\begin{array}{ccc} 0 & 0 & 0 \\ \vee & \wedge & \vee \end{array}$$

$$D\left(-\tfrac{1}{2},-1,-1\right) = \begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -6 \end{pmatrix}$$

$$\boxed{\begin{array}{ll} f\left(\frac{1}{2},1,1\right) & \text{MINIMO LOKALA} \\ f\left(-\frac{1}{2},-1,-1\right) & \text{MAXIMO LOKALA} \end{array}}$$

.13

$$\cdot \left[\frac{2y^3}{x^3} - \frac{y^2}{x^4} + \frac{y^2}{x^3} \right] +$$

$$+ \cdot \cancel{\left[\frac{4y^2}{x^3} - \frac{4y^2}{x^3} \right]} +$$

$$+ \frac{\partial v}{\partial w} \cdot \left[2y + \frac{4y^2}{x^2} - y - \frac{3y^2}{x^2} \right] +$$

$$\partial^2 v \quad [2y^3 \quad y^3 \quad y^2]$$

---

9. ARIKETA

ii) $f(x, y) = 2x + y^2$

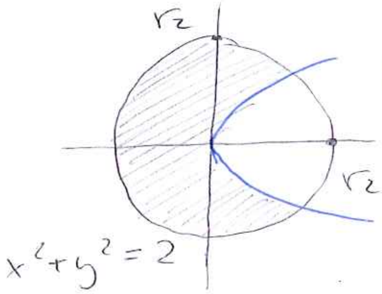
$D = \{ (x, y) \in \mathbb{R}^2 \ / \ x^2 + y^2 \leq 2 \ , \ y^2 - x \geq 0 \}$ → PARABOLA

$\downarrow$

ZIRKUNF. ← $x^2 + y^2 = 2$

$(x-0)^2 + (y-0)^2 = (\sqrt{2})^2$

$y^2 = x$



$(0,0) \rightarrow 0^2 + 0^2 \leq 2 \Rightarrow$ Barruan

$(1,0) \rightarrow 0^2 - 1 \geq 0 \Rightarrow$ Ez

1) D-ren barnean

Ez dago baldintzarik

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 2 \neq 0 \\ \dfrac{\partial f}{\partial y} = 2y = 0 \end{cases} \Rightarrow \text{Ez dago punturik}$$

2) D-ren muga

2.1) Zirkunferentzian

$x^2 + y^2 = 2$

$x^2+y^2=2$

$g_1(x,y)=x^2+y^2=\overset{c}{\overset{\shortparallel}{2}}$

$h(\lambda,x,y)=f(x,y)-\lambda(x^2+y^2-2)$

$$\nabla h=\bar{0}\quad\begin{cases}-x^2-y^2+2=0\\ 2-2x\lambda=0\\ 2y-2y\lambda=0 \longrightarrow 2y(1-\lambda)=0 \begin{cases}\to y=0\\ \to \lambda=1\end{cases}\end{cases}$$

$y=0 \Rightarrow -x^2+2=0 \Rightarrow x=\pm\sqrt{2} \Rightarrow \cancel{(\sqrt{2},0)}, (-\sqrt{2},0)$

Ez dago barne ∂D

$\lambda=1 \Rightarrow 2-2x=0 \Rightarrow x=1$

$\Rightarrow -1-y^2+2=0 \Rightarrow (1,1),(1,-1)$

- $\nabla g_1=\bar{0}? \Rightarrow \nabla g_1=(2x,2y)=\bar{0} \Rightarrow (0,0)\in\partial D$

2.2) Parabolan

$y^2-x=0 \longrightarrow g_2(x,y)=y^2-x=\overset{c}{\overset{\shortparallel}{0}}$

$h(\lambda,x,y)=f(x,y)-\lambda(y^2-x)$

$$\nabla h=\bar{0}\Rightarrow\begin{cases}2-\lambda=0\\ 2y+2y\lambda=0\\ -y^2+x=0\end{cases}\Rightarrow (0,0)\in\partial D$$

- $\nabla g_2=\bar{0}? \Rightarrow \nabla g_2=(-1,2y)=\bar{0} \Rightarrow$ Ez dago puntirik

2.3) Gure ∂D mugan kurba edo gainazal bat baino gehiagoz osatua badago, ebakidurako puntuetan artertu behar da.

- $h(\lambda,\mu,x,y)=f-\lambda(x^2+y^2-2)-\mu(x^2+y^2-0)$

- $\begin{cases}x^2+y^2=2\\ y^2-x=0\end{cases} \Rightarrow \begin{matrix}(1,1)\\(1,-1)\end{matrix}$

3) Ebaluatu

$f(1,1) = 3 = f(1,-1)$

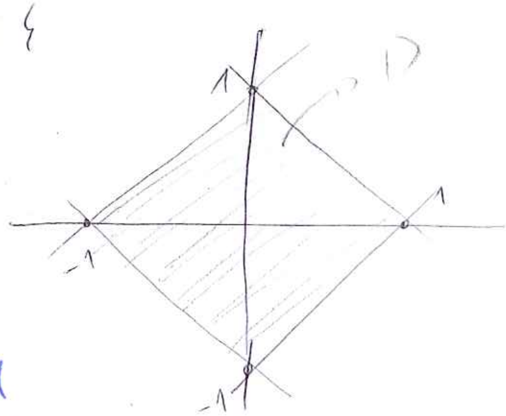
$f(-\sqrt{2},0) = -2\sqrt{2}$

$f(0,0) = 0$

$\Rightarrow$ $(1,1)$ eta $(1,-1)$ maximo absolutuak
$(-\sqrt{2},0)$ minimo absolutuak.

iii) $f(x,y) = x^2 - xy + y^2$

$D = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$



$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

1. KOADRANTEAN $\Rightarrow x + y \leq 1$
2. KOADRANTEAN $\Rightarrow -x + y \leq 1$
3. KOADRANTEAN $\Rightarrow -x - y \leq 1$
4. KOADRANTEAN $\Rightarrow x - y \leq 1$

1) D-ren barnean

$\nabla f = \bar{0} \Rightarrow \begin{cases} 2x - y = 0 \\ -x + 2y = 0 \end{cases} \Rightarrow (0,0) \in D$

2) ∂D

1.1) $y = x + 1$ muga

$h(\lambda, x, y) = f - \lambda(y - x - 1) \rightarrow \nabla h = \bar{0} \Rightarrow (-\frac{1}{2}, \frac{1}{2}) \in \partial D$

$\nabla g_1 = \bar{0}? \Rightarrow \nabla g_1 = (-1,1) \neq \bar{0}$

2.2) $y = 1 - x$ muga

$h(\lambda, x, y) = f - \lambda(1 - x - y) \rightarrow \nabla h = \bar{0} \Rightarrow (\frac{1}{2}, \frac{1}{2}) \in \partial D$

$\nabla g_2(x,y) = 1 - x - y$

$\nabla g_2 = \bar{0}? \Rightarrow \nabla g_2 = (-1,-1) \neq \bar{0}$

2.3) $y = -1 - x$ muga

$g_3(x,y) = -1 - x - y$

$h(\lambda, x, y) = f - \lambda(-1 - x - y)$

$\nabla h = \bar{0} \Rightarrow \left(-\frac{1}{2}, -\frac{1}{2}\right) \in \partial D$

$\nabla g_3 = (-1, -1) \neq \bar{0}$

2.4). $y = x - 1$ muga

$g_4(x,y) = x - \cancel{y}\, y - 1$

$h(\lambda, x, y) = f - \lambda(x - y - 1)$

$\nabla h = \bar{0} \Rightarrow \left(\frac{1}{2}, -\frac{1}{2}\right) \in \partial D$

$\nabla g_4 = (1, -1) \neq \bar{0}$

2.5) Ebakidura puntuak

$(1,0), (0,1), (-1,0), (0,-1)$

3) Ebaluatu

$f(0,0) = 0$ $\qquad f\left(-\frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{3}{4}$

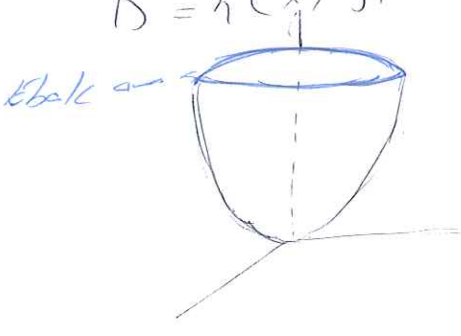
$f(0,1) = f(1,0) = f(-1,0) = f(0,-1) = 1$

$f\left(\frac{1}{2}, \frac{1}{2}\right) = f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$

$\Rightarrow \begin{cases} (0,1), (1,0), (-1,0), (0,-1) \text{ maximo absolutuak} \\ (0,0) \text{ minimo absolutua} \end{cases}$

v) $f(x,y,z) = x + y + z$ $\qquad$ parabola $\qquad$ planoa

$D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z \leq 1\}$



1) D-ren barnean

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 1 \neq 0 \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 1 \neq 0 \end{cases} \Rightarrow \text{Ez dago punturik}$$

2) ∂D

2.1) $x^2+y^2=z$ mugan

$h(\lambda, x, y, z) = f - \lambda(x^2+y^2-z) \Rightarrow \nabla h = \bar{0}$

$\Rightarrow \left(\frac{-1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \in \partial D$

$\nabla g_1 = (2x, 2y-1) \neq \bar{0}$

2.2) $z=1$ mugan

$g_2(x,y,z) = 1-z = 0 = c_2$

$h(\lambda, x, y, z) = f - \lambda(1-z) \Rightarrow \nabla h = \bar{0} \Rightarrow$ Ez dago

$\nabla g_2 = (0,0,-1) \neq \bar{0}$

2.3) Ebakidura

$g_1(x,y,z) = x^2+y^2-z$

$g_2(x,y,z) = 1-z$

$h(\lambda, \mu, x, y, z) = f - \lambda\cdot(x^2+y^2-z) - \mu(1-z)$

$\nabla h = \bar{0} \Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) \wedge \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1\right) \in \partial D$

3) Ebaluatu

$f\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}$

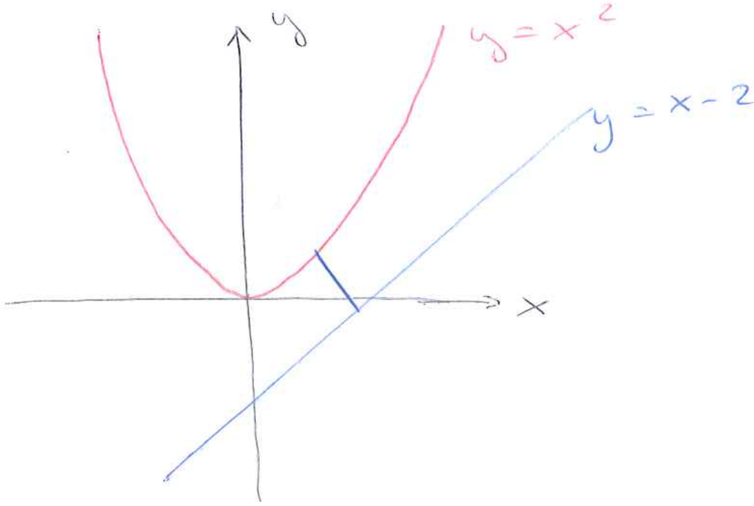
$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) = \sqrt{2}+1$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1\right) = -\sqrt{2}+1$

$\left(-\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}\right)$ minimo absolutua

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$ maximo absolutuak

10. ARIKETA



$(x_0, y_0) \rightarrow y_0 = x_0 - 2 \rightarrow g_1(x_0, y_0) = x_0 - 2 - y_0 = 0$

$(x_1, y_1) \rightarrow y_1 = x_1^2 \rightarrow g_2(x_1, y_1) = x_1^2 - y_1 = 0$

$d((x_0, y_0),(x_1, y_1)) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$ minimoa

$f(x_0, y_0, x_1, y_1) = (x_0 - x_1)^2 + (y_0 - y_1)$ minimoa

$h(\lambda, \mu, x_0, y_0, x_1, y_1) =$

$= f - \lambda(x_0 - 2 - y_0) - \mu(x_1^2 - y_1)$

$\nabla h = \bar{0} \Rightarrow \left(\underset{x_0}{\frac{11}{8}}, \underset{y_0}{-\frac{5}{8}}, \underset{x_1}{\frac{1}{2}}, \underset{y_1}{\frac{1}{4}}\right)$ minimoa

12. ARIKETA

Baldintzak

$(x, y, z) \in C \iff \begin{cases} 2z = 16 - x^2 - y^2 \\ x + y = 4 \end{cases}$

$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$ min a max

$f(x, y, z) = x^2 + y^2 + z^2$ min a max

$h(\lambda, \mu, x, y, z) = x^2 + y^2 + z^2 - \lambda(16 - x^2 - y^2 - 2z) - \mu(x + y - 4)$

$\nabla h = \bar{0} \iff \begin{cases} -x - y + 4 = 0 \\ -16 + x^2 + y^2 + 2z = 0 \\ 2x + 2\lambda x - \mu = 0 \\ 2y + 2\lambda\mu - \mu = 0 \end{cases} \quad 2z + 2\lambda = 0$

$2(x-y)+2\lambda(x-y)=0$

$2(1-\lambda)(x-y)=0 \quad \begin{cases} \to \lambda=-1 \\ \to x=y \end{cases}$

$\lambda=-1 \Rightarrow (2-\sqrt{3},\ 2+\sqrt{3},1),\ (2+\sqrt{3},\ 2-\sqrt{3},1)$

$x=y \Rightarrow (2,2,4)$

$\nabla g_i=\bar{0}$?

$\nabla g_1=(-2x,-2y,-2)\neq\bar{0}$

$\nabla g_2=(1,1,0)\neq\bar{0}$

$f(2,2,4)=24$

$f(2-\sqrt{3},2+\sqrt{3},1)=15$

$f(2+\sqrt{3},2-\sqrt{3},1)=15$

$\Rightarrow (2-\sqrt{3},\ 2+\sqrt{3},1)$ eta $(2+\sqrt{3},2-\sqrt{3},1)$ min abs
eta jatorritik hurbilen ↵

$\Rightarrow (2,2,4)$ max abs, jatorritik urrunen

Ordezkatu puntuak f-n ⇒ tarte absolutuak

13. ARIKETA

Baldintzak $\begin{cases} x^2+y^2+z^2=54 \\ 2x+y-z=2 \end{cases}$

$g_1(x,y,z)=x^2+y^2+z^2-54=0 \quad =c_1$

$g_2(x,y,z)=2x+y-z-2=0 \quad =c_2$

$f(x,y,z)=\overset{\text{Altuera}}{z} \longrightarrow$ MAXIMOA

$h(\lambda,\mu,x,y,z)=f-\lambda(g_1-c_1)-\mu(g_2-c_2)$

$\nabla h=\bar{0} \Rightarrow \left(\frac{10}{3},\frac{5}{3},\frac{19}{3}\right) \Rightarrow$ MAXIMOA

$(-2,-1,-7)^{z_1 z_2}$ altuera

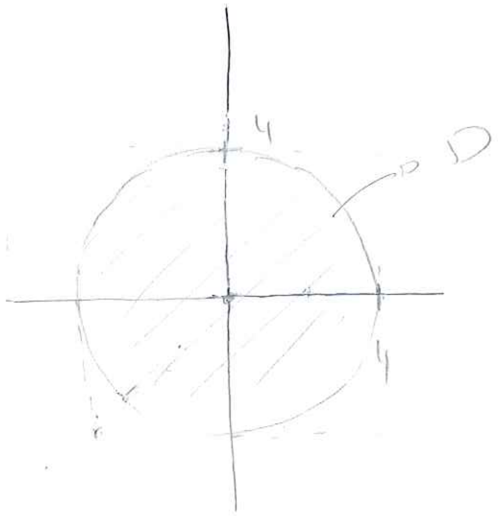
20

9. ARIKETA

$(x-0)^2+(y-0)^2=4$ Zirkunferentzia

1) $f(x,y)=x^2-y^2$

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 4\}$ (0,0)-n $r=4$



3) EBALUATU

$$\boxed{\begin{array}{l} \left.\begin{array}{l} f(0,2)=-4 \\ f(0,-2)=-4 \end{array}\right\} \to \text{min abs} \\ \left.\begin{array}{l} f(2,0)=4 \\ f(-2,0)=4 \end{array}\right\} \to \text{MAX ABS} \end{array}}$$

1) D-ren barnean

$$\nabla f=\bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x}=2x=0 \Rightarrow x=0 \\ \dfrac{\partial f}{\partial y}=-2y=0 \Rightarrow y=0 \end{cases} \Rightarrow (0,0)\in D$$

2) ∂D

$x^2+y^2=4$

$g(x,y)=x^2+y^2=4=c$

$h(\lambda,x,y)=f(x,y)-\lambda(x^2+y^2-4)=$

$=x^2-y^2-\lambda x^2-\lambda y^2+4\lambda$

$$\nabla h=\bar{0} \Rightarrow \begin{cases} 2x-2\lambda x=0 \\ -2y-2\lambda y=0 \\ -x^2-y^2+4=0 \end{cases} \qquad 2x-2\lambda x=0 \to x=0,\ \lambda=1$$

$x=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$

$\lambda=1 \Rightarrow y=0 \Rightarrow x=\pm 2$

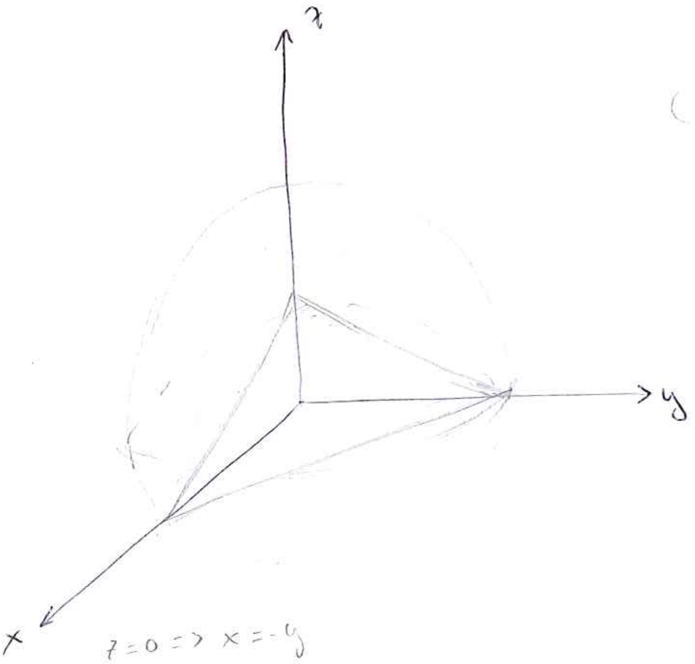
$\Rightarrow (0,2), (0,-2)$
$(2,0), (-2,0)$

$$\nabla g_1=\bar{0}? \quad \nabla g_1 \equiv \begin{cases} \dfrac{\partial g}{\partial x}=2x=0 \\ \dfrac{\partial g}{\partial y}=2y=0 \end{cases} \Rightarrow (0,0)\in D$$

iv) $f(x,y,z) = xyz$ $\quad (x-0)^2+(y-0)^2+(z-0)^2=1$

esfera $\qquad$ Planoa

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1,\ x+y+z=0\}$$



$(0,0) \notin D$

$z=0 \Rightarrow x=-y$

1) D-ren barnean

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = yz = 0 \\ \frac{\partial f}{\partial y} = xz = 0 \\ \frac{\partial f}{\partial z} = xy = 0 \end{cases} \Rightarrow (0,0,0) \notin D$$

2) D-ren mugan

2.1) Esferan

$x^2+y^2+z^2=1$

$g_1 = x^2+y^2+z^2=1$ "c_1"

$h(\lambda,x,y,z) = f(x,y,z) - \lambda(x^2+y^2+z^2-1) =$

$= xyz - \lambda x^2 - \lambda y^2 - \lambda z^2 + \lambda$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} yz - 2\lambda x = 0 \\ xz - 2\lambda y = 0 \\ xy - 2\lambda z = 0 \\ -x^2-y^2-z^2+1=0 \end{cases} \Rightarrow \begin{matrix} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{matrix}$$

$\nabla g_1 = \bar{0}$?

$(2x + y^2 + z^2, \ x^2 + 2y + z^2, \ x^2 + y^2 + 2z) = 0 \Rightarrow (0,0,0) \notin D$

2.2) Planoan

$x + y + z = 0$

$g_2(x,y,z) = x + y + z = 0^{c_2}$

$h(\lambda, x, y, z) = f(x,y,z) - \lambda(x + y + z) =$

$= xyz - \lambda x - \lambda y - \lambda z = 0$

$$\nabla h = \bar{0} = \begin{cases} yz - \lambda = 0 \\ xz - \lambda = 0 \\ xy - \lambda = 0 \\ -x - y - z = 0 \end{cases} \Rightarrow (0,0,0) \notin D$$

$\nabla g_2 = \bar{0}$?

$(y+z, \ x+z, \ x+y) = \bar{0} \Rightarrow (0,0,0) \notin D$

2.3) Ebaki puntuak

- $h(\lambda, \mu, x, y, z) = f - \lambda(x^2 + y^2 + z^2 - 1) - \mu(x + y + z)$

- $\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y + z = 0 \end{cases}$

3) Ebaluatu

$f(1,0,0) = f(0,1,0) = f(0,0,1) = (0,0,0)$!

2. GAIA: FUNTZIO INPLIZITUAK

$F(x,y)=0$ adierazpena emanik, noiz lortu daiteke $y=f(x)$ bezala adieraztea non $F(x,f(x))=0$ den $\forall x \in J$?

ADIBIDEAK

1) $F(x,y)=2x-y=0 \xrightarrow{y=f(x)} y=2x=f(x)$

2) $F(x,y)=x^2y+\sin y+e^y=0 \xrightarrow{y=f(x)}$?

- $y=f(x) \longrightarrow y$ x-en funtzio ESPLIZITUA da
- $F(x,y)=0 \longrightarrow y$ x-en funtzio INPLIZITUA da

TEOREMA 2.1: FUNTZIO INPL. TMA-REN KASU PARTIKULARRA

$F:\mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ C^1 klasekoa

Idatz dezagun $(\bar{x},z)\in\mathbb{R}^{n+1}$ non $\bar{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$ $z\in\mathbb{R}$. Izan bedi $(\bar{x}_0,z_0)\in\mathbb{R}^{n+1}$ non

$$\Rightarrow \begin{cases} 1)\ F(\bar{x}_0,z_0)=0 \\ 2)\ \dfrac{\partial F}{\partial z}(\bar{x}_0,z_0)\neq 0 \end{cases}$$

$\Longrightarrow$ $\exists$ dira $U\subset\mathbb{R}^n$ $\bar{x}_0$-ren ingurune bat, $V\subset\mathbb{R}$ z_0-ren ingurune bat eta $g:U\longrightarrow V$ funtzio BAKAR bat non $F(\bar{x},\underbrace{g(\bar{x})}_{z})=0$

Gainera, g diferentziagarria da, bere deribatu partzialak jarraituak dira eta

$$\frac{\partial g}{\partial x_i}=\frac{-\dfrac{\partial F}{\partial x_i}}{\dfrac{\partial F}{\partial z}} \quad \forall i=1,\dots,n$$

OHARRA:

Behin $z=g(x)$ funtzioa existitzen dela jakinda eta diferentziagarria dela frogatuta, g-ren deribatu partzialak kalkulatzeko **DIFERENTZIAZIO INPLIZITUA** erabiliko dugu

$F(\bar{x}, g(x)) \overset{\circledast}{=} 0$ izanik, $\frac{\partial g}{\partial x_i}$ kalkulatzeko $\circledast$ adierazpena x_i-rekiko deribatuko dugu katearen erregela erabiliz

$$F \left\langle \begin{array}{l} x_i \\ z = g \longrightarrow x_i \end{array} \right. \qquad F(\bar{x}, g(\bar{x})) = 0$$

$$\frac{\partial}{\partial x_i}\left(F(\bar{x}, g(\bar{x}))\right) = \frac{\partial}{\partial x_i} \cdot 0$$

$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x_i} = 0$$

$$\Rightarrow \frac{\partial g}{\partial x_i} = \frac{-\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial z}}$$

ADIBIDEA

$x^2 + z^2 = 1 \longrightarrow z = g(x)$ bilatu eta $\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x}$

$F(x, z) = x^2 + z^2 - 1 = 0 \longrightarrow F$ C^1 klasekoa (polinomioa)

$F : \mathbb{R}^{1+1} \longrightarrow \mathbb{R}$

1) $F(x_0, z_0) = x_0^2 + z_0^2 - 1 = 0$

2) $\frac{\partial F}{\partial z} = 2z \Big|_{(x_0, z_0)} = 2z_0 \neq 0 \Rightarrow z_0 \neq 0$

Ondorioz, (x_0, z_0) puntuak $x_0^2 + z_0^2 = 1 \wedge z_0 \neq 0$ betetzean $\overset{TEOR}{\Longrightarrow} \exists g$ funtzio bakarra $x_0 \in U \subset \mathbb{R}$ puntuaren U ingurunean non $z = g(x)$ eta $F(x, g(x)) = 0$

$x^2 + z^2 = 1 \Rightarrow z = \pm\sqrt{1 - x^2}$

- $z_0 > 0 \longrightarrow V_1$ ingurunean $g_1(x) = \sqrt{1 - x^2}$ $\Rightarrow U \to |x_0| \leq 1$
- $z_0 < 0 \longrightarrow V_2$ ingurunean $g_2(x) = -\sqrt{1 - x^2}$

$\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial x}$ KALKULATU

Deribatu inplizituki

$z = g(x) \qquad F(x,z) = 0 \qquad x^2 + z^2 = 0$

$$2x + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-2x}{2z} = -\frac{x}{z}$$

ADIBIDEA

$$x^3 + 8xz^2 + 3y^2 - 3z^3y = 1$$

Noiz adierazi daiteke gainazala $z = g(x,y)$ funtzio diferentziagarri baten grafikoa bezala?

polinomioa

- $F(x,y,z) = x^3 + 8xz^2 + 3y^2 - 3z^3y - 1$ C^1 klasekoa
- $F(x_0,y_0,z_0) = 0 \wedge \frac{\partial F}{\partial z} = 16xz - 9z^2y \Big|_{(x_0,y_0,z_0)} \neq 0$

$z_0 \neq 0 \quad 16x_0 - 9z_0y_0 \neq 0 \wedge F(x_0,y_0,z_0) = 0$

$\Rightarrow \exists U \subseteq \mathbb{R}^2$ (x_0,y_0)-ren ingurune bat

$V \subset \mathbb{R}$ z_0-ren ingurune bat eta $g: U \to V$

bakar bat non $F(x,y,\underbrace{g(x,y)}_{z}) = 0$

TEOREMA 2.2: FUNTZIO INPLIZITUEN TEOREMA OROKORRA

Kontsidera dezagun honako sistema

$$(*) \begin{cases} F_1(x_1,\dots,x_n,z_1,\dots,z_m) = 0 \\ \vdots \\ F_m(x_1,\dots,x_n,z_1,\dots,z_m) = 0 \end{cases}$$ non F_i C^1 den $\forall i = 1,\dots,m$

$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_m} \end{vmatrix}$$

Izan bitez $\bar{x}_0 \in \mathbb{R}^n$, $\bar{z}_0 \in \mathbb{R}^m$ eta demagun

$F_i(\bar{x}_0, \bar{z}_0) = 0$ dela $\forall i = 1,\dots,m \wedge \Delta(\bar{x}_0,\bar{z}_0) \neq 0$

$\Rightarrow (\bar{x}_0, \bar{z}_0)$ puntuko ingurune batean $\exists h_1,\dots,h_m$

C^1 klasekoak non $z_i = h_i(x_1,\dots,x_n)$

⊛ Sistemaren soluzioak diren, bi funtzioak bakarrak dira $(\bar{x}_0, \bar{z}_0)$ puntuaren ingurune batean eta beraien deribatu partzialak kalkulatzeko deribatu inplizitua erabili daiteke.

ADIBIDEA:

Frogatuko dugu ondorengo sistema $(x_0, y_0, u_0, v_0) = (1,1,1,1)$ puntuaren ingurune batean ebatzi daitekeela,

$$* \begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$$ non $u = u(x,y) \wedge v = v(x,y)$ funtzio bakarrak izanik

Ere kalkulatuko dugu $\frac{\partial u}{\partial x}(1,1)$

[TEOREMA 2.2]

$$F_1(x,y,u,v) = xu + yvu^2 - 2$$
$$F_2(x,y,u,v) = xu^3 + y^2v^4 - 2$$ ⇒ C^1 klasekoak, polinomioak

$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \begin{vmatrix} x + 2yvu & yu^2 \\ 3xu^2 & 4y^2v^3 \end{vmatrix}$$

$$F_1(1,1,1,1) = 0 \qquad \Delta(1,1,1,1) = \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} = 9 \neq 0$$
$$F_2(1,1,1,1) = 0$$

$\underset{\Longrightarrow}{\text{Teor}}$ $\exists u, v \in C^1$ $u = u(x,y) \wedge v = v(x,y)$ bakarrak sistemaren* soluzio direnak $(1,1,1,1)$ puntuaren ingurune batean.

$$\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases} \xrightarrow[x\text{-ekiko}]{\text{deribatu}} \begin{cases} u + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x}u^2 + 2yvu\frac{\partial u}{\partial x} = 0 \\ 3xu^2\frac{\partial u}{\partial x} + 4y^2v^3\frac{\partial v}{\partial x} = 0 \end{cases}$$

$$\xrightarrow[\text{ordezkatu}]{\text{puntua}} \begin{cases} 1 + \frac{\partial u}{\partial x}(1,1) + 1 \cdot \frac{\partial v}{\partial x}(1,1) \cdot 1 + 1 \cdot 1 \cdot 2 \cdot 1 \cdot \frac{\partial u}{\partial x}(1,1) \overset{(1)}{=} 0 \\ 1 + 3\frac{\partial u}{\partial x}(1,1) + 4\frac{\partial v}{\partial x}(1,1) \overset{(2)}{=} 0 \end{cases}$$

(2) $\Rightarrow \frac{\partial v}{\partial x}(1,1) = -\frac{1}{4} - \frac{3}{4}\frac{\partial u}{\partial x}(1,1)$

(1) $1 + \frac{\partial u}{\partial x}(1,1) - \frac{1}{4} - \frac{3}{4}\cdot\frac{\partial u}{\partial x}(1,1) + 2\frac{\partial u}{\partial x}(1,1) = 0$

$\frac{3}{4} + \frac{9}{4}\frac{\partial u}{\partial x} = 0$

$\Rightarrow \frac{\partial u}{\partial x}(1,1) = -\frac{1}{3} \Rightarrow \frac{\partial v}{\partial x}(1,1) = 0$

[Adibidea 43. orrialdean]

2.2- ALDERANTZIZKO FUNTZIOAREN TEOREMA

· FUNTZIO INPLIZITUAREN KASU PARTIKULARRA

$\left[\begin{array}{l} f: \mathbb{R} \to \mathbb{R} \\ y = f(x) \Rightarrow \exists\, g \text{ funtzioa } (f\text{-ren alderantzizkoa}), \\ \qquad x = g(y), \text{ non } y = f(g(y)) ? \end{array}\right]$

$\to F(x,y) = y - f(x) = 0$

ADIBIDEA

$y = e^x = f(x) \Rightarrow \underbrace{\ln y}_{g(y)} = x$

TEOREMA 2.3: ALDERANTZIZKO FUNTZIOAREN TEOREMA

Izan bitez,

- $U \subset \mathbb{R}^n$ irekia
- $f: U \subset \mathbb{R}^n \to \mathbb{R}$ C^1 KLASEKOA
- $\bar{x}_0 \in U$
- $\bar{f} = (f_1, \dots, f_n)$

Baldin eta $J(\bar{f})(x_0) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{x}_0) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(\bar{x}_0) & \cdots & \frac{\partial f_n}{\partial x_n}(\bar{x}_0) \end{vmatrix} \neq 0$

$$\Rightarrow \begin{cases} y_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ y_n = f_n(x_1, \dots, x_n) \end{cases}$$ Sistema ebatzi deitzeko

$\bar{x} = \bar{g}(\bar{y})$ funtzioaren bidez, $\bar{x}$ $\bar{x}_0$-tik hurbil eta $\bar{y}$ $f(\bar{x}_0)$-tik hurbil daudenean

Gainera, soluzioa bakarra da eta $\bar{g} = (g_1, \dots, g_n)$ C^1 klasekoa da,

$$\Rightarrow \begin{cases} x_1 = g_1(y_1, \dots, y_n) \\ \vdots \\ x_n = g_n(y_1, \dots, y_n) \end{cases}$$

ARIBIDEA: Egiaztatuko dugu $f(x,y) = (\underbrace{x^2 + xy - y^3}_{f_1}, \underbrace{3xy^2 - x^5 y + 1}_{f_2})$ funtzioak alderantzizkoa duela $(1,0)$ puntuaren ingurune batean eta emango dugu alderantzizkoaren lehen mailako hurbilketa polinomiko bat.

$$\begin{cases} f_1 = x^2 + xy - y^3 \\ f_2 = 3xy^2 - x^5 y + 1 \end{cases}$$ C^1 klasekoak (polinomioak)

$$J(\bar{f})(1,0) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 3x + y & x - 3y^2 \\ 3y^2 - 5x^4 y & 6xy - x^5 \end{vmatrix}_{(1,0)} = -2 \neq 0$$

$\overset{\text{Teor}}{\Longrightarrow} \exists x = x(u,v) \wedge y = y(u,v)$ C^1 klasekoak $(1,1)$ puntuaren ingurune batean definituta

$f_1(1,0) = 1$, $f_2(1,0) = 1$

$$\text{non} \begin{cases} (x(u,v))^2 + x(u,v) \cdot y(u,v) - (y(u,v))^2 = u \\ 3x(u,v) \cdot (y(u,v))^2 - (x(u,v))^5 \cdot y(u,v) + 1 = v \end{cases}$$

x eta y C^1 klasekoak direnez $(\overset{u_0}{1},\overset{v_0}{1})$ puntuaren inguruune batean, Taylorren lehen mailako polinomioak kalkulatu ahal ditugu.

$u = u(x,y)$

$v = v(x,y)$

$$x(u,v) = x(u_0,v_0) + \frac{\partial x}{\partial u}(u_0,v_0)\cdot(u-u_0) + \frac{\partial x}{\partial v}(u_0,v_0)(v-v_0) + R_x$$

$$y(u,v) = y(u_0,v_0) + \frac{\partial y}{\partial u}(u_0,v_0)\cdot(u-u_0) + \frac{\partial y}{\partial v}(u_0,v_0)(v-v_0) + R_x$$

$$\begin{cases} u = x^2 + xy - y^3 \\ v = 3xy^2 - x^5y + 1 \end{cases}$$

u-rekiko $\Rightarrow$
$$\begin{cases} 1 = 2x\frac{\partial x}{\partial u} + \frac{\partial x}{\partial u}y + x\frac{\partial y}{\partial u} - 3y^2\frac{\partial y}{\partial u} \\ \overset{(1)}{0} = 3\frac{\partial x}{\partial u}y^2 + 3x2y\frac{\partial y}{\partial u} - 5x^4\frac{\partial x}{\partial u}y - x^5\frac{\partial y}{\partial u} + 0 \end{cases}$$

v-rekiko $\Rightarrow$
$$\begin{cases} 0 = 2x\frac{\partial x}{\partial v} + \frac{\partial x}{\partial v}y + x\frac{\partial y}{\partial v} - 3y^2\frac{\partial y}{\partial v} \\ \overset{(2)}{1} = 3\frac{\partial x}{\partial v}y^2 + 3x2y\frac{\partial y}{\partial v} - 5x^4\frac{\partial x}{\partial v}y - x^5\frac{\partial y}{\partial v} + 0 \end{cases}$$

$$(1)\ \xrightarrow[u=1,\ v=1]{x=1,\ y=0} \begin{cases} 2\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} = 1 \\ -\frac{\partial y}{\partial u} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial y}{\partial u} = 0 \\ \frac{\partial x}{\partial u} = \frac{1}{2} \end{cases}$$

$$(2)\ \xrightarrow[v=1,\ u=1]{x=1,\ y=0} \begin{cases} 2\frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} = 0 \\ -\frac{\partial y}{\partial v} = 1 \end{cases} \Rightarrow \begin{cases} \frac{\partial y}{\partial v} = -1 \\ \frac{\partial x}{\partial v} = \frac{1}{2} \end{cases}$$

ordezkatu

$$x(u,v) \sim 1 + \frac{1}{2}(u-1) + \frac{1}{2}(v-1) = \boxed{\frac{u+v}{2}}$$

$$y(u,v) \sim 0 + 0\cdot(u-1) - 1\cdot(v-1) = \boxed{1-v}$$

ANALISI BEKTORIALA ETA KONPLEXUA

2. Gaia: FUNTZIO INPLIZITUAK

Ariketak

1. Izan bedi $f(x, y, z) = \sin z + (1 + x^2)^y + z + y^2 - 2y$.

   (i) Frogatu $f(x, y, z) = 0$ ekuazioak funtzio inplizitu bat definitzen duela, $z = g(x, y)$, $(0, 1, 0)$ puntuaren inguruan.

   (ii) Kalkulatu g-ren lehen eta bigarren ordenako deribatuak $(0, 1)$ puntuan.

   Em.: $g_x(0, 1) = g_y(0, 1) = g_{xy}(0, 1) = 0$, $g_{xx}(0, 1) = g_{yy}(0, 1) = -1$.

2. Izan bitez $x^2+y^2+z^2-3xyz = 0$ ekuazioa, $f(x, y, z) = xy^2z^3$ funtzioa eta $(x_0, y_0, z_0) = (1, 1, 1)$ puntua.

   (i) Frogatu emandako ekuazioak $z = z(x, y)$ funtzioa inplizituki definitzen duela (x_0, y_0, z_0) puntuaren ingurune batean. Kalkula ezazu $\dfrac{\partial h}{\partial x}(1, 1)$ baldin eta $h(x, y) = f(x, y, z(x, y))$ bada.

   (ii) Frogatu emandako ekuazioak $y = y(x, z)$ funtzioa inplizituki definitzen duela (x_0, y_0, z_0) puntuaren ingurune batean. Kalkula ezazu $\dfrac{\partial h}{\partial x}(1, 1)$ baldin eta $h(x, z) = f(x, y(x, z), z)$ bada.

   Em.: (i) -2, (ii) -1.

3. $z = z(x, y)$ funtzio inplizitua $F\left(x + \dfrac{z}{y}, y + \dfrac{z}{x}\right) = 0$ ekuazioaren bidez definitzen da. Kalkula ezazu

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}.$$

   Em.: $z - xy$.

4. Izan bedi $x^2 + y^2 + z^2 = yf\left(\dfrac{z}{y}\right)$ ekuazioak definitutako $z = z(x, y)$ funtzio inplizitua. Froga ezazu ondorengo berdintza:

$$(x^2 - y^2 - z^2)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = 2xz.$$

5. $z^3 - 2xz + y = 0$ ekuazioak $z = z(x, y)$ funtzio inplizitua definitzen du $(1, 1, 1)$ puntuaren ingurune batean. Kalkulatu funtzio horren Taylorren bigarren mailako garapena $(1, 1)$ puntuan.

   Em.: $z(x, y) = 1 + 2(x - 1) - (y - 1) - 8(x - 1)^2 + 10(x - 1)(y - 1) - 3(y - 1)^2 + R_2$.

6. Aztertu itzazu $x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0$ ekuazioaren bidez definitutako $z(x, y)$ funtzio inplizituen mutur lokalak.

$y = f(x)$

$f\ (1+y^2)^x$

+ 7. Izan bedi $f\colon \mathbf{R}^5 \longrightarrow \mathbf{R}^2$ ondoren definitutako funtzioa:

$$f(x,y,z,u,v) = \left(u + v + x^2 - y^2 + z^2, u^2 + v^2 + u - 2xyz\right).$$

(i) Frogatu $f(x,y,z,u,v) = (0,0)$ sistemak $(u,v) = (h_1(x,y,z), h_2(x,y,z))$ funtzio inplizitu diferentziagarri bat definitzen duela (0,0,0,-1/2,1/2) puntuaren ingurune batean.

(ii) Kalkulatu $(h_1(x,y,z), h_2(x,y,z))$ funtzioaren deribatua (0,0,0) puntuan.

$$Em.: \frac{\partial h_i}{\partial x}(0,0,0) = \frac{\partial h_i}{\partial y}(0,0,0) = \frac{\partial h_i}{\partial z}(0,0,0) = 0,\ i = 1,2.$$

+ 8. Izan bedi hurrengo sistema:

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0, \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0. \end{cases}$$

Froga ezazu sistema honek $u = u(x,y)$ eta $v = v(x,y)$ funtzioak definitzen dituela $(2,-1)$ puntuaren ingurune batean, $u(2,-1) = 2$ eta $v(2,-1) = 1$ izanik. Kalkula itzazu $\dfrac{\partial u}{\partial x}$ eta $\dfrac{\partial u}{\partial y}$ $(2,-1)$ puntuan.

$$Em.: \frac{\partial u}{\partial x}(2,-1) = \frac{13}{32},\ \frac{\partial u}{\partial y}(2,-1) = \frac{5}{32}.$$

+ 9. Aztertu ea ondorengo sistemak $x(u,v,w)$, $y(u,v,w)$, $z(u,v,w)$ funtzioak definitzen dituen $(0,0,0)$ puntuaren inguruan :

$$\begin{cases} u(x,y,z) = x + xyz, \\ v(x,y,z) = y + xy, \\ w(x,y,z) = z + 2x + 3z^2. \end{cases}$$

10. Izan bedi $T\colon \mathbf{R}^2 \to \mathbf{R}^2$, $T(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$. Noiz existitzen da transformazio honen alderantzizkoa, $T^{-1}\colon \mathbf{R}^2 \to \mathbf{R}^2$, $T^{-1}(x,y) = (r(x,y), \theta(x,y))$?

11. Izan bedi $T\colon \mathbf{R}^3 \to \mathbf{R}^3$, $T(\rho,\theta,\varphi) = (x(\rho,\theta,\varphi), y(\rho,\theta,\varphi), z(\rho,\theta,\varphi))$,

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \\ z = \rho\cos\varphi. \end{cases}$$

Azter ezazu noiz existitzen den T-ren alderantzizkoa, $T^{-1}\colon \mathbf{R}^3 \to \mathbf{R}^3$,

$$T^{-1}(x,y,z) = (\rho(x,y,z), \theta(x,y,z), \varphi(x,y,z)).$$

2. Funtzio inplizituak — Ariketak

2. Ariketa

$z = g(x,y) = 0$

$z = 0$

$$x^2+y^2+z^2-3xyz=0$$

$$f(\overline{x,y},\overline{z}) = xy^2z^3$$

$$(x_0,y_0,z_0) = (1,1,1)$$

$f(x,y,z) = f(x, y, g(x,y))$

0

i) $z = z(x,y)$

$$F(\underbrace{x,y}_{\bar{x}},\underbrace{z}_{z}) = x^2+y^2+z^2-3xyz=0$$

$g(x,y)=0$

$$F: \mathbb{R}^{2+1} \longrightarrow \mathbb{R}, \quad F \in C^1$$

$$\bar{x} = (x,y) \in \mathbb{R}^2 \wedge z \in \mathbb{R}$$

1) $F(1,1,1) = 0$

2) $\frac{\partial F}{\partial z}(1,1,1) = 2z - 3xy\Big|_{(1,1,1)} = -1 \neq 0$

Teorema $\Longrightarrow$ $\exists$ dira $U \subset \mathbb{R}^2$ $(1,1)$-en ingurune bat, $V \subset \mathbb{R}$ $z_0 = 1$-en ingurune bat eta $g: U \to V$ bakar bat non $F(x,y,g(x,y)) = 0$

$$h(x,y) = f(x,y,z(x,y))$$

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = y^2z^3 + 3xy^2z^2\cdot\frac{\partial z}{\partial x}$$

$\frac{\partial z}{\partial x} \longrightarrow$ Deribatu inplizituki $x^2+y^2+z^2-3xy = 0$ $\xrightarrow[x\text{-ekiko}]{\text{deribatu}}$

$$2x + 2z\frac{\partial z}{\partial x} - 3yz - 3xy\frac{\partial z}{\partial x} = 0 \xrightarrow[(1,1,1)]{\text{ordezkatu}}$$

$$2 + 2\frac{\partial z}{\partial x}(1,1) - 3 - 3\frac{\partial z}{\partial x}(1,1) = 0$$

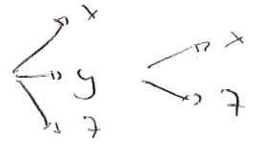
$$\frac{\partial z}{\partial x}(1,1) = -1$$

$$\frac{\partial h}{\partial x}(1,1) \overset{(1,1,1)}{=} 1 + 3\cdot(-1) = -2$$

ii) $y = y(x, z)$

1) $F(1,1,1) = 0$ TEOR... $\Rightarrow$

2) $\frac{\partial F}{\partial y}(1,1,1) \neq 0$

$h(x,z) = f(x, y(x,z), z) \Rightarrow h \to f$ 

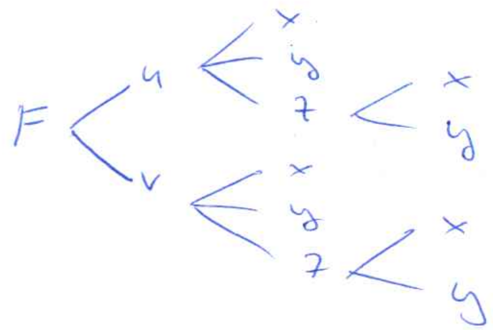
3. ARIKETA

$z = z(x,y)$ funtzio INPLIZITUA [Eskatzen ez bada, ez egin frogapenik]

$F(\underbrace{x + \frac{z}{y}}_{u}, \underbrace{y + \frac{z}{x}}_{v}) = 0$

KALKULATU

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$



x-rekiko

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial u} + \frac{\partial F}{\partial u}\cdot\frac{1}{y}\cdot\frac{\partial z}{\partial x} + \frac{\partial F}{\partial v}\left(-\frac{z}{x^2}\right) + \frac{\partial F}{\partial v}\cdot\frac{1}{x}\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial F}{\partial u} - \frac{z}{x^2}\frac{\partial F}{\partial v}}{-\frac{1}{y}\frac{\partial F}{\partial u} - \frac{1}{x}\frac{\partial F}{\partial v}}$$

y-rekiko

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial F}{\partial u}\left(-\frac{z}{y^2}\right) + \frac{\partial F}{\partial u}\frac{1}{y} + \frac{\partial F}{\partial v}\cdot 1 + \frac{\partial F}{\partial v}\cdot\frac{1}{x}\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{\frac{\partial F}{\partial u}\left(\frac{-z}{y^2}\right) + \frac{\partial F}{\partial v}}{-\frac{1}{y}\frac{\partial F}{\partial u} - \frac{1}{x}\frac{\partial F}{\partial v}}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

$$x\cdot\left(\frac{\frac{\partial F}{\partial u} - \frac{z}{x^2}\frac{\partial F}{\partial v}}{-1/y\frac{\partial F}{\partial u} - \frac{1}{x}\frac{\partial F}{\partial v}}\right) + y\cdot\left(\frac{-\frac{z}{y^2}\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}}{-\frac{1}{y}\frac{\partial F}{\partial u} - \frac{1}{x}\frac{\partial F}{\partial v}}\right) =$$

$$= [\ldots] = z - xy$$

5. ARIKETA

$z^3 - 2xz + y = 0$ $\quad z = z(x,y)$ funtzio INPLIZITUA

$(1,1,1)$ puntuaren ingurune bateen T_2 $(1,1)$-ean

$$z(x,y) = z(1,1) + \frac{\partial z}{\partial x}(1,1)(x-1) + \frac{\partial z}{\partial y}(1,1)(y-1) +$$
$$+ \frac{1}{2}\frac{\partial^2 z}{\partial x^2}(1,1)\cdot(x-1)^2 + \frac{1}{2}\frac{\partial^2 z}{\partial y^2}(1,1)\cdot(y-1)^2 +$$
$$+ \frac{\partial^2 z}{\partial x \partial y}(1,1)\cdot(x-1)(y-1) + R_2$$

x-rekiko deribatu

$$3z^2\cdot\frac{\partial z}{\partial x} - 2z - 2x\frac{\partial z}{\partial x} + 0 = 0 \qquad (1)$$

$$\xrightarrow{(1,1,1)} 3\frac{\partial z}{\partial x}(1,1) - 2 - 2\frac{\partial z}{\partial x}(1,1) = 0 \Rightarrow \underline{\frac{\partial z}{\partial x}(1,1) = 2}$$

y-rekiko deribatu

$$3z^2\frac{\partial z}{\partial y} - 2x\frac{\partial z}{\partial y} + 1 = 0 \qquad (2)$$

$$\xrightarrow{(1,1,1)} \frac{\partial z}{\partial y}(1,1) = -1$$

(1) x-rekiko

$$6z\frac{\partial z}{\partial x}\frac{\partial z}{\partial x}+3z^2\cdot\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}-2x\frac{\partial^2 z}{\partial x^2}=0$$

$(1,1,1) \to$
$$6\cdot 2\cdot 2+3\frac{\partial^2 z}{\partial x^2}(1,1)-2\cdot 2-2\cdot 2-2\frac{\partial^2 z}{\partial x^2}(1,1)=0$$

$$\frac{\partial^2 z}{\partial x^2}(1,1)=-16$$

(1) y-rekiko

$$6z\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}-2\frac{\partial z}{\partial y}-2x\frac{\partial^2 z}{\partial x\partial y}=0 \to \text{Gaizki}$$

$(1,1,1) \to$
$$6\cdot(-1)\cdot 2-2\cdot(-1)-2\frac{\partial^2 z}{\partial x\partial y}(1,1)=0$$

$$\frac{\partial^2 z}{\partial x\partial y}(1,1)=10$$

(2) y-rekiko

[...] $\quad\frac{\partial^2 z}{\partial y^2}(1,1)=-6$

$$\boxed{z(x,y)=1+2(x-1)-(y-1)-8(x-1)^2+10(x-1)(y-1)-3(y-1)^2+R^2}$$

6. ARIKETA

$x^2+y^2+z^2-2x+2y-4z=0 \qquad \underline{z(x,y) \text{ INPL.}}$

· z-ren muturrak [1. Gaian]

Deribatu x eta y-rekiko kontuan hartuz

$$\nabla z=\bar{0} \Rightarrow \begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{1-x}{z-2}\\[2ex] \dfrac{\partial z}{\partial y}=\dfrac{-1-y}{z-2}\end{cases}$$

$x=1 \wedge y=-1 \Rightarrow (1,-1)$ PUNTU KRITIKOA

Berriro deribatu

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

+ aurrekoko hasierako ek. (1,-1) ordezkatu

4

$$z^2-4z-12=0 \quad \begin{cases} \to z=6 \\ \to z=-2 \end{cases}$$

$$\frac{\partial^2 z}{\partial x^2}(1,-1) \quad \begin{cases} \xrightarrow{z=6} -1/4 \\ \xrightarrow{z=-2} -1/4 \end{cases}$$

[...]

$$\frac{\partial^2 z}{\partial y^2}(1,-1) \quad \begin{cases} \xrightarrow{z=6} -1/4 \\ \xrightarrow{z=-2} 1/4 \end{cases}$$

$$\frac{\partial^2 z}{\partial x \partial y}(1,-1) \quad \begin{cases} \xrightarrow{z=6} 0 \\ \xrightarrow{z=-2} 0 \end{cases}$$

$\boxed{z=6}$

$$|H(1,-1)| = \begin{vmatrix} \overset{<0}{-1/4} & 0 \\ 0 & -1/4 \end{vmatrix} > 0 \Rightarrow \text{MAXIMO LOKALA } (1,-1)\text{-n}$$

$\boxed{z=-2}$

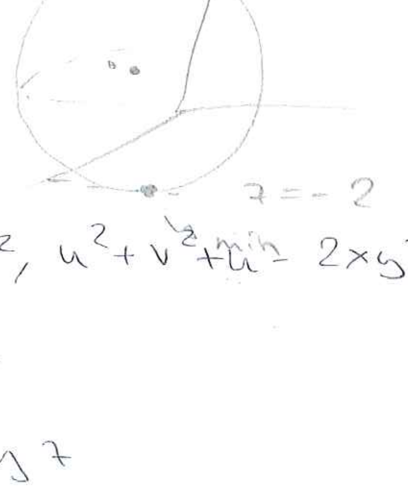
$$|H(1,-1)| = \begin{vmatrix} \overset{>0}{1/4} & 0 \\ 0 & 1/4 \end{vmatrix} > 0 \Rightarrow \text{MINIMO LOKALA } (1,-1)\text{-n}$$

Nolatan?

$$x^2+y^2+z^2-2x+2y-4z-10=0$$

$$(x-1)^2-1+(y+1)^2-1+(z-2)^2-4-10=0$$

$$(x-1)^2+(y+1)^2+(z-2)^2=16$$



$$\text{ESFERA} \quad \begin{cases} \to \text{ZENTROA } (1,-1,2) \\ \to \text{ERRADIOA } 4 \end{cases}$$

7. ARIKETA

$$f(x,y,z,u,v) \overset{(1)}{=} (u+v+x^2-y^2+z^2,\ u^2+v^2+u-2xyz)$$

i) $F_1(x,y,z,u,v) \overset{(2)}{=} u+v+x^2-y^2+z^2$

$F_2(x,y,z,u,v) = u^2+v^2+u-2xyz$

$$\left(\underset{x_0}{0}, \underset{y_0}{0}, \underset{z_0}{0}, \underset{u_0}{-\frac{1}{2}}, \underset{v_0}{\frac{1}{2}}\right)$$

$\rightarrow C^1$ KLASEKOAK

1) $F_1(0,0,0,-\frac{1}{2},\frac{1}{2})=0$ $\qquad u=h_1(x,y,z)$

$F_2(0,0,0,-1/2,1/2)=0$ $\qquad v=h_2(x,y,z)$

2) $$\Delta=\begin{vmatrix}\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v}\\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2u+1 & 2v\end{vmatrix}_{(0,0,0,-1/2,1/2)}=1\neq 0$$

$\overset{\text{TEOR}}{\underset{2.2}{\Longrightarrow}}\ \exists h_1,h_2\ C^1$ bakoitzan $(0,0,0,-1/2,1/2)$ inguruan

bakoitzan non $\begin{cases}F_1=0\\F_2=0\end{cases}$ sistema betetzen da

ii) $\frac{\partial h_i}{\partial x},\frac{\partial h_i}{\partial y},\frac{\partial h_i}{\partial z}\ (0,0,0)\quad i=1,2$

(1) $\rightarrow$ x-rekiko deribatu

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0 \longrightarrow \frac{\partial u}{\partial x}=0$$

(2) $\rightarrow$ x-rekiko deribatu

$$\frac{\partial v}{\partial x}=0$$

$$\boxed{\frac{\partial h_1}{\partial x}(0,0,0)=0 \qquad \frac{\partial h_2}{\partial x}(0,0,0)=0}$$

9. ARIKETA

$$\begin{cases}u(x,y,z)=x+xyz=f_1\\ v(x,y,z)=y+xy=f_2\\ w(x,y,z)=z+2x+3z^2=f_3\end{cases}\xrightarrow{(0,0,0)}\begin{cases}x(u,v,w)\\ y(u,v,w)\\ z(u,v,w)\end{cases}$$

- $\bar f(x,y,z)=(f_1,f_2,f_3)$ $\quad f_i\ C^1$ klasekoak $\mathbb{R}$-n (polinomioak)

- $\bar x_0=(0,0,0)$

$$J(\bar f)(\bar x_0)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}1+yz & xy & xy\\ y & 1+x & 0\\ 2 & 0 & 1+6z\end{vmatrix}$$

6

ordezkatu $(0,0,0)$ $= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

ALD. FUN. TEOR $\xrightarrow[2.3]{\text{TEOR}}$ Hasierako sistema ebatzi daiteke $x(u,v,w)$, $y(u,v,w)$ eta $z(u,v,w)$ funtzioen bidez $\bar{x}=(x,y,z)$ puntura $\bar{x}_0=(0,0,0)$-tik gertu eta $\bar{f}$ $\bar{f}(0,0,0)$-tik gertu dagoenean

11. ARIKETA

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(\rho,\theta,\varphi) = (\underbrace{x(\rho,\theta,\varphi)}_{\rho\cos\theta\sin\varphi}, \underbrace{y(\rho,\theta,\varphi)}_{\rho\sin\theta\sin\varphi}, \underbrace{z(\rho,\theta,\varphi)}_{\rho\cos\varphi})$

$T^{-1}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T^{-1}(x,y,z) = (\rho(x,y,z), \theta(x,y,z), \varphi(x,y,z))$ Non?

- $x = \rho\cos\theta\sin\theta = T_1$
  $y = \rho\sin\theta\cos\varphi = T_2$ $\Rightarrow C^1$ klasekoak $\mathbb{R}^2$ osoan (polinomio eta trigonometrikoak)
  $z = \rho\cos\varphi = T_3$
  $\bar{T} = (T_1, T_2, T_3)$

- $$J(\bar{T})(\bar{x}_0) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} = [\dots] = -\rho^2\sin\varphi$$

$-\rho^2\sin\varphi \neq 0 \xRightarrow[2.3]{\text{TEOR}} \exists\, \bar{T}^{-1} \Rightarrow \boxed{\begin{cases} \rho \neq 0 \\ \wedge \\ \sin\varphi \neq 0 \iff \varphi \neq k\pi,\ k \in \mathbb{Z} \end{cases}}$

7

1. ARIKETA

$$f(x,y,z) = \sin z + (1+x^2)^y + z + y^2 - 2y$$

i) Frogatu $f(x,y,z)=0$ ek. funtzio inplizitu bat definitzen duela $z=g(x,y)$ $(0,1,0)$ puntuan f C^1 klasekoa ✓

(1) $f(0,1,0) = \sin 0 + (1+0^2)^1 + 0 + 1^2 - 2\cdot 1 = 0$ ✓

(2) $\frac{\partial F}{\partial z} = \cos z + 1 \Big|_{(0,1,0)} = 1 \neq 0$ ✓

TEOR 2.1

$\Longrightarrow \exists$ da $U \in \mathbb{R}^3$ $(0,1)$-en ingurune bat, $V \in \mathbb{R}$ $z_0=0$-ren ingurune bat eta $g: U \to V$ funtzio bakar bat non $F(\bar{x}, g(\bar{x}))=0$ den

i) Kalkulatu g-ren 1. a 2. deribatuak $(0,1)$-en

$$f(x,y,g(x,y)) = \sin(g(x,y)) + (1+x^2)^y + g(x,y) + y^2 - 2y$$

- $\frac{\partial f}{\partial x} = \cos(g(x,y)) \cdot \frac{\partial z}{\partial x} + y \cdot (1+x^2)^{y-1} \cdot 2x + \frac{\partial z}{\partial x}$

$\underset{(0,1)}{\Longrightarrow} \cos 0 \frac{\partial z}{\partial x} + 0 + \frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x} = 0}$

- $\frac{\partial f}{\partial y} = \cos(g(x,y)) \cdot \frac{\partial z}{\partial y} + (1+x^2)^y \cdot \ln(1+x^2) + \frac{\partial z}{\partial y} + 2y - 2$

$\underset{(0,1)}{\Longrightarrow} \cos 0 \cdot \frac{\partial z}{\partial y} + 0 + \frac{\partial z}{\partial y} + 2y - 2 = 0 \Rightarrow \boxed{\frac{\partial z}{\partial y} = 0}$

- $\frac{\partial^2 f}{\partial x^2} = -\sin(g(x,y)) \cdot \frac{\partial^2 z}{\partial x^2} + \cos(g(x,y)) \frac{\partial^2 z}{\partial x^2} + y \cdot (y-1) \cdot (1+x^2) \cdot 4x + y \cdot (1+x^2)^{y-1} \cdot 2 + \frac{\partial^2 z}{\partial x^2}$

$\underset{(0,1)}{\Longrightarrow} 0 + \frac{\partial^2 z}{\partial x^2} + 0 + 2 + \frac{\partial^2 z}{\partial x^2} = 0 \Rightarrow \boxed{\frac{\partial^2 z}{\partial x^2} = -1}$

- $\frac{\partial^2 f}{\partial x^2} = -\sin(g(x,y)) \cdot \frac{\partial^2 z}{\partial y^2} + \cos(g(x,y)) \cdot \frac{\partial^2 z}{\partial y^2} + (1+x^2)^y \cdot \ln^2(1+x^2) + \frac{\partial^2 z}{\partial y^2} + 2$

$$\underset{(0,1)}{\Longrightarrow} \cos 0 \cdot \frac{\partial^2 z}{\partial y^2} + 0 + \frac{\partial^2 z}{\partial y^2} + 2 = 0 \Rightarrow \boxed{\frac{\partial^2 z}{\partial y^2} = -1}$$

8. ARIKETA

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases}$$

Froga ezazu $u = u(x,y) \wedge v = v(x,y)$ definituta dituela $(2,-1)$ puntuaren inguruan batean

$u(2,-1) = 2 \wedge v(2,-1) = 1$ izanik

$f(x,y,u,v) = x^2 - y^2 - u^3 + v^2 + 4$

$g(x,y,u,v) = 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$

- f, g C^1 klasekoak (polinomioak)

- $F(2,-1,2,1) = 0$?

$f(2,-1,2,1) = 4 - 1 - 8 + 1 + 4 = 0$ ✓

$g(2,-1,2,1) = \underbrace{2 \cdot 2 \cdot (-1)}_{-4} \underbrace{+1}_{+1} \underbrace{-2 \cdot 4}_{-8} \underbrace{+3 \cdot 1}_{+3} + 8 = 0$ ✓

- $\frac{\partial F}{\partial z}(2,-1,2,1) \neq 0$

$\frac{\partial f}{\partial u}(2,-1,2,1) = -3u^2 = -12 \neq 0$

$\frac{\partial f}{\partial v}(2,-1,2,1) = 2v = 2 \neq 0$

$\frac{\partial g}{\partial u}(2,-1,2,1) = -4u = -8 \neq 0$

$\frac{\partial g}{\partial v}(2,-1,2,1) = 12v^3 = 12 \neq 0$

$\overset{\text{TEOR 2.1}}{\Longrightarrow}$ $\exists$ dira $u \in \mathbb{R}^n$ $(2,-1)$-en inguruneko bat, $V \in \mathbb{R}$ $(u,v) = (2,1)$-en ingurune bat non $F(\bar{x}, u(\bar{x}), v(\bar{x})) = 0$

$$f(x,y,u,v) = x^2 - y^2 - u^3 + v^2 + 4$$

$$g(x,y,u,v) = 2xy + y^2 - 2u^2 + 3v^4 + 8$$

$$\frac{\partial f}{\partial x} = 2x - 3\underset{2}{u}^2 \cdot \frac{\partial u}{\partial x} + 2\underset{1}{v} \cdot \frac{\partial v}{\partial x}$$

$$\xRightarrow{(2,-1)} 4 - 12 \cdot \frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = 0 \Rightarrow -6\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + 2 = 0$$

$$\frac{\partial f}{\partial y} = -2y - 3u^2\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

$$\xRightarrow{(2,-1)} +2 - 12\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = 0 \Rightarrow -6\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + 1 = 0$$

$$\frac{\partial g}{\partial x} = 2y - 4u\frac{\partial u}{\partial x} + 12v^3\frac{\partial v}{\partial x}$$

$$\xRightarrow{(2,-1)} -2 - 8\frac{\partial u}{\partial x} + 12\frac{\partial v}{\partial x} = 0 \Rightarrow 4\frac{\partial u}{\partial x} - 6\frac{\partial v}{\partial x} + 1 = 0$$

$$\frac{\partial g}{\partial y} = 2x + 2y - 4u\frac{\partial u}{\partial y} + 12v^3\frac{\partial v}{\partial y}$$

$$\xRightarrow{(2,-1)} 4 - 2 - 8\frac{\partial u}{\partial y} + 12\frac{\partial v}{\partial y} = 0 \Rightarrow -4\frac{\partial u}{\partial y} + 6\frac{\partial v}{\partial y} + 1 = 0$$

$$\begin{array}{r} 6\cdot \\ + \\ 1\cdot \end{array}\left\{\begin{array}{l} -6\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + 2 = 0 \\ 4\frac{\partial u}{\partial x} - 6\frac{\partial v}{\partial x} + 1 = 0 \end{array}\right.$$

$$-32\frac{\partial u}{\partial x} + 13 = 0 \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{13}{32}}$$

$$\begin{array}{r} 6\cdot \\ - \\ 1\cdot \end{array}\left\{\begin{array}{l} -6\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + 1 = 0 \\ -4\frac{\partial u}{\partial y} + 6\frac{\partial v}{\partial y} + 1 = 0 \end{array}\right.$$

$$-32\frac{\partial u}{\partial y} + 5 = 0 \Rightarrow \boxed{\frac{\partial u}{\partial y} = \frac{5}{32}}$$

10. ARIKETA

$$T: \mathbb{R}^2 \to \mathbb{R}^2, \quad T(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$$

Noiz existitzen da alderantzizkoa?

$T_1 = r\cos\theta$

$T_2 = r\sin\theta$ $\Rightarrow C^1$ Klasekoak (trigonometrikoak)

$$J(\bar{T})(r,\theta) = \begin{vmatrix} \frac{\partial T_1}{\partial r} & \frac{\partial T_1}{\partial \theta} \\ \frac{\partial T_2}{\partial r} & \frac{\partial T_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} =$$

$$= r\cos^2\theta + r\sin^2\theta = r \neq 0$$

$$r \neq 0 \overset{\text{TEOR 2.3}}{\Rightarrow} \exists T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2 : T^{-1}(x,y) = (r(x,y), \theta(x,y))$$

11. ARIKETA

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(\rho,\theta,\varphi) = (x(\rho,\theta,\varphi), y(\rho,\theta,\varphi), z(\rho,\theta,\varphi))$

$$\begin{cases} x = \rho\cos\theta\sin\varphi = T_1 \\ y = \rho\sin\theta\sin\varphi = T_2 \\ z = \rho\cos\varphi = T_3 \end{cases}$$

C^1 Klasekoak (trigonometrikoak)

$U \in \mathbb{R}^3$ irekia

$\bar{x}_0 \in U$

$\bar{F} = (f_1, \dots, f_n)$

$$J(T)(\rho,\theta,\varphi) = \begin{vmatrix} \frac{\partial T_1}{\partial \rho} & \frac{\partial T_1}{\partial \theta} & \frac{\partial T_1}{\partial \varphi} \\ \frac{\partial T_2}{\partial \rho} & \frac{\partial T_2}{\partial \theta} & \frac{\partial T_2}{\partial \varphi} \\ \frac{\partial T_3}{\partial \rho} & \frac{\partial T_3}{\partial \theta} & \frac{\partial T_3}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} =$$

$$= -\cos^2\theta\sin^3\varphi\,\rho^2 - \rho^2\cos^2\varphi\sin\varphi\sin^2\theta - \rho^2\cos^2\theta\cos^2\varphi\sin\varphi - \rho^2\sin^2\theta\sin^3\varphi =$$

$$= \rho^2.$$

3. GAIA: INTEGRAL BIKOITZA

1. INTEGRAL BIKOITZA ERREKTANGELU BATEN GAINEAN

$f: [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$

Partiketa

$\{x_0, x_1, \dots, x_n\}$, $x_0 = a$, $x_n = b$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1} f(p_i)\cdot(x_i - x_{i-1})$$

DEFINIZIOA

Izan bitez $a,b,c,d \in \mathbb{R}$, $D = [a,b]\times[c,d]$ eta $f: D \to \mathbb{R}$ bornatua eta kontsidera ditzagun $a = x_0 < x_1 < \dots < x_n = b$ eta $c = y_0 < y_1 < \dots < y_n = d$ partiketak.

Izan bedi $p_{ij} \in [x_{i-j}, x_i]\times[y_{j-1}, y_j]$

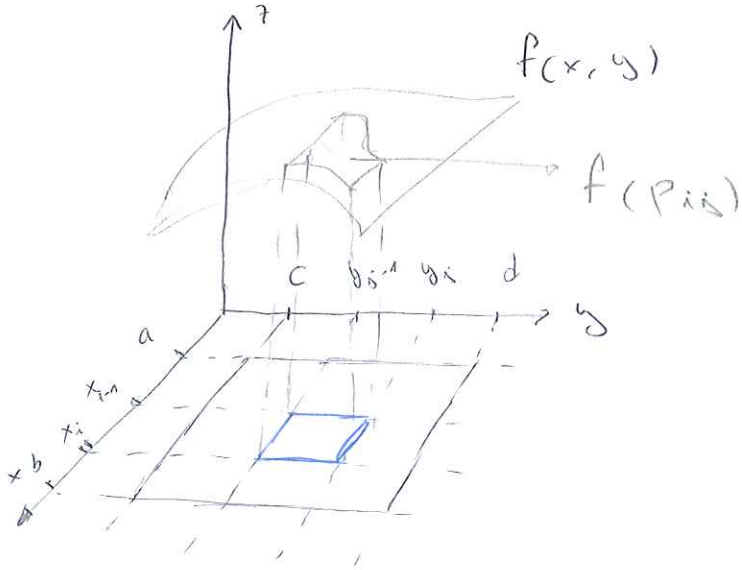
$i = 1, \dots, n$ eta $j = 1, \dots n$ bakoitzerako

Baldin eta existitzen bada

$$\lim_{n\to\infty} \sum_{i,j=1}^{n} f(p_{ij})\cdot(x_i - x_{i-1})(y_j - y_{j-1}) = L$$

eta finitua bada $\Longrightarrow$ f integragarria da D-n

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy = L$$



TEOREMA 3.1

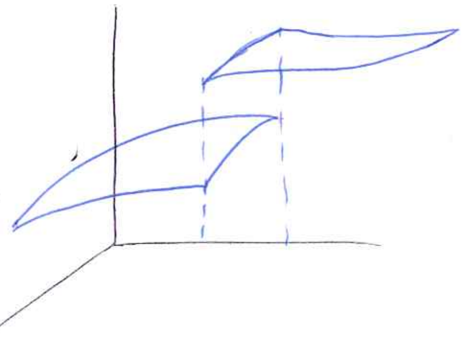
$f: D = [a,b]\times[c,d] \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ jarraitua

$\Longrightarrow$ f integragarria

TEOREMA 3.2: Izan bedi $f: D = [a,b]\times[c,d] \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ bornatua eta demagun f-ren etenguneak funtzio jarraituan bildura finitu batean kokatzen direla

$\Longrightarrow$ f integragarria D eremuan.

- $f(x) \geq 0 \longrightarrow \int_c^b f(x)\,dx \equiv$ azalera
- $f(x,y) \geq 0 \longrightarrow \int_a^b\int_c^d f(x,y)\,dy\,dx \equiv$ bolumena



CAVALIERIREN PRINTZIPIOA BOLUMENAK KALKULATZEKO

$f: [a,b]\times[c,d] \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ jarraitua eta $f(x,y) \geq 0$ $\forall (x,y) \in D$. f-ren azpian geratzen den bolumena:

1) Gorputza $x = x_0$ planoarekin ebakitzean $z = f(x_0, y)$ aldagai bateko funtzioaren grafikoaren azpian geratzen den azalera $A(x)$ deitu eta x a-tik b-ra mugitu,

$$B = \int_a^b A(x)\,dx = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

2) Berdin $y = y_0$ planoarekin eginez, $z = f(x, y_0)$ aldagai bateko funtzioaren grafikoaren azpian geratzen den azalera $\hat{A}(y)$ deitu eta y c-tik d-ra mugitu

$$B = \int_c^d \hat{A}(x)\,dy = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

ADIBIDEA

$f(x,y) = x^2+y$ Kalkulatu $\iint_D f(x,y)\,dx\,dy$

$D = [0,1]\times[0,1]$

f jarraitua (polinomioa) eta $f(x,y)\geq 0$

- $\int_0^1\int_0^1 x^2+y\,dy\,dx = \int_0^1 \left[x^2y+\frac{1}{2}y^2\right]_0^1 dx = \int_0^1 x^2+\frac{1}{2}\,dx =$

$= \frac{1}{3}x^3+\frac{1}{2}x\Big|_0^1 = \frac{1}{3}+\frac{1}{2} = \frac{5}{6}$

- $\int_0^1\int_0^1 x^2+y\,dx\,dy = \int_0^1\left[\frac{1}{3}x^3+yx\right]_0^1 dy = \int_0^1 \frac{1}{3}+y\,dy =$

$= \frac{1}{3}y+\frac{1}{2}y^2\Big|_0^1 = \frac{1}{3}+\frac{1}{2} = \frac{5}{6}$

DEFINIZIOA: Izan bedi $f: [a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ bornatua. f-ren INTEGRAL ITERATUAK dira:

$\int_a^b\int_c^d f(x,y)\,dy\,dx$ eta $\int_c^d\int_a^b f(x,y)\,dx\,dy$.

TEOREMA 3.3: FUBINIREN TEOREMA

$f: D=[a,b]\times[c,d]\to\mathbb{R}$ bornatua eta bere etenguneen multzoa funtzio jarraituen grafikoen bildura finitua

i) $\int_c^d f(x,y)\,dy$ existitzen bada $\forall x\in[a,d]$

$\Rightarrow \int_a^b\int_c^d f(x,y)\,dy\,dx$ existituko da eta

$$\iint_D f\,dA = \int_a^b\int_c^d f\,dy\,dx$$

ii) $\int_a^b f(x,y)\,dx$ $\exists$ bada $\forall y\in[a,b]$ $\Rightarrow$

$\int_c^d\int_a^b f(x)\,dy$ $\exists$ da eta $\iint_D f\,dA = \int_c^d\int_a^b f\,dx\,dy$

Baldintza denak betetzen badira

$$\iint_D f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

ADIBIDEA

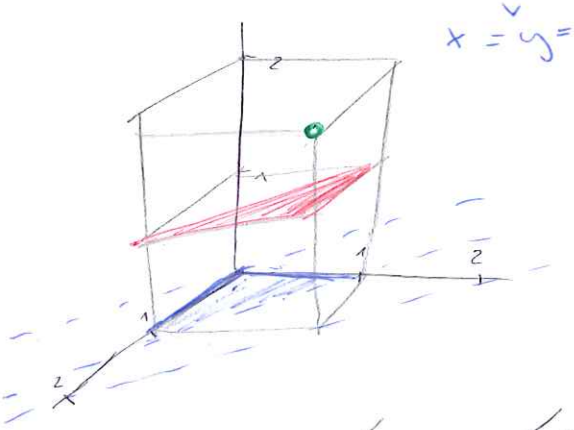
$\iint_D f(x,y)\,dA$ Kalkulatu non $f(x,y) = [x+y] = \max\{n\in\mathbb{Z}$ non $n\le x+y\}$

eta $D = [0,1]\times[0,1] \Rightarrow 0\le x+y\le 2$

$$\iint_D f(x,y)\,dA \qquad f(x,y) = [x+y] = \begin{cases} 0, & 0\le x+y<1 \\ 1, & 1\le x+y<2 \\ 2, & x+y=2 \end{cases}$$

$$x+y=2 \Leftrightarrow x=y=1$$

$$\begin{cases} 0, & -x\le y<1-x \\ 1, & 1-x\le y<2-y \\ 2, & (x,y)=(1,1) \end{cases}$$



f ez da jarraitua (1,1) puntuan baina gaindigarria da, eta $y=1-x$ zuzena $f(x)=1-x$ funtzio jarraituaren grafikoa dela pentsatu daiteke.

TEOR 3.2 $\Longrightarrow$ f integragarria D-n

$$\iint_D f(x,y)\,dA = \int_0^1\int_0^1 [x+y]\,dy\,dx =$$

$$= \int_0^1\left[\cancel{\int_0^{1-x} 0\,dy} + \int_{1-x}^1 1\,dy\right]dx = \int_0^1\left[\int_{1-x}^1 1\,dy\right]dx =$$

$$= \int_0^1 [y]_{1-x}^1\,dx = \int_0^1 1-(1-x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\iint_D f(x,y)\,dA = \int_0^1\int_0^1 [x+y]\,dx\,dy = \int_0^1\left[\int_0^{1-y} 0\,dx + \int_{1-y}^1 1\,dx\right]dy$$

OHARRA:

$$\iint_D f(x)g(y)\,dy\,dx = \int_a^b\int_c^d f(x)g(y)\,dy\,dx = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right)$$

3.2. INTEGRAL BIKOITZA EREMU OROKORRAGOETAN

$a, b, c, d \in \mathbb{R}$ $\quad a<b \wedge c<d$

EREMU ELEMENTALAK

i) Izan bitez $\phi_1, \phi_2 : [a,b] \longrightarrow \mathbb{R}$ jarraituak

$\phi_1(x) \leq \phi_2(x) \quad \forall x \in [a,b]$

$\Longrightarrow D = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ \phi_1(x) \leq y \leq \phi_2(x)\}$

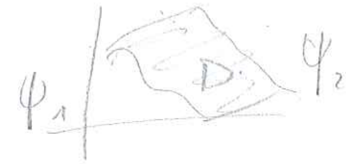
1. motako eremua

ii) Izan bitez $\psi_1, \psi_2 : [c,d] \longrightarrow \mathbb{R}$ jarraituak

$\psi_1(y) \leq \psi_2(y) \quad \forall y \in [c,d]$

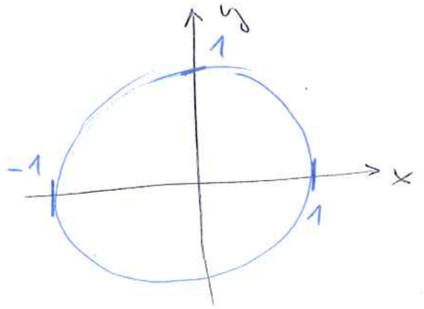
$\Longrightarrow D = \{(x,y) \in \mathbb{R}^2 : y \in [c,d],\ \psi_1(y) \leq x \leq \psi_2(y)\}$

2. motako eremua



iii) $D \subset \mathbb{R}^2$ 3. motako eremua da 1 $\wedge$ 2 motakoa bada

ADIBIDEA



1. motakoa $\longrightarrow x \in [-1,1]$

$\phi_1(x) = -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} = \phi_2(x)$

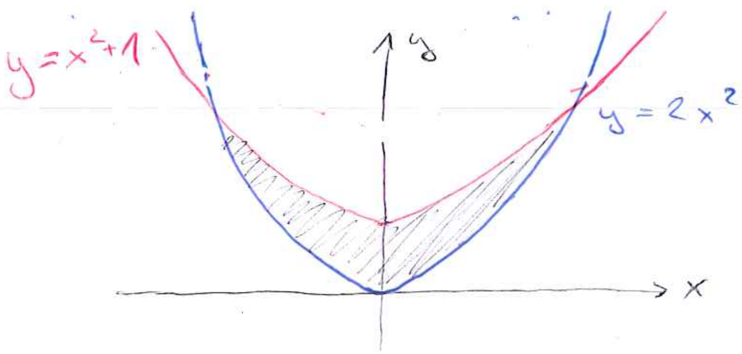
2. motakoa $\longrightarrow y \in [-1,1]$

$-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$

3. MOTAKOA

ADIBIDEAK

1) $f(x,y) = x + 2y$-ren integrala kalkulatu $y = 2x^2$ eta $y = 1-x^2$ parabolen bidez mugatutako A eremua



$$\begin{cases} y=2x^2 \\ y=1+x^2 \end{cases} \Rightarrow x=\pm 1 \quad y=2$$

1. ПОТАКОА:

$$x\in[-1,1] \Rightarrow \phi_1 = 2x^2 \le y \le 1+x^2 = \phi_2$$

$$\iint_A f(x,y)\,dA = \int_{-1}^{1}\int_{2x^2}^{x^2+1}(x+2y)\,dy\,dx =$$

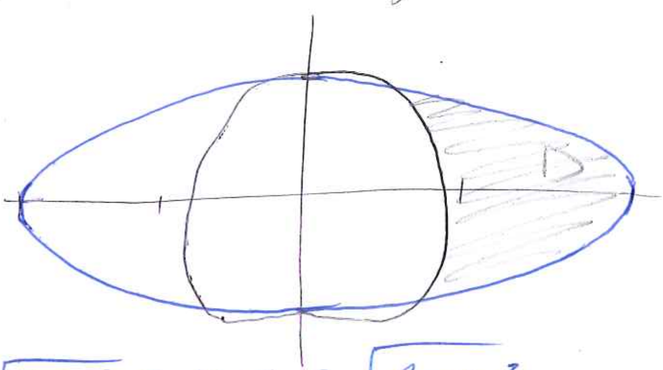
$$=\int_{-1}^{1}\left[xy+y^2\right]_{2x^2}^{x^2+1}dx = \int_{-1}^{1} x(x^2+1)+(x^2+1)^2+$$

$$-x2x^2-(2x^2)^2dx=\int_{1}^{1}x^3+x+x^4+2x^2+1-2x^3-4x^4\,dx=$$

$$=\int_{-1}^{1}1+x+2x^2-x^3-3x^4\,dx=[\dots]=\frac{32}{15}$$

2) $\iint_D x\,dx\,dy$

D eremua $x^2+y^2=1$ ekuazioko zirkunferentziak eta $\frac{x^2}{4}+y^2=1$ ekuazioko elipseak eskuineko planoerdian mugatzen duten eremua da.



$$\begin{cases} x^2+y^2=1 \\ \frac{x^2}{4}+y^2=1 \end{cases} \quad x=0 \quad y=\pm 1$$

2. ПАІКАКОА $y\in[-1,1]$

$$\sqrt{1-y^2}\le x\le 2\sqrt{1-y^2}$$

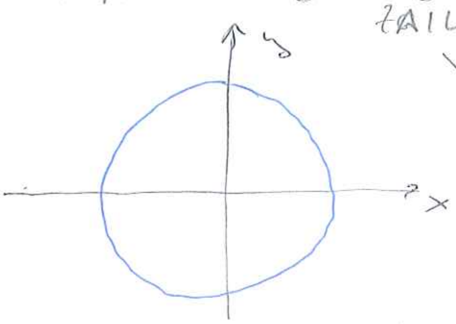
$$\iint_D x\,dx\,dy=\int_{-1}^{1}\int_{\sqrt{1-y^2}}^{2\sqrt{1-y^2}}x\,dx\,dy=\int_{-1}\left[\frac{x^2}{2}\right]_{\sqrt{1-y^2}}^{2\sqrt{1-y^2}}dx=\int_{-1}^{1}\frac{3}{2}-\frac{3}{2}y^2dy$$

$$=[\dots]=\boxed{2}$$

6

3) $\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\, dy\, dx$ Kalkulatu

ZAILA



$\int_0^a \int_0^{\sqrt{a^2-y^2}} (a^2-y^2)^{1/2}\, dx\, dy = \dots = \frac{2c^3}{3}$

PROPOSIZIOA 3.4: INTEGRAL BIKOITZEN PROPIETATEAK

Izan bitez $D \subset \mathbb{R}^2$ eremu elementala:

$f, g : D \to \mathbb{R}$ integragarriak D-n

i) $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ integragarria eta

$$\iint_D \alpha f + \beta g \, dA = \alpha \iint_D f\, dA + \beta \iint_D g\, dA$$

ii) $f(x,y) \geq g(x,y) \quad \forall (x,y) \in D$

$$\Rightarrow \iint_D f\, dA \geq \iint_D g\, dA$$

iii) $D_i \subset \mathbb{R}^2$ eremu elementalak $\forall i = 1, \dots, m$

$$D_i \cap D_j = \phi \quad \wedge \quad D = \bigcup_{i=1}^m D_i$$

$$\iint_D f\, dA = \sum_{i=1}^m \iint_{D_i} f\, dA$$

iv) $|f|$ integragarria eta $\left| \iint_D f\, dA \right| \leq \iint_A |f|\, dA$

TEOREMA 3.5: BATAZ BESTEKO BALIOAREN TEOREMA

Izan bitez $D \subset \mathbb{R}^2$ eremu elementala eta

$f : D \to \mathbb{R}$ jarraitua.

$$\Rightarrow \exists\, \bar{x}_0 \in D \text{ non } \iint_D f\, dA = f(x_0, y_0) \cdot A(D)$$

PROPOSIZIOA 3.6: INTEGRAL BIKOITZA ETA SIMETRIA

Izan bitez D OX (OY) ardatzarekiko simetrikoa

den 1 motako (2. motako) eremu elementala eta D^+ $y \geq 0$ $(x \geq 0)$ planoerdian agertzen den D-ren zatia.

i) f y (x) aldagaiaren bikoitia $f(x,-y) = f(x,y)$

$(f(-x,y) = f(x,y))$

$\Rightarrow \iint_D f\,dA = 2 \cdot \iint_{D^+} f\,dA$

ii) f y (x) aldagaiaren bakoitia $f(x,-y) = -f(x,y)$

$(f(-x,y) = -f(x,y))$

$\Rightarrow \iint_D f\,dA = 0$

ADIBIDEAK

$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

1) $f(x,y) = x^2 + 2y$

x aldagaiaren bikoitia

$f(-x,y) = x^2 + 2y = f(x,y)$

D OY ardatzarekiko simetrikoa $\rightarrow D^+$

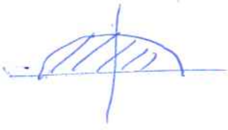
$\hookrightarrow \iint f\,dA = 2 \iint_D f\,dA$

2) $g(x,y) = 3y$

y aldagaiaren bakoitia

$g(x,-y) = -g(x,y)$

D OX-rekiko simetrikoa $\Rightarrow D^+$



$\iint_D f\,dA = 0$

3.3. ALDAGAI-ALDAKETAK INTEGRAL BIKOITZETAN

Gogoratu aldagai bakarreko funtzio integralak:

$$T = g : [a,b] \subset \mathbb{R} \longrightarrow [g(a), g(b)] \subset \mathbb{R}$$
$$t \longrightarrow g(t) = x$$

$$\int_{g(a)}^{g(b)} f(x)\,dx \underset{\substack{\uparrow \\ x=g(t) \\ dx=g'(t)dt}}{=} \int_a^b f(g(t))\cdot g'(t)\,dt$$

- $\iint_D f(x,y)\,dA$ integralean aldagai-aldaketa aplikatzeko

$\rightarrow T: D^* \subset \mathbb{R}^2 \longrightarrow D \subset \mathbb{R}^2$

$$(u,v) \longrightarrow T(u,v) = (x(u,v), y(u,v))$$

$$(x,y) \in D$$

D emanik, D^* ateratzeko $T^{-1}(D)$ egin beharko dugu eta horretarako T^{-1} existitu beharko da

DEFINIZIOA: Izan bedi $T: D^* \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ C^1 klasekoa non $T(u,v) = (x(u,v), y(u,v))$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Rightarrow$$ T-ren DETERMINANTE JAKOBIARRA

OHARRA: $J \neq 0 \underset{\substack{\uparrow \\ \text{ALD. FUNTZ. TEOR.}}}{\Rightarrow} \exists T$ funtzioa

TEOREMA 3.7: ALDAGAI-ALDAKETA INTEGRAL BIKOITZETAN

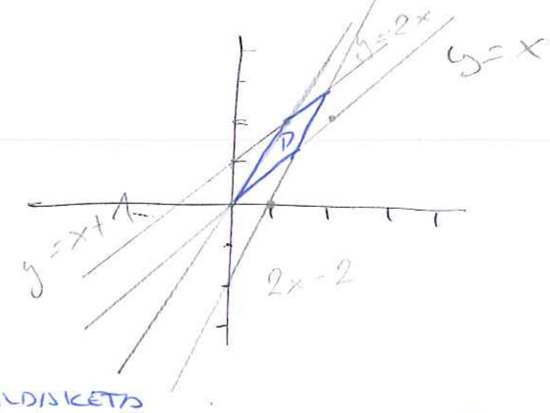
Izan bitez D eta D^* planoko bi eremu elemental eta $T: D^* \longrightarrow D$ C^1 klasekoa TRANSFORMAZIO INJEKTIBOA $T(D^*) = D$ izanik. Orduan f integragarria D eremuan da

$$\iint_D f(x,y)\,dA = \iint_{D^*} f(x(u,v), y(u,v))\cdot \underset{\substack{\text{BALIO ABSOLUTUA} \\ \downarrow}}{|J|}\cdot du\,dv$$

ADIBIDEAK

D eremua $y = 2x$, $y = 2x-2$, $y = x+1$ eta $y = x$ mugatutako eremua da. Kalkulatu

$$\iint_D \underbrace{x \cdot y}_{f(x,y)} \, dx\,dy$$



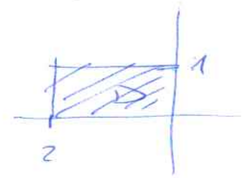
$$y = 2x \Rightarrow \quad y - 2x \le 0$$

$$y = 2x - 2 \Rightarrow -2 \le y - 2x$$

ALDAGAI-ALDAKETA

$$\begin{matrix} -2 \le y - 2x \le 0 \\ 0 \le y - x \le 1 \end{matrix} \Longrightarrow * \begin{cases} u = y - 2x \\ v = y - x \end{cases} \rightarrow D^* \begin{cases} u \in [-2, 0] \\ v \in [0, 1] \end{cases}$$

$$T : D^* \longrightarrow D$$



$$T(u,v) = (x(u,v), y(u,v))$$

$$* \quad u - v = -2x + x = -x \Rightarrow \boxed{x = v - u}$$

$$v = y - x = y - v + u \Rightarrow \boxed{y = 2v - u}$$

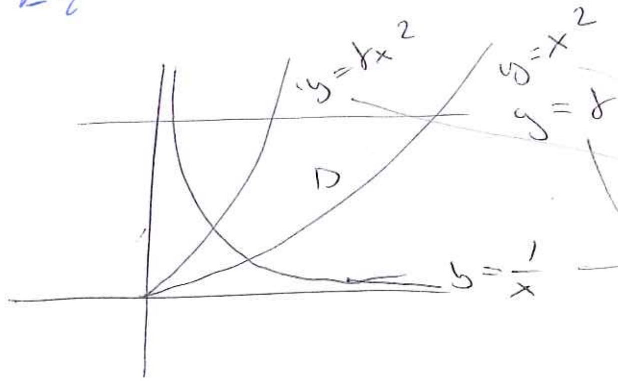
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} = -1$$

$$\iint_D x\,dy\,dx\,dy \underset{ALD}{\overset{ALD}{=}} \iint_{D^*} (u-v)\cdot(2v-u)\cdot|-1|\,du\,dv =$$

$$= \int_{-2}^{0}\int_0^1 2v^2 - 3uv + u^2\,dv\,du = \int_{-2} \left[\frac{2v^3}{3} - \frac{3uv^2}{2} + u^2 v\right]_0^1 du =$$

$$= \int_2^0 \frac{2}{3} - \frac{3u}{2} + u^2\,du = [\ldots] = 7$$

2)



KALKULATU $\iint_D dx\,dy = A(D)$

$$1 \le \frac{y}{x^2} \le 8$$

$$1 \le xy$$

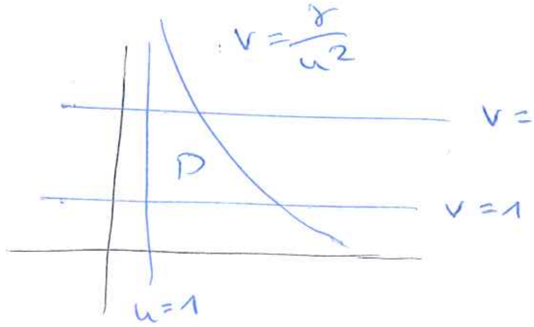
$$8 \ge y$$

ALDAGAI ALDAKETA

$$\begin{cases} u^3 = xy \\ v^3 = \frac{y}{x^2} \end{cases} \Rightarrow \begin{matrix} x = u/v \\ y = vu^2 \end{matrix}$$

$$J = \begin{vmatrix} 1/v & u/v^2 \\ 2uv & u^2 \end{vmatrix} = \dots = \frac{3u^2}{v}$$

$$\begin{cases} u \in [1,2] \\ u \geq 1 \\ v \leq \dfrac{8}{u^2} \end{cases}$$

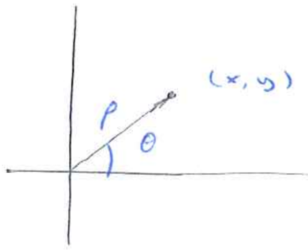


$$A(D) = \iint_D dx\,dy = \int_1^2 \int_1^{\sqrt{8/v}} \frac{3u^2}{v}\,du\,dv = [\dots] = -\frac{16}{7} - \ln 2 + \frac{32\sqrt{2}}{3}$$

DEFINIZIOA:

$(x,y) \in \mathbb{R}^2$. (x,y) puntuko KOORDENATU POLARRAK (ρ, θ) moduan adierazten dira.

$\rho \in [0, +\infty) \wedge \theta \in [0, 2\pi)$



$$T(\rho, \theta) \underset{\uparrow}{=} (\rho\cos\theta, \rho\sin\theta)$$

ALDAGAI - ALDAKETA $(0,0)$-n ZENTRATUA

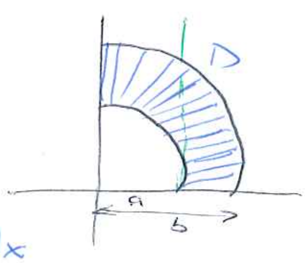
$$T(\rho, \theta) \underset{\uparrow}{=} (x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)$$

ALDAGAI - ALDAKETA (x_0, y_0)-n ZENTRATUA

$$J = \frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \rho\sin\theta & \rho\cos\theta \end{vmatrix} = \dots = \rho$$

ADIBIDEA:

$\iint_D \ln(x^2+y^2)\,dx\,dy$ non D eremua $x^2+y^2=a^2$ eta $x^2+y^2=b^2$ zirkunferentzien arteko eremua den lehen koadrantean eta $0<a<b$



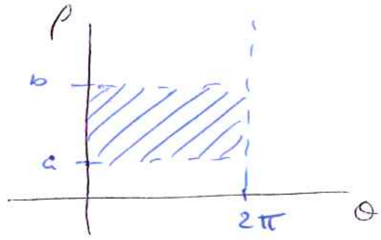
KARTESIARRETAN

$$I = \int_0^a \int_{\sqrt{a^2-x^2}}^{\sqrt{b^2-x^2}} \ln(x^2+y^2)\,dy\,dx + \int_a^b \int_0^{\sqrt{b^2-x^2}} \ln(x^2+y^2)\,dy\,dx$$

↳ KONPLIKATUA

ALDAGAI ALDAKETA
(KOORDENATU POLARRAK)

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad J = \rho \qquad \rho \in [a,b] \quad \theta \in [0, \tfrac{\pi}{2}]$$



$$I = \int_0^{\pi/2}\int_a^b \ln\left((\rho\cos\theta)^2 + (\rho\sin\theta)^2\right) \cdot \underbrace{\rho}_{|J|}\, d\rho\, d\theta =$$

$$= \int_0^{\pi/2}\int_a^b \ln(\rho^2)\cdot\rho\, d\rho\, d\theta = \int_a^b\int_0^{\pi/2} \ln(\rho^2)\rho\, d\theta\, d\rho =$$

$$= \int_a^b \ln(\rho^2)\rho \cdot [\theta]_0^{\pi/2} d\rho = \frac{\pi}{2}\int_a^b \ln(\rho^2)\cdot\rho\, d\rho =$$

$$u = \ln\rho^2 \qquad du = \frac{2}{\rho^2}\rho\, d\rho = \frac{2}{\rho} d\rho$$

$$dv = \rho\, d\rho \qquad v = \frac{1}{2}\rho^2$$

$$= \left[\ln\rho^2 \cdot \frac{1}{2}\rho^2\right]_a^b - \int \frac{1}{\cancel{2}}\rho^{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{\rho}} d\rho \cdot \frac{\pi}{2} =$$

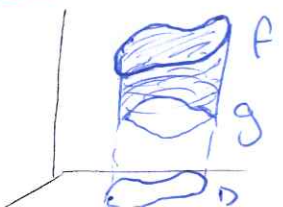
$$= \frac{\pi}{2}\left[\frac{1}{\cancel{2}}\ln\rho^{\cancel{2}}\cdot\rho^2 - \frac{1}{2}\rho^2\right]_a^b = \left[\ln b \cdot b^2 - \frac{1}{2}b^2 - \ln a \cdot a^2 + \frac{1}{2}a^2\right]\frac{\pi}{2} =$$

$$= \frac{\pi}{2}\left[b^2\ln b - a^2\ln a - \left(\frac{b^2}{2} - \frac{a^2}{2}\right)\right]$$

3.4. INTEGRAL BIKOITZEN APLIKAZIOAK

i) $A(D) = \iint_D 1\, dx\, dy \to$ D-ren AZALERA

ii) $z = f(x,y)$ $\wedge$ $z = g(x,y)$ gainazalen arteko BOLUMENA $D \in \mathbb{R}^2$ eremuaren gainean



$$B = \iint_D [f(x,y) - g(x,y)]\, dx\, dy$$

iii) $m(D) = \iint_D \rho(x,y)\,dx\,dy \longrightarrow$ D-ren MASA

ρ: DENTSITATEA

iv) D-ren MASA ZENTRUA $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\iint_D x\,\rho(x,y)\,dx\,dy}{m(D)} \quad ; \quad \bar{y} = \frac{\iint_D y\,\rho(x,y)\,dx\,dy}{m(D)}$$

v) $D \subset \mathbb{R}^2$ ERREMU ELEMENTALA eta $f: D \to \mathbb{R}$ jarraitua. f-ren BATAZBESTEKO BALIOA D-n

$$[f]_m = \frac{\iint_D f\,dx\,dy}{A(D)}$$

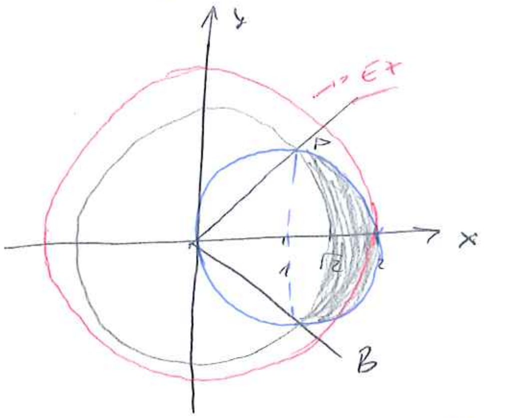
ADIBIDEA:

$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2x \ , \ x^2+y^2 \ge 2\}$

$\rho(x,y) = K$ Konstantea. Kalkulatu MASA ZENTRUA

$x^2+y^2 = 0 \Rightarrow (x-1)^2+y^2 = 1$

$x^2+y^2 = 2 \Rightarrow x^2+y^2 = (\sqrt{2})^2$



Ordezkatu puntuak jakiteko ze parte den

$m(D) \overset{def}{=} \iint_D \rho(x,y)\,dx\,dy =$

$= \iint_D K\,dx\,dy = K \cdot \iint_D dx\,dy \overset{polarrak}{=} *$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow \rho \in [\sqrt{2}, \rho_0] \quad \theta \in [B, A]$

$\to A, B$

$\begin{cases} x^2+y^2 = 2x \\ x^2+y^2 = 2 \end{cases} \Rightarrow \begin{array}{l} 0 = 2x-2 \\ x = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \end{array}$

$A(1,1) \wedge B(1,-1)$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \xrightarrow{A} \begin{matrix} 1 = \sqrt{2}\cos\theta \\ 1 = \sqrt{2}\sin\theta \end{matrix} \Rightarrow \cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta_+ = \frac{\pi}{4}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \xrightarrow{B} \begin{matrix} 1 = \sqrt{2}\cos\theta \\ -1 = \sqrt{2}\sin\theta \end{matrix} \Rightarrow \theta_- = -\frac{\pi}{4}$$

$$\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

$\longrightarrow \rho_0$ CORTU

$$(x-1)^2 + y^2 = 1 \longrightarrow x^2 + y^2 = 2x$$

pdemek

$$\Rightarrow \rho^2 = 2\rho\cos\theta \Rightarrow \rho(\rho - 2\cos\theta) = 0$$

$$\rho = 0 \vee \rho_0 = 2\cos\theta$$

$$* = k \cdot \int_{-\pi/4}^{\pi/4} \int_{\sqrt{2}}^{2\cos\theta} 1 \cdot \underbrace{\rho}_{|J|} \cdot d\rho\, d\theta = k \int_{-\pi/4}^{\pi/4} \underbrace{2\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} - 1 \, d\theta = \dots = k$$

$$m(D) = k \qquad \rho(x,y) = k$$

$$\bar{x} = \frac{\iint_D x\,\rho(x,y)\,dx\,dy}{m(D)} = \frac{\iint_D x\,k\,dx\,dy}{k} = \iint_D x\,dx\,dy =$$

$$= \int_{-\pi/4}^{\pi/4} \int_{\sqrt{2}}^{2\cos\theta} \rho\cos\theta \underbrace{\rho}_{\text{JACOB}}\, d\rho\, d\theta = \int_{-\pi/4}^{\pi/4} \cos\theta \left[\frac{1}{3}\rho^3\right]_{\sqrt{2}}^{2\cos\theta} d\theta =$$

$$= \int_{-\pi/4}^{\pi/4} \frac{8}{3}\underbrace{\cos^4\theta}_{\left(\frac{1+\cos 2\theta}{2}\right)^2} - \frac{2}{3}\sqrt{2}\cos\theta\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{2}{3}(1+\cos 2\theta)^2 d\theta - \left[\frac{2}{3}\sqrt{2}\sin\theta\right]_{-\pi/4}^{\pi/4} =$$

$$= \int_{-\pi/4}^{\pi/4} \frac{2}{3}(1 + 2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\frac{1+\cos 4\theta}{2}})\, d\theta - \frac{2}{3}\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$$

$$\left[\frac{2}{3}\theta + \frac{4}{3}\frac{\sin 2\theta}{2} + \frac{1}{3}\left(\theta + \frac{\sin 4\theta}{4}\right)\right]_{-\pi/4}^{\pi/4} - \frac{4}{3} = \cdots = \frac{\pi}{2}$$

$$\bar{y} = \frac{\iint_D y \overset{k}{\cancel{\rho}}\, dx\, dy}{\cancel{m(D) = k}} = \iint_D y\, dx\, dy \overset{2\text{ AUKERA}}{=} \begin{cases} \bullet \text{ LEHEN } \bar{x}\text{-EKIN BEZALA (LUZEA)} \\ \bullet\ * \end{cases}$$

* D eremua OX-rekiko simetrikoa eta $f(x,y)=y$ y aldagaiarekiko bakoitia $(f(x,-y)=-f(x,y))$

$\overset{\text{PROP 3.6}}{\Longrightarrow} \iint_D y\, dx\, dy = 0$

ANALISI BEKTORIALA ETA KONPLEXUA

3. Gaia: INTEGRAL BIKOITZA

Ariketak

1. Kalkula errektangeluen gaineko integral bikoitz hauek:

   (i) $\displaystyle\iint_R (x^2y^2 + x)\, dA$, non $R = [0,2] \times [-1,0]$. *Em.:* 26/9

   (ii) $\displaystyle\iint_\Omega x^2\, dxdy$ non $\Omega = \{(x,y) \,:\, 0 \le x \le 1,\ 0 \le y \le 3\}$. *Em.:* 1

   (iii) $\displaystyle\iint_D \sin(x+y)\, dxdy$ non $D = \{(x,y) \,:\, 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}\}$. *Em.:* 2

   (iv) $\displaystyle\int_0^2 \int_0^1 (5 - 2x - y)\, dy\, dx$. *Em.:* 5

2. Idatz itzazu f funtzio integragarriaren bi integral iteratuak, jarraian ematen diren eremuen gainean:

   (i) $D = \{(x,y) \in \mathbf{R}^2 \,:\, x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

   (ii) $D = \{(x,y) \in \mathbf{R}^2 \,:\, x + y \le 1,\ x - y \le 1,\ x \ge 0\}$.

   (iii) $D = \{(x,y) \in \mathbf{R}^2 \,:\, y \ge x^2,\ y \le \sqrt{x}\}$.

   (iv) $(0,0)$ puntuan zentratutako sektore zirkularra, bere arkuaren muturrak $(-1,1)$ eta $(1,1)$ puntuak izanik.

   (v) $x^2 + y^2 \le x$ desberdintza betetzen dituzten puntuek osatutako multzoa.

   (vi) $D = \{(x,y) \in \mathbf{R}^2 \,:\, 0 \le x \le 3,\ -4 \le 3x - 2y \le -1\}$.

   (vii) D $y \le x \le y + 2a$, $0 \le y \le a$ desberdintzek determinatuta dago.

3. Integral iteratua emanda, marraztu ezazu integrazio-eremua eta idatzi beste integral iteratua:

   (i) $\displaystyle\int_0^1 \int_0^y f(x,y)\, dx\, dy$

   (ii) $\displaystyle\int_0^2 \int_{x^2}^{4x-x^2} f(x,y)\, dy\, dx$

   (iii) $\displaystyle\int_0^1 \int_y^1 f(x,y)\, dx\, dy$

   (iv) $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} f(x,y)\, dx\, dy$

   (v) $\displaystyle\int_{1/2}^1 \int_{x^3}^x f(x,y)\, dy\, dx$

   (vi) $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\, dx\, dy$

4. Kalkula itzazu honako eremu hauen azalerak:

(i) $y = x$, $y = 5x$, $x = 1$ zuzenek mugatzen duten eremuaren azalera.

(ii) x ardatzaren gainean dagoen eta $y = 0$, $y^2 = 4ax$, $x + y = 3a$ kurbek mugatzen duten eskualdearen azalera.

(iii) $xy = 2$, $y = 1$ eta $y = x + 1$ kurben bidez mugatutako lehenengo koadranteareneremuaren azalera.

(iv) $y = x^2$ eta $y = 2x - x^2$ parabolek mugatzen duten eskualdearen azalera.

(v) $4y = x^2$, $2y = x^2$, $y = x$ eta $x = 1$ kurbek mugatzen duten azalera.

Em.: (i) 2; (ii) $\dfrac{10a^2}{3}$; (iii) $2\ln 2 - \dfrac{1}{2}$; (iv) $\dfrac{1}{3}$; (v) $\dfrac{23}{12}$.

5. Kalkula itzazu integral bikoitz hauek:

(i) $\displaystyle\iint_\Omega x^3 y\, dxdy$ non $\Omega = \{(x, y) \ : \ 0 \leq x \leq 1,\ 0 \leq y \leq x\}$ den.

(ii) $\displaystyle\iint_D \frac{x^2}{y^2}\, dxdy$, non D eremua $x = 2$, $x = y$ zuzenez eta $xy = 1$ hiperbolaz mugatutako eremua den.

(iii) $\displaystyle\iint_D \cos(x + y)\, dxdy$, D eremua $x = 0$, $x = y$, eta $y = \pi$ zuzenez mugatutakoa denean.

(iv) $\displaystyle\iint_A (x + y)^{-4}\, dxdy$, non $A = \{(x, y) \in \mathbf{R}^2 \ : \ x \geq 1,\ y \geq 1,\ x + y \leq 4\}$ den.

(v) $\displaystyle\iint_A e^{x/y}\, dxdy$, non A multzoa $y^3 \leq x \leq y^2$ desberdintzak betetzen dituzten lehenengo koadranteko puntuek osatzen dutena den.

Em.: (i) $\dfrac{1}{12}$; (ii) $\dfrac{9}{4}$; (iii) -2; (iv) $\dfrac{1}{48}$; (v) $\dfrac{3-e}{2}$.

6. Kalkula itzazu honako eremu hauen azalerak, kasu bakoitzean aldagai-aldaketa egokia erabiliz:

(i) $y^2 = 2px$, $y^2 = 2qx$, $x^2 = 2ry$, $x^2 = 2sy$ parabolez bornatutako eremua, $0 < p < q$ eta $0 < r < s$ izanik.

(ii) $xy = a^2$, $xy = 2a^2$, $y = x$, $y = 2x$ kurbek mugatutako eremua $x > 0$, $y > 0$ denean .

(iii) $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ lemniskatak mugatutako azalera.

(iv) $(x^2 + y^2)^2 \leq 2a^2(x^2 - y^2)$, $x^2 + y^2 \geq a^2$ eremua.

(v) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ elipseak mugatutako azalera.

Em.: (i) $\dfrac{4}{3}(q - p)(s - r)$; (ii) $\dfrac{a^2 \ln 2}{2}$; (iii) $2a^2$; (iv) $a^2\left(\sqrt{3} - \dfrac{\pi}{3}\right)$; (v) πab.

7. Jarraian ematen diren integralak kalkulatu, aldagai-aldaketa egokien bidez:

(i) $\displaystyle\iint_A \frac{x-y}{x+y}\,dxdy$, non A erpinak $(0,2)$, $(1,1)$, $(2,2)$ eta $(1,3)$ puntuetan dituen laukia den.

(ii) $\displaystyle\iint_D y^3(x^2+y^2)^{-3/2}\,dxdy$, $D=\{(x,y)\in\mathbf{R}^2\ :\ 1/2\le y\le 1,\ x^2+y^2\le 1\}$.

(iii) $\displaystyle\iint_D (x^2+y^2)\,dxdy$, non D $x^2+y^2\le 2ax$ eremua den, $a>0$ izanik.

(iv) $\displaystyle\iint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy$, non $D=\Big\{(x,y)\in\mathbf{R}^2\ :\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1\Big\}$ den, a eta b zenbaki positiboak direlarik.

(v) $\displaystyle\iint_D \sin\big(\sqrt{x^2+y^2}\big)\,dxdy$ non D zirkulu unitarioa den.

(vi) $\displaystyle\iint_A (x+y)\,dxdy$ non $A=\{(x,y)\in\mathbf{R}^2\ :\ 1\le x^2+y^2\le 4,\ x\ge 0,\ y\ge 0\}$ den.

Em.: (i) $\ln\dfrac{1}{2}$; (ii) $\dfrac{\sqrt{3}}{4}$; (iii) $\dfrac{3\pi}{2}a^4$; (iv) $\dfrac{2\pi ab}{3}$; (v) $2\pi(\sin 1-\cos 1)$; (vi) $\dfrac{14}{3}$.

8. Kalkulatu emandako gainazalez mugatutako solidoen bolumenak:

(i) $z=0$ eta $z=\cos x\cos y$, $x+y=\pi/2$, $x+y=-\pi/2$, $x-y=\pi/2$, $x-y=-\pi/2$.

(ii) $x+y-z+1=0$, $x+y=1$, $x=0$, $y=0$ eta $z=0$ planoak.

(iii) $x+y+z=2R$ planoa, $x^2+y^2=R^2$ zilindroa eta $x=0$, $y=0$, $z=0$ plano koordenatuak, lehenengo oktantean.

(iv) $z=x^2+y^2$ paraboloidea eta $x^2+y^2=1$ zilindroa, $z\ge 0$ espazioerdian.

(v) $x^2+y^2+z^2=1$ esfera.

(vi) $z=1-(x^2+y^2)$ gainazala, $z=0$ planoa eta $x^2+y^2-x=0$ zilindroa.

Em.: (i) π; (ii) $5/6$; (iii) $\dfrac{3\pi-4}{6}R^3$; (iv) $\pi/2$; (v) $4\pi/3$; (vi) $\dfrac{5\pi}{32}$.

9. Urrezko plaka batek $0\le x\le 2\pi$, $0\le y\le\pi$ forma du, eta bere dentsitatea $y^2\sin^2(4x)+2$ g/cm^2 da. Urreak 7 dolar/g balio badu, zenbat balioko du urrezko plakak ?

Em.: $\dfrac{7\pi^4}{3}+28\pi^2$.

10. Izan bedi D $y\ge\dfrac{-x}{\sqrt{3}}$, $y\le x$, $x^2+y^2\ge x$, eta $x^2+y^2\le 4x$ ekuazioek definitzen duten eremua. Kalkula itzazu eremuaren azalera eta masa, dentsitatea $\rho(x,y)=\dfrac{y}{x^2+y^2}$ bada.

Em.: $A=\dfrac{15}{4}\Big(\dfrac{5\pi}{12}+\dfrac{2+\sqrt{3}}{4}\Big)$, $M=\dfrac{3}{8}$.

11. Kalkula ezazu zirkulu unitarioaren lehenengo koadrantea betetzen duen lamina baten masa-zentrua, dentsitatea $\rho(x,y)=\sqrt{1-(x^2+y^2)}$ bada.

Em.: $(3/8,3/8)$.

m(D) ✓

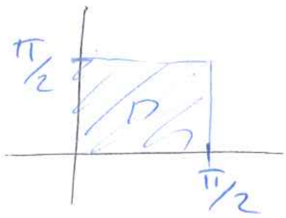
12. Lamina batek triangelu angeluzuzenaren forma du, altuera h eta oinarriaren luzera b izanik. Laminaren dentsitatea angelu zuzenerako distantziaren karratuarekiko proportzionala baldin bada, aurki ezazu laminaren masa-zentrua.

Em.: $\left(\dfrac{b\,(3\,b^2+h^2)}{5\,(b^2+h^2)}, \dfrac{h\,(b^2+3\,h^2)}{5\,(b^2+h^2)}\right)$.

3. INTEGRAL BIKOITZA
ARIKETAK

1. ARIKETA

iii) $\iint_D \sin(x+y)\,dx\,dy =$ non $D = \{(x,y) : 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}\}$



$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy =$$

$$= \int_0^{\pi/2} \Big[-\cos(x+y)\Big]_0^{\pi/2} dy =$$

$$= \int_0^{\pi/2} \Big[-\cos\left(\frac{\pi}{2}+y\right) + \cos(0+y)\Big] dy =$$

$$= \Big[-\sin\left(\frac{\pi}{2}+y\right) + \sin y\Big]_0^{\pi/2} = -\sin\pi + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} - \sin 0 =$$

$$= 1 + 1 = \boxed{2}$$

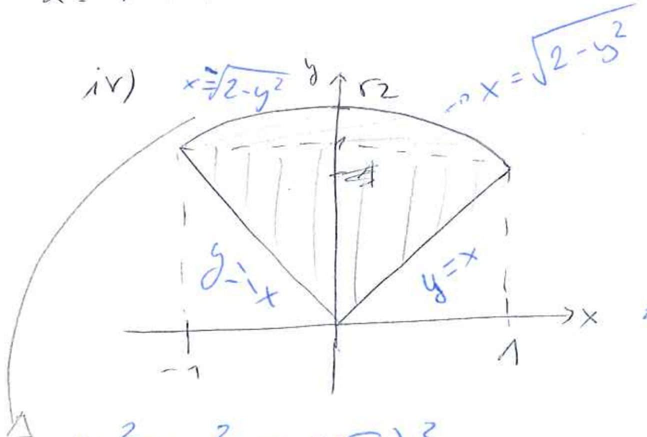
iv) $\int_0^2 \int_0^1 (5 - 2x - y)\,dy\,dx =$

$$= \int_0^2 \Big[5y - 2xy - \frac{1}{2}y^2\Big]_0^1 dx =$$

$$= \int_0^2 \left(5 - 2x - \frac{1}{2}\right) dx = \Big[5x - x^2 - \frac{1}{2}x\Big]_0^2 = [\dots] = 5$$

2. ARIKETA

iv)



Proiektatu OY ardatzean

$$\iint_D f\,dx\,dy \overset{?}{=} \int_0^1 \int_{-y}^{y} f\,dx\,dy + \int_1^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f\,dx\,dy =$$

$$= \iint_D f\,dx\,dy = \int_{-1}^{0} \int_{-x}^{\sqrt{2-x^2}} f\,dy\,dx + \int_0^1 \int_x^{\sqrt{2-x^2}} f\,dy\,dx$$

Proiektatu OX ardatzean

$x^2 + y^2 = (\sqrt{2})^2$

(zirkunferentziaren ek.)

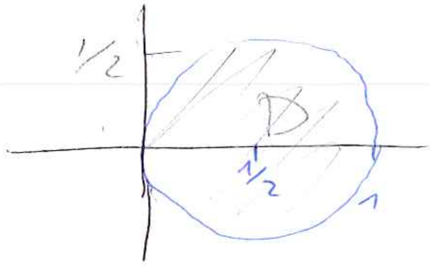
v) $x^2+y^2 \le x$

$x^2+y^2=x$

$x^2-x+y^2=0$

$(x-\frac{1}{2})^2-\frac{1}{4}+y^2=0$

$(x-1/2)^2+y^2=\frac{1}{4} \Rightarrow \begin{cases} (1/2,0) & \text{ZENTRUA} \\ R=1/2 & \end{cases}$ ZIRKUNFERENTZIA



Bakandu x ∧ y

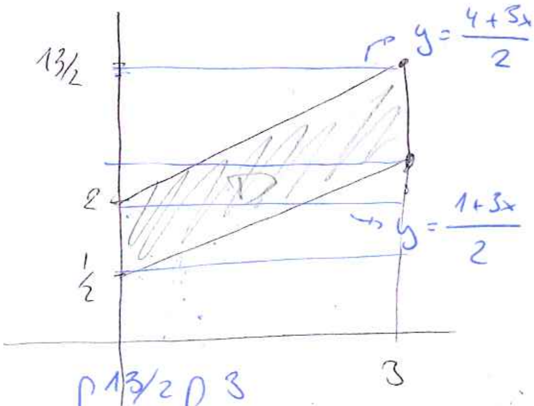
- OX ardatzean proiektatu

$$\int_0^1 \int_{-\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}}^{\sqrt{1/4-(x-1/2)^2}} f\,dy\,dx$$

- OY ardatzean proiektatu

$$\int_{-1/2}^{1/2} \int_{\frac{1}{2}-\sqrt{\frac{1}{4}-y^2}}^{\frac{1}{2}+\sqrt{1/4-y^2}} f\,dx\,dy$$

vi) $D=\{(x,y)\in\mathbb{R}^2 : 0\le x<3,\ -4\le 3x-2y\le -1\}$



- OX-rekiko proiektatu

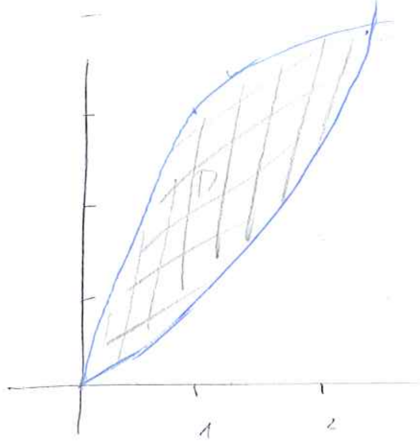
$$\int_0^3 \int_{\frac{1+3x}{2}}^{\frac{4+3x}{2}} f\,dy\,dx$$

- OY-rekiko proiektatu

$$\int_{1/2}^{2} \int_0^{\frac{2y-1}{3}} f\,dx\,dy + \int_2^5 \int_{\frac{2y-4}{3}}^{\frac{2y-1}{3}} f\,dx\,dy + \int_5^{13/2} \int_{\frac{2y-4}{3}}^{3} f\,dx\,dy$$

3. ARIKETA

iii) $\int_0^2 \int_{x^2}^{4x-x^2} f(x,y)\,dy\,dx = I$



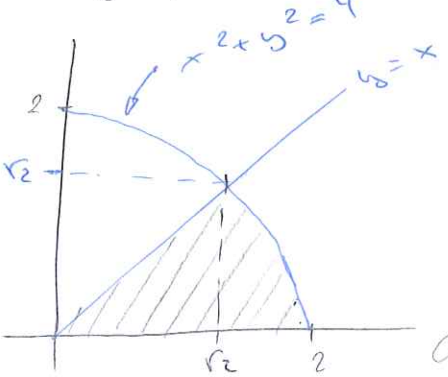
$y = 4x - x^2 \Rightarrow x^2 - 4x + y = 0$

$x = \frac{4 \pm \sqrt{16-4y}}{2} = 2 \pm \sqrt{4-y}$

$$I = \int_0^4 \int_{\sqrt{y}}^{2-\sqrt{4-y}} f(x,y)\,dx\,dy$$

OY-n PROIEKTATU

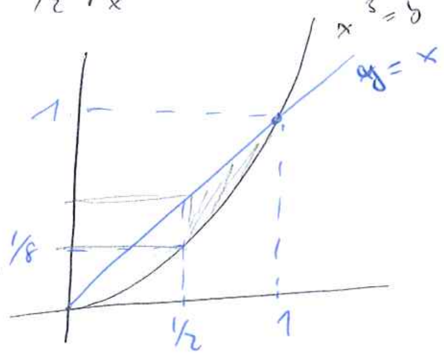
iv) $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$



OX-en proiektatu

$$I = \int_0^{\sqrt{2}} \int_0^x f(x,y)\,dy\,dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

v) $\int_{1/2}^1 \int_{x^3}^x f(x,y)\,dy\,dx$



$$I = \int_{1/8}^{1/2} \int_{1/2}^{\sqrt[3]{y}} f(x,y)\,dx\,dy + \int_{1/2}^1 \int_y^{\sqrt[3]{y}} f(x,y)\,dx\,dy$$

OY-n

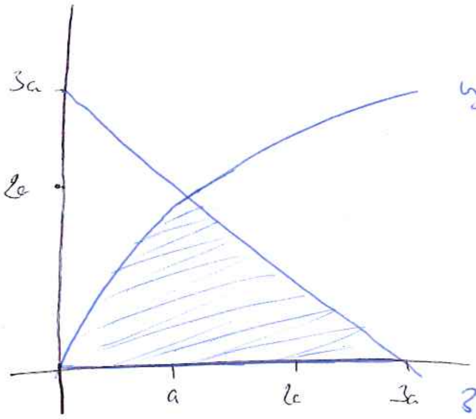
3

4. ARIKETA

ii) $y = 0$

$y^2 = 4ax$ x-en gainean

$x + y = 3a$



$y = \sqrt{4ax} \Rightarrow y^2 \cdot \frac{1}{4a} = x$

OY-n proiektatu (erraztasunagatik)

$3a - x = y$

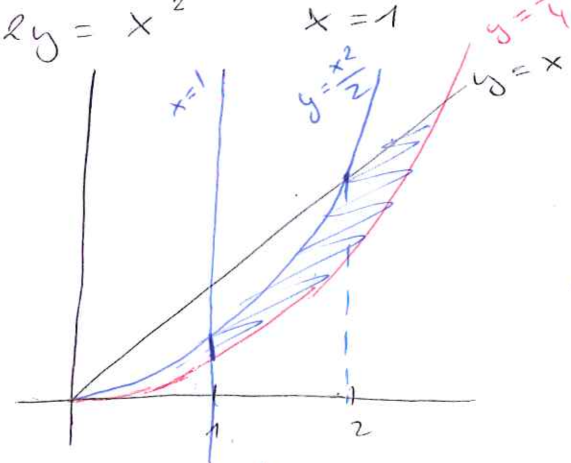
$$A(D) = \iint_D 1\,dx\,dy = \int_0^{2a} \int_{\frac{y^2}{4a}}^{3a-y} 1\,dx\,dy =$$

$$= \int_0^{2a} \left[x \right]_{y^2/4a}^{3a-y} dy = \int_0^{2a} \left(3a - y - \frac{y^2}{4a}\right) dy =$$

$$= \left[3ay - \frac{1}{2}y^2 - \frac{1}{3}\frac{y^3}{4a} \right]_0^{2a} = 6a^2 - 2a^2 - \frac{2}{3}a^2 = \boxed{\frac{10}{3}a^2}$$

v) $4y = x^2$ $y = x$

$2y = x^2$ $x = 1$



$$A(D) = \iint_D 1\,dx\,dy$$

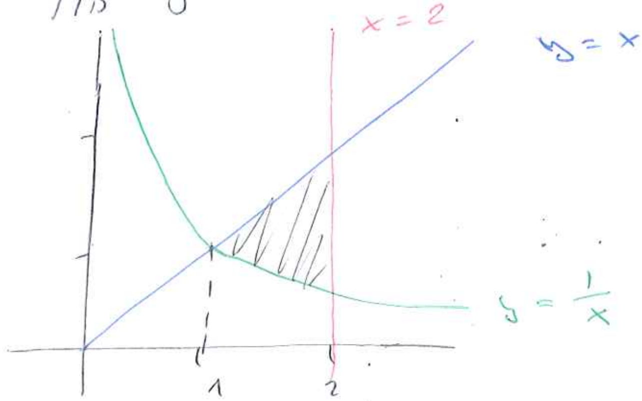
$$A(D) = \int_1^2 \int_{x^2/4}^{x^2/2} 1\,dy\,dx + \int_2^4 \int_{x^2/4}^{x} 1\,dy\,dx =$$

$$= \int_1^2 \left[y \right]_{x^2/4}^{x^2/2} dx + \int_2^4 \left[y \right]_{x^2/4}^{x} dx = \int_1^2 \frac{x^2}{2} - \frac{x^2}{4}\,dx + \int_2^4 x - \frac{x^2}{4}\,dx = \dots = \frac{23}{12}$$

4

5. ARIKETA

iii) $\iint_D \frac{x^2}{y^2}\,dx\,dy \qquad D \Rightarrow \begin{cases} x=2 \\ x=y \end{cases} \quad xy=1$



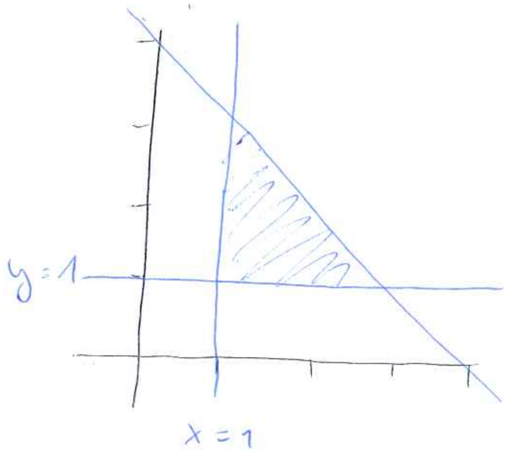
$$\iint_D \frac{x^2}{y^2}\,dx\,dy = \int_1^2\int_{1/x}^{x} \frac{x^2}{y^2}\,dy\,dx = \int_1^2 x^2 \int_{1/x}^{x} \frac{1}{y^2}\,dy\,dx =$$

$$= \int_1^2 x^2 \left[\frac{y^{-1}}{-1}\right]_{1/x}^{x} dx = \int_1^2 x^2 \cdot \left[\frac{-1}{x} + x\right] dx =$$

$$= \int_1^2 -x + x^3\,dx = \dots = \boxed{\frac{9}{4}}$$

iv) $\iint_A (x+y)^{-4}\,dx\,dy \qquad A = \{(x,y)\in\mathbb{R}^2 \mid x\geq 1, y\geq 1, x+y\leq 4\}$

$x=1$, $y=1$, $y=4-x$



$$\iint_A (x+y)^{-4}\,dx\,dy = \int_1^3\int_1^{4-x} (x+y)^{-4}\,dy\,dx =$$

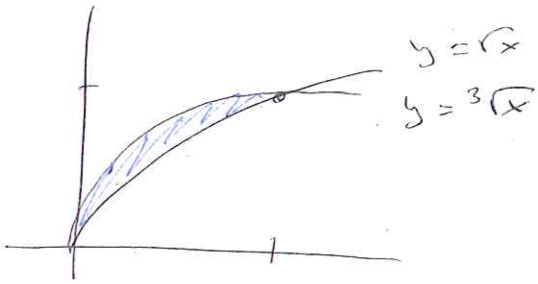
$$= \int_1^3 \left[\frac{-1}{3}(x+y)^{-3}\right]_1^{4-x} dx = \int_1^3 -\frac{1}{3}\left[(x+4-x)^{-3} - (x+1)^{-3}\right] dx =$$

$$= -\frac{1}{3}\int_1^3 4 - (x+1)^{-2}\,dx = -\frac{1}{3}\left[4x - \frac{1}{2}(x+1)^{-2}\right]_1^3 = \dots = \frac{1}{48}$$

5

v) $\iint_A e^{x/y}\,dx\,dy$

$A: y^3 \le x \le y^2$ 1. KOADRANTEAN



Buteen
$\sqrt{\ }$ agrikka liketike

$$\iint_A e^{x/y}\,dx\,dy = \int_0^1 \int_{y^3}^{y^2} e^{x/y}\,dx\,dy =$$

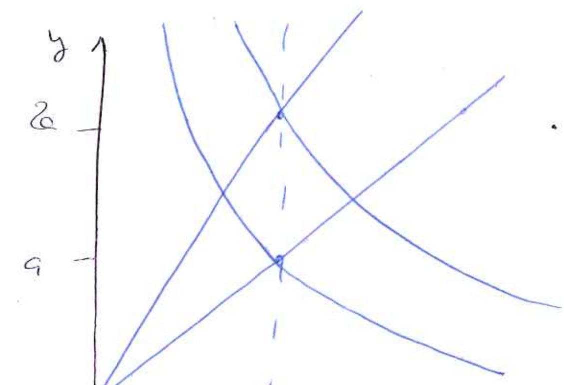
$$= \int_0^1 \int_{y^3}^{y^2} e^{\frac{1}{y}x}\,dx\,dy = \int_0^1 \left[y e^{\frac{x}{y}}\right]_{y^3}^{y^2} dy =$$

$$= \int_0^1 y e^{y} - y e^{y^2}\,dy = \left[y e^{y}\right]_0^1 - \int_0^1 e^{y}\,dy - \left[\tfrac{1}{2} e^{y^2}\right]_0^1 =$$

$$= \dots = -\frac{e}{2} + \frac{3}{2}$$

6. ARIKETA

ii) $xy = a^2$
$xy = 2a^2$
$y = x$
$y = 2x$
$x, y > 0$



$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \qquad \begin{aligned} a^2 \le u &\le 2a^2 \\ 1 \le v &\le 2 \end{aligned}$$

$$A(D) = \iint_D 1\,dx\,dy \overset{\text{ALD-ALD}}{\overset{\downarrow}{=}} \iint 1 \cdot |J|\,du\,dv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} = \dots = \frac{1}{2v}$$

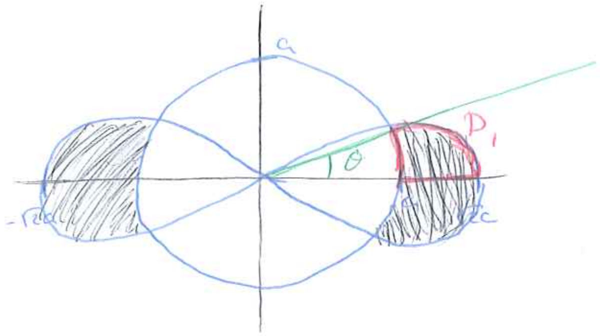
$x = \frac{u^{1/2}}{v^{1/2}}$ $y = u^{1/2} v^{1/2}$

$$= \iint_{D^*} \frac{1}{2v}\, du\, dv = \int_{a^2}^{2a^2} \int_1^2 \frac{1}{2v}\, dv\, du = \dots = \frac{a^2}{2} \ln 2$$

→ ZIRKUNFERENTZIA

iv) $(x^2+y^2)^2 \leq 2a^2(x^2-y^2)$, $x^2+y^2 \geq a^2$

$(x^2+y^2)^2 \leq 2a^2(x^2-y^2) \Rightarrow$ LEMNISKATA



$$A(D) = \iint_D 1\, dx\, dy \underset{\uparrow}{=} 4 \cdot \iint_{D_1} 1\, dx\, dy = *$$

SIMETRIKOA

ALD-ALD ⇒ POLARRAK

$$\begin{cases} x = 0 + \rho \cos\theta \\ y = 0 + \rho \sin\theta \end{cases} \qquad |J| = \rho$$

$$\theta_0 \to A \quad \begin{cases} x^2+y^2 = a^2 \\ (x^2+y^2)^2 = 2a^2(x^2-y^2) \end{cases}$$

$$a^4 = 2a^2(a^2-2y^2) \Rightarrow y = \pm \frac{a}{2} \to y = \frac{a}{2}$$

$$y = \frac{a}{2} \underset{\text{ALDALD}}{=} \overset{a}{\rho} \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta_0 = \frac{\pi}{6}$$

$$\theta \in [0, \frac{\pi}{6}]$$

$\rho_2 \to$ LEMNISKATA $\Rightarrow (x^2+y^2)^2 = 2a^2(x^2-y^2)$

$\downarrow \begin{matrix} x = \rho\cos x \\ y = \rho \sin x \end{matrix}$

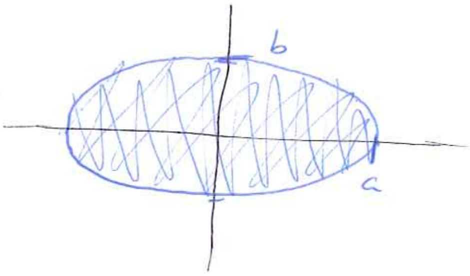
$$\rho^4 = 2a^2(\rho^2\cos^2\theta - \rho^2\sin^2\theta) = 2a^2\rho^2\cos 2\theta$$

$$\Rightarrow \rho^2 = 2a^2 \cos 2\theta \Rightarrow \rho = \sqrt{2}a\sqrt{\cos 2\theta}$$

$$\rho \in [a, \sqrt{2}a\sqrt{\cos 2\theta}]$$

$$* = 4 \cdot \int_0^{\pi/6} \int_a^{\sqrt{2}a\sqrt{\cos 2\theta}} \rho\, d\rho\, d\theta = \dots = a^2\sqrt{3} - \frac{a^2\pi}{3}$$

v) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$A(D) = \iint_D 1\,dx\,dy =$

ALD-ALD

$\begin{cases} x = a\rho\cos\theta + 0 \\ y = b\rho\sin\theta + 0 \end{cases}$

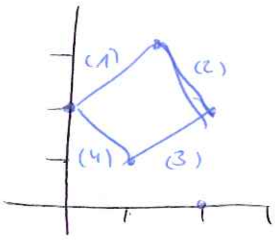
$J = \frac{\partial(x,y)}{\partial(\rho,\theta)} = \boxed{ab\rho}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \xrightarrow[ALD]{ALD} \rho^2 = 1 \Rightarrow \rho = 1 \Rightarrow \begin{cases} \theta \in [0, 2\pi] \\ \rho \in [0, 1] \end{cases}$

$= \int_0^{2\pi} \int_0^1 ab\rho\, d\rho\, d\theta = [\ldots] = ab\pi$

7. ARIKETA

i) $\iint_A \frac{x-y}{x+y}\,dx\,dy$ ERPINAK $(0,2),\ (1,1)$
$(2,2)\ (1,3)$



(1) $\to y = x + 2$

(2) $\to y = -x + 4$

(3) $\to y = x$

(4) $\to y = -x + 2$

ALDAGAI-ALDAKETA

$\begin{cases} u = x + y \\ v = x - y \end{cases}$

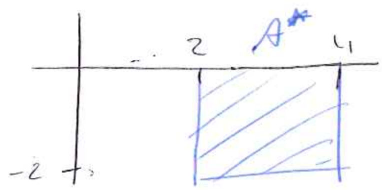
$x = \frac{u+v}{2}$ $\quad y = \frac{u-v}{2}$

(1) $\to v = -2$

(2) $\to u = 4$

(3) $\to v = 0$

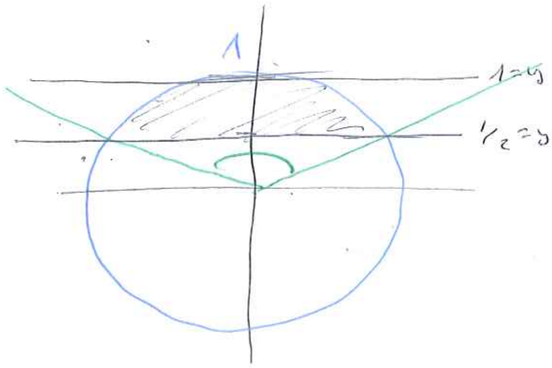
(4) $\to u = 2$



$J = \frac{\partial(x,y)}{\partial(u,v)} = \ldots = -\frac{1}{2}$

$= \iint_{A^*} \frac{v}{u} \cdot \left| -\frac{1}{2} \right| du\,dv = \frac{1}{2} \int_1^4 \int_{-2}^0 \frac{v}{u}\, dv\, du = \ldots = \ln\frac{1}{2}$

ii) $\iint_D y^3(x^2+y^2)^{-3/2}\,dx\,dy =$



ALD-ALD : POLARRAK

$\begin{cases} x = \rho\cos\theta + 0 \\ y = \rho\sin\theta + 0 \end{cases} \rightarrow J = \rho$

$y = \frac{1}{2} = \overset{\overset{1}{\shortparallel}}{\rho}\sin\theta \Rightarrow \theta = \frac{\pi}{6} \wedge \frac{5\pi}{6}$

$\theta \in [\frac{\pi}{6}, \frac{5\pi}{6}]$

$\rho \in [$zuzena, zirkunf$]$

$y = \frac{1}{2} = \rho\sin\theta \Rightarrow \rho = \frac{1}{2\sin\theta} \Rightarrow \rho \in [\frac{1}{2\sin\theta}, 1]$

ordezkatu aldald

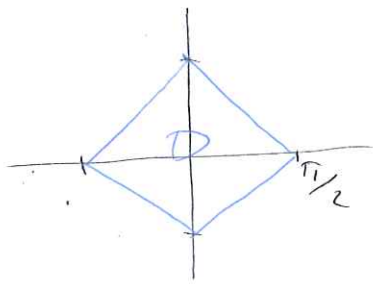
$= \int_{\pi/6}^{5\pi/6}\int_{\frac{1}{2\sin\theta}}^{1} \cancel{\rho^3}\sin^3\theta\,\cancel{(\rho^2)^{-3/2}}\cdot\rho\,d\rho\,d\theta = [\ldots]^{*} = \frac{\sqrt{3}}{4}$

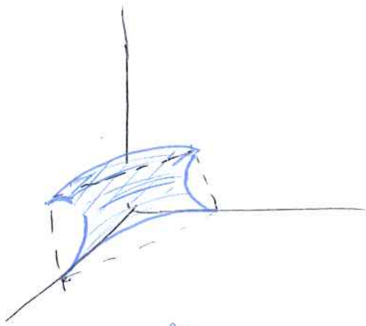
$* \int \sin^3\theta\,d\theta = \int \sin^2\theta\sin\theta\,d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta =$

$\underset{t=\cos\theta}{=} \int (1-t^2)(-dt)$

8. ARIKETA

i) $0 \le z \le \cos x - \cos y$





$B = \iint_D \cos x\cos y - 0\,dx\,dy = 2\cdot\iint_{D_1} \cos x\cos y\,dx\,dy =$

cosxcosy x ardatzarekiko bikoitia eta D OY ardatzarekiko simetrikoa

$= 2\int_0^{\pi/2}\int_{y-\pi/2}^{\pi/2-y} \cos x\cos y\,dx\,dy = [\ldots] = \pi$

$\sin(\frac{\pi}{2}-y) = \cos y \wedge \cos^2 y = \frac{1+\cos 2y}{2}$

1. ARIKETA

i) $\iint_R (x^2y^2+x)\,dA$ non $R=[0,2]\times[-1,0]$

$$\int_{-1}^{0}\int_0^2 (x^2y^2+x)\,dx\,dy = \int_{-1}^{0}\left[\frac{x^3}{3}y^2+\frac{x^2}{2}\right]_0^2 dy =$$

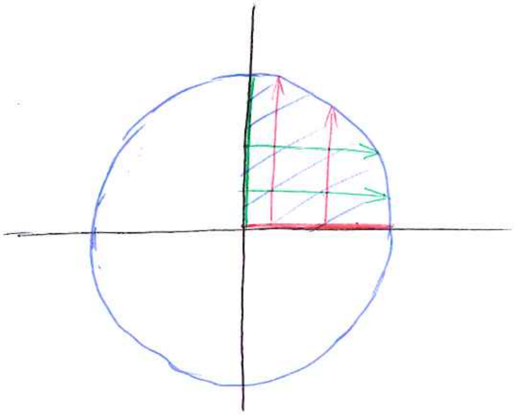
$$= \int_{-1}^{0} \frac{8}{3}y^2+2\,dy = \left[\frac{8}{3}\frac{y^3}{3}+2y\right]_{-1}^{0} = \frac{8}{9}+2 = \boxed{\frac{26}{9}}$$

ii) $\iint_\Omega x^2\,dx\,dy$ non $\Omega=\{(x,y):\ 0\le x\le 1,\ 0\le y\le 3\}$

$$\int_0^3\int_0^1 x^2\,dx\,dy = \int_0^3\left[\frac{x^3}{3}\right]_0^1 dy = \int_0^3 \frac{1}{3}\,dy = \left[\frac{y}{3}\right]_0^3 = \boxed{1}$$

2. ARIKETA

i) $D=\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\le 1,\ x\ge 0,\ y\ge 0\}$



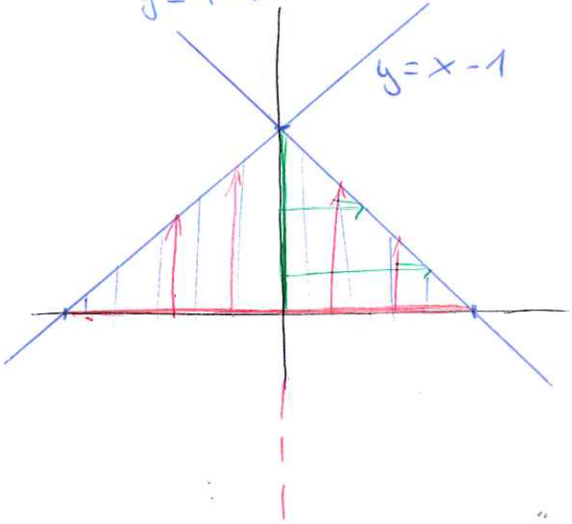
$\boxed{OX}$

$$\int_0^1\int_0^{\sqrt{1-x^2}} f\,dy\,dx$$

$\boxed{OY}$

$$\int_0^1\int_0^{\sqrt{1-y^2}} f\,dx\,dy$$

ii) $D=\{(x,y)\in\mathbb{R}^2:\ x+y\le 1\quad x-y\le 1\quad x\ge 0\}$



$\boxed{OX}$

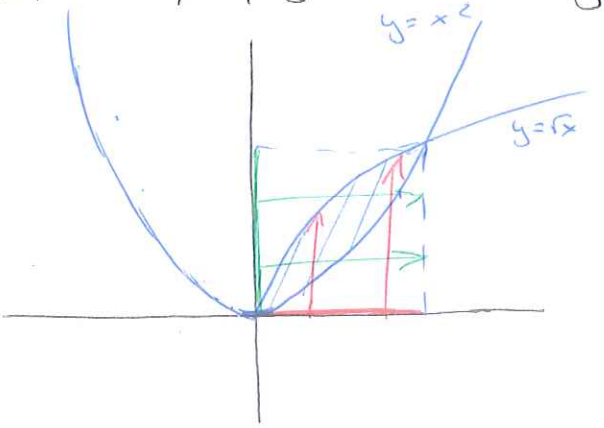
$$\int_{-1}^{0}\int_0^{x-1} f\,dy\,dx + \int_0^1\int_0^{1-x} f\,dy\,dx$$

$\boxed{OY}$

$$2\int_0^1\int_0^{1-y} f\,dx\,dy$$

10

iii) $D = \{(x,y) \in \mathbb{R}^2 : \; y \geq x^2 \; , \; y \leq \sqrt{x} \}$



$\boxed{OX}$

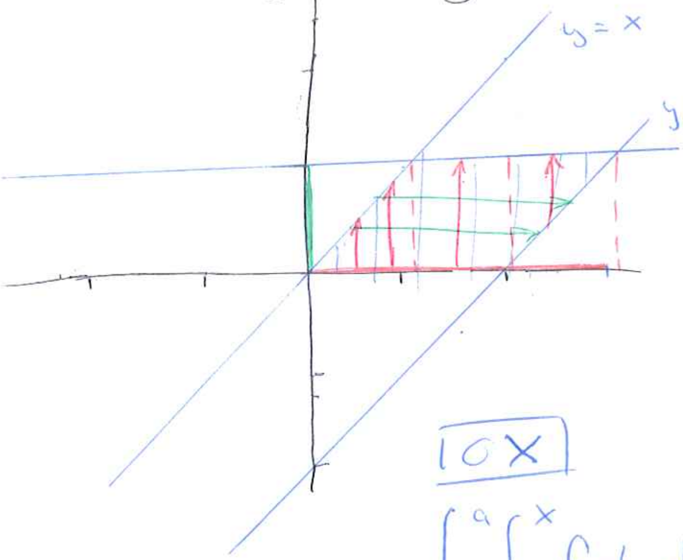
$$\int_0^1 \int_{x^2}^{\sqrt{x}} f \, dy \, dx$$

$\boxed{OY}$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} f \, dx \, dy$$

vii)

$D : \; y \leq x \leq y + 2a \; , \; 0 \leq y \leq a$



$\boxed{OX}$

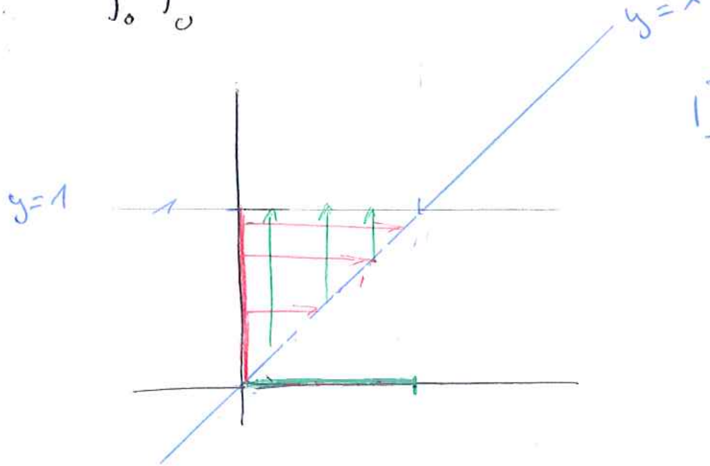
$$\int_0^a \int_y^{y+2a} f \, dx \, dy$$

$\boxed{OX}$

$$\int_0^a \int_0^x f \, dy \, dx + \int_a^{2a} \int_0^a f \, dy \, dx + \int_{2a}^{3a} \int_{x-2a}^{a} f \, dy \, dx$$

3. ARIKETA

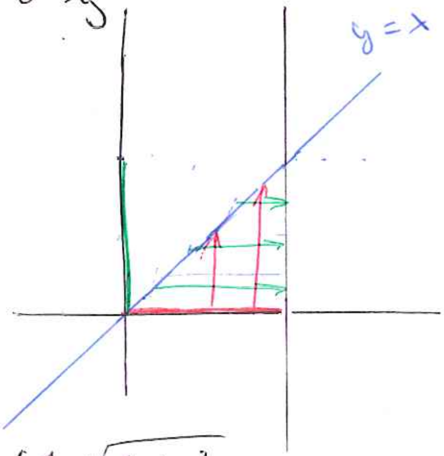
i) $\displaystyle\int_0^1 \int_0^y f(x,y) \, dx \, dy$



$\boxed{OX}$

$$\int_0^1 \int_x^1 f(x,y) \, dy \, dx$$

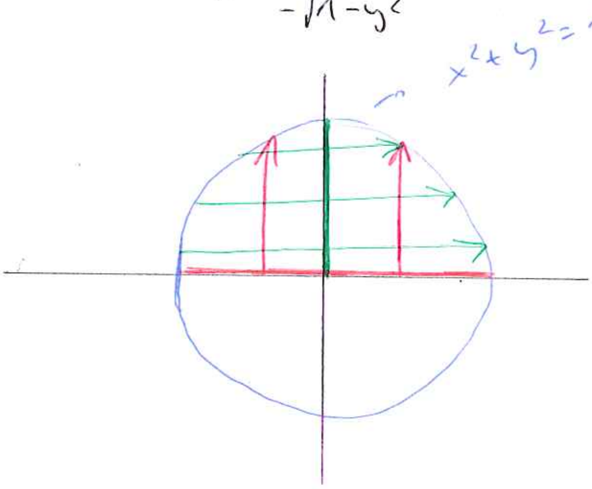
iii) $\int_0^1 \int_y^1 f(x,y)\,dx\,dy$



$\boxed{OX}$

$$\int_0^1 \int_0^x f(x,y)\,dy\,dx$$

vi) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$



$\boxed{OX}$

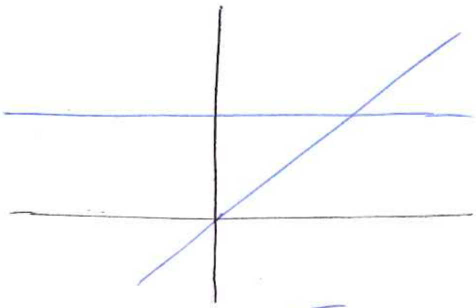
$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

5. ARIKETA

i) $\iint_\Omega x^3 y\,dx\,dy$ non $\Omega = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\}$

$$\int_0^1 \int_0^x x^3 y\,dy\,dx = \int_0^1 \left[x^3 \frac{y^2}{2}\right]_0^x dx = \int_0^1 \frac{x^5}{2}\,dx = \frac{x^6}{12} = \boxed{\frac{1}{12}}$$

iii) $\iint_D \cos(x+y)\,dx\,dy$ $\quad x=0,\ x=y,\ y=\pi$



$$\int_0^\pi \int_0^y \cos(x+y)\,dx\,dy =$$

$$= \int_0^\pi \left[\sin(x+y)\right]_0^y dy = \int_0^\pi \sin 2y\,dy = \left[\frac{-1}{2}\cos 2x\right]_0^\pi =$$

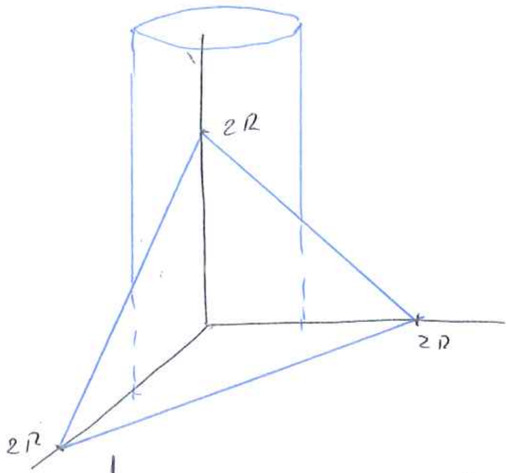
$$= -\frac{1}{2} -$$

8. ARIKETA

iii) $x+y+z=2R$

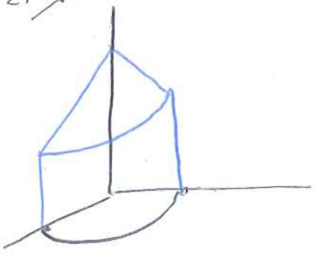
$x^2+y^2=R^2$

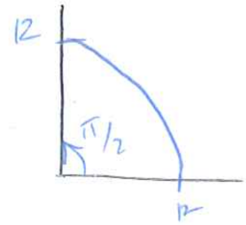
$x=0,\ y=0,\ z=0$



OXY planoan proiektatu

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$$





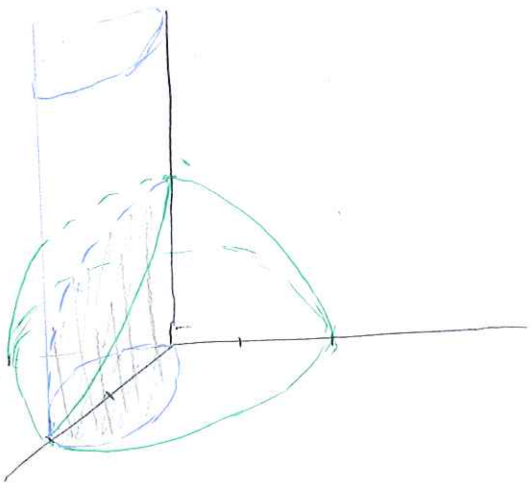
$$B=\int_0^{\pi/2}\int_0^{R}\left[(2R-\rho\sin\theta-\rho\cos\theta)-(\overset{z}{\overset{\shortparallel}{0}})\right]\rho\,d\rho\,d\theta=[\dots]=\frac{3\pi-4}{6}R^2$$

vi) $z=1-(x^2+y^2)$ ⇒ PARABOLOIDEA.

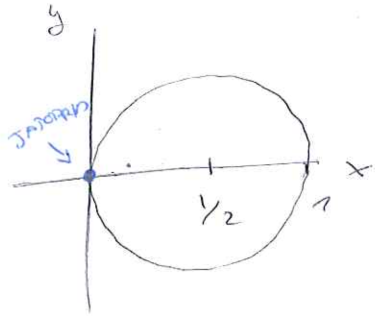
$z=0$

$x^2+y^2-x=0$ ⇒ ZILINDROA

↳ $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$ ⟨ $(\frac{1}{2},0)$ ZENTRUA ; $r=\frac{1}{2}$



PROIEKTATU OYX



$$B=\iint_D (z_{paraboloid}-z_{planoa})\,dx\,dy$$

B) AUKERA

1) Ald-ald POLARRAK

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \qquad |J|=\rho$$

$\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\rho\in[0,\ zirkunf]$

13

zirkunf $\Rightarrow x^2+y^2-x=0$

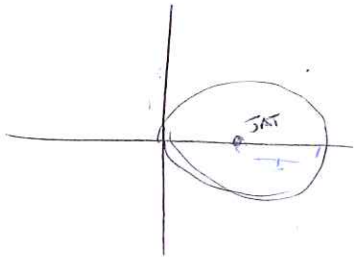
$$\rho^2-\rho\cos\theta=0$$

$$\rho(\rho-\cos\theta)=0 \begin{cases} \rho=0 \\ \rho=\cos\theta \end{cases}$$

$$B=\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}(1-\rho^2)\cdot\overset{J}{\rho}\,d\rho\,d\theta=[\dots]=\frac{5}{32}\pi$$

2) ALD-ALD POLARRAK.

$$\begin{cases} x=\rho\cos\theta+\frac{1}{2} \\ y=\rho\sin\theta+0 \end{cases} \quad |J|=\rho$$



$\theta\in[0,2\pi]$

$\rho\in[0,\frac{1}{2}]$

$$B=\int_0^{2\pi}\int_0^{1/2}\left[1-\left[(\rho\cos\theta+\tfrac{1}{2})^2+(\rho\sin\theta)^2\right]\right]\rho\,d\rho\,d\theta=\frac{5}{32}\pi$$

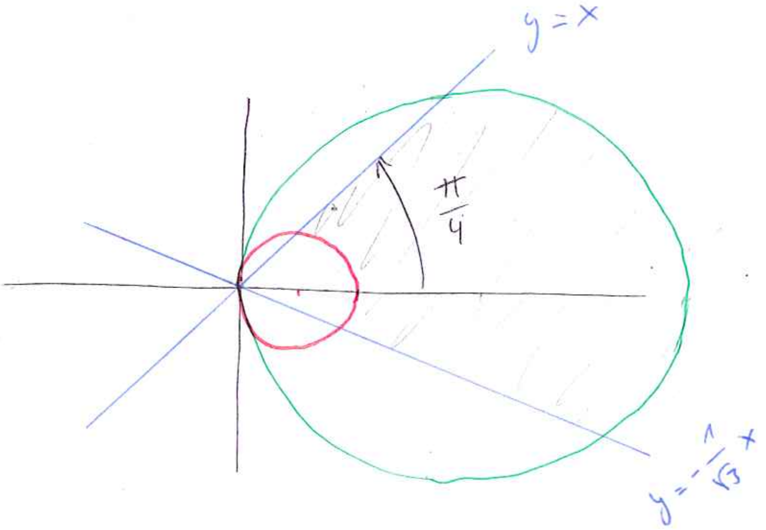
10. ARIKETA

$y\geq\frac{-x}{\sqrt{3}} \rightarrow y=\frac{-x}{\sqrt{3}}$

$y\leq x$

$x^2+y^2\geq x \Rightarrow (x-\frac{1}{2})^2+y^2=\frac{1}{4}$

$x^2+y^2\leq 4x \Rightarrow (x-2)^2+y^2=4$



$$A(D)=\iint_D 1\,dx\,dy$$

ALD-ALD

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$$

$\theta\in[\frac{\pi}{4},\theta_0]=[\frac{-\pi}{6},\frac{\pi}{4}]$

$\rho\in[\text{zirk TXIKI}_{(1)},\text{zirk HANDI}_{(2)}]$

$\theta_0 \rightarrow y=-\frac{1}{\sqrt{3}}x$

$$\rho\sin\theta=-\frac{1}{\sqrt{3}}\rho\cos\theta \Rightarrow \tan\theta=-\frac{1}{\sqrt{3}} \Rightarrow \theta=-\frac{\pi}{6}$$

(1) $x^2+y^2=x$

$$\rho^2=\rho\cos\theta$$

$$\rho(\rho-\cos\theta)=0 \quad \begin{cases} \rho=0 \\ \rho=\cos\theta \end{cases}$$

(2) $x^2+y^2=4x$

$$\rho^2=4\rho\cos\theta$$

$$\rho(\rho-4\cos\theta)=0 \quad \begin{cases} \rho=0 \\ \rho=4\cos\theta \end{cases}$$

$$A(D)=\int_{-\pi/6}^{\pi/4}\int_{\cos\theta}^{4\cos\theta}\rho\, d\rho\, d\theta=$$

$$=\int_{-\pi/6}^{\pi/4}\left[\frac{16\cos^2\theta}{2}-\frac{\cos^2\theta}{2}\right]d\theta=[\dots]=\frac{15}{4}\left[\frac{5}{12}\pi+\frac{2+\sqrt{3}}{4}\right]$$

$$m(D)=\iint_D \tilde{\rho}(x,y)\,dx\,dy=$$

$$=\int_{-\pi/6}^{\pi/4}\int_{\cos\theta}^{4\cos\theta}\frac{\rho\sin\theta}{\rho^2}\rho\, d\rho\, d\theta=[\dots]=\frac{3}{8}$$

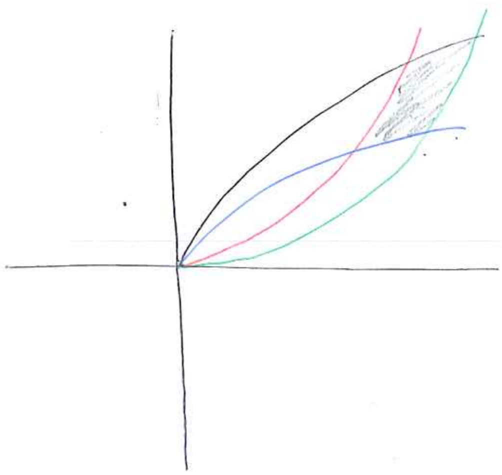
6. ARIKETA

i) $y^2=2px \Rightarrow y=\sqrt{2px} \Rightarrow \frac{y^2}{x}=2p$

$y^2=2qx \Rightarrow y=\sqrt{2qx} \Rightarrow \frac{y^2}{x}=2q$

$x^2=2ry \Rightarrow y=\frac{x^2}{2r} \Rightarrow \frac{x^2}{y}=2r$

$x^2=2sy \Rightarrow y=\frac{x^2}{2s} \Rightarrow \frac{x^2}{y}=2s$



$$\Rightarrow \quad u=\frac{y^2}{x} \qquad 2p\le u\le 2q$$

$$v=\frac{x^2}{y} \qquad 2r\le v\le 2s$$

$$x = \frac{y^2}{u}$$

$$x = \sqrt{vy} \Rightarrow vy = \frac{y^4}{u^2} \Rightarrow y^3 = v u^2 \Rightarrow y = v^{1/3} u^{2/3}$$

$$x = \frac{v^{2/3} \cdot u^{4/3}}{u} = v^{2/3} \cdot u^{1/3} \Rightarrow x = v^{2/3} \cdot u^{1/2}$$

$$\frac{\partial x}{\partial u} = \frac{1}{3} v^{2/3} u^{-2/3} \qquad \frac{\partial x}{\partial v} = \frac{2}{3} v^{-1/3} u^{1/3}$$

$$\frac{\partial y}{\partial u} = \frac{2}{3} v^{1/3} \cdot u^{-1/3} \qquad \frac{\partial y}{\partial v} = \frac{1}{3} u^{2/3} \cdot v^{-2/3}$$

$$J = \begin{vmatrix} \frac{1}{3} v^{2/3} u^{-2/3} & \frac{2}{3} u^{1/3} v^{-1/3} \\ \frac{2}{3} v^{1/3} u^{-1/3} & \frac{1}{3} u^{2/3} v^{-2/3} \end{vmatrix} =$$

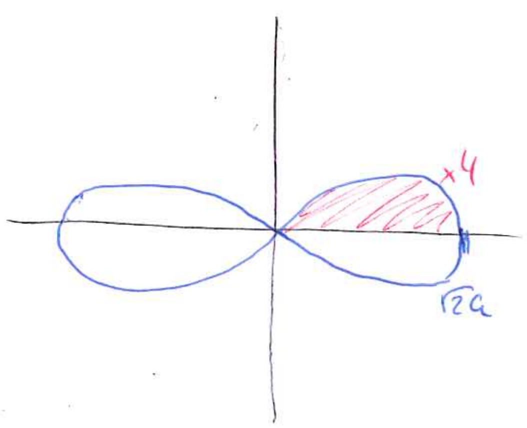
$$= \frac{1}{9} v^{2/3 - 2/3} u^{-2/3 + 2/3} - \frac{4}{9} u^{1/3 - 1/3} v^{-1/3 + 1/3} = -\frac{3}{9} = -\frac{1}{3} \Rightarrow J = -\frac{1}{3}$$

$$A(D) = \int_{2p}^{2q} \int_{2r}^{2s} \left| -\frac{1}{3} \right| dv\, du = \int_{2p}^{2q} \left| -\frac{1}{3} \right| \left[v \right]_{2r}^{2s} du =$$

$$= \int_{2p}^{2q} \left| -\frac{1}{3} \right| 2 (s - r)\, du = \left| -\frac{2}{3} \right| (s - r) \cdot \left[u \right]_{2p}^{2q} \Rightarrow$$

$$\Rightarrow \boxed{A(D) = \frac{4}{3} (s - r)(q - p)}$$

iii) $(x^2 + y^2)^2 = 2a^2 (x^2 - y^2)$



ALDAGAI-ALDAKETA

– POLARRAK –

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad J = \rho$$

$\theta \in [0, \frac{\pi}{4}]$

$\rho \in [0, \rho_0]$

$\rho_0 = ?$

$$(x^2+y^2)^2 = 2a^2(x^2-y^2) \overset{ALD}{\underset{ALD}{=}}> \rho^4 = 2a^2(\rho^2(\cos^2\theta - \sin^2\theta))$$

$$\rho^2 = 2a^2\cos 2\theta \Rightarrow \rho_0 = \sqrt{2}\, a\sqrt{\cos 2\theta}$$

$$A = 4 \cdot \int_0^{\pi/4}\int_0^{a\sqrt{2\cos 2\theta}} \rho\, d\rho\, d\theta = 4\int_0^{\pi/4}\left[\frac{\rho^2}{2}\right]_0^{a\sqrt{2\cos 2\theta}} d\theta =$$

$$= 4\int_0^{\pi/4} \frac{2\cos 2\theta\, a^2}{2} d\theta = 4a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} \Rightarrow \boxed{A_{(D)} = 4a^2}$$

7. ARIKETA

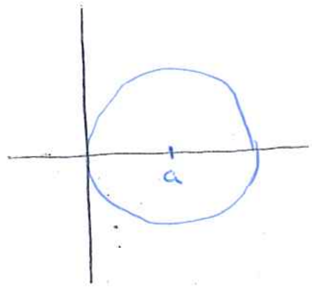
iii) $\iint_D (x^2+y^2)\,dx\,dy$ non $D \equiv x^2+y^2 \le 2ax \quad a>0$

$$x^2+y^2 = 2ax \Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$$

$$(x-a)^2 + y^2 = a^2$$

$(a,0)$ ZENTRUA $\wedge$ $r = a$

ALD - ALD [POLARRAK]



$$\begin{cases} x = \rho\cos\theta + a & \rho \in [0,a] \\ y = \rho\sin\theta & \theta \in [0, 2\pi] \end{cases}$$

$$\int_0^a\int_0^{2\pi} (\rho^2\cos^2\theta + 2a\cos\theta + a^2 + \rho^2\sin^2\theta)\,\rho\, d\theta\, d\rho =$$

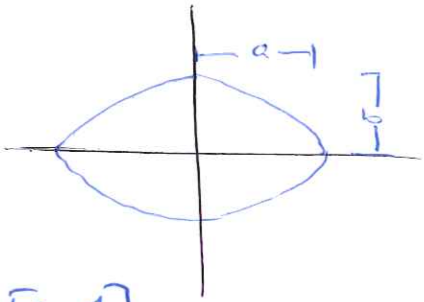
$$= \int_0^a \rho\left[\rho^2\theta + \cancel{2a\sin\theta}^{\,0} + a^2\theta\right]_0^{2\pi} d\rho = 2\pi\int_0^a \rho^3 + \rho a^2\, d\rho =$$

$$= 2\pi \cdot \left[\frac{1}{4}\rho^4 + \frac{a^2}{2}\rho^2\right]_0^a = 2\pi \cdot \left(\frac{a^4}{4} + \frac{a^4}{2}\right) =$$

$$= \boxed{\frac{3\pi}{2}a^4}$$

iv) $\iint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$ non $D=\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$

ALD - ALD [POLARRAK]



$$\begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases} \Rightarrow J = ab\rho$$

$\theta\in[0,\frac{\pi}{2}]$ $\qquad \rho\in[0,\rho_0]=[0,1]$

ρ_0?

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{a^2\rho^2\cos^2\theta}{a^2}+\frac{b^2\rho^2\sin^2\theta}{b^2}=1$$

$$\Rightarrow \frac{\cancel{a^2b^2}\rho^2\cos^2\theta+\cancel{a^2b^2}\rho^2\sin^2\theta}{\cancel{a^2b^2}}=\rho^2=1\Rightarrow\rho_0=1$$

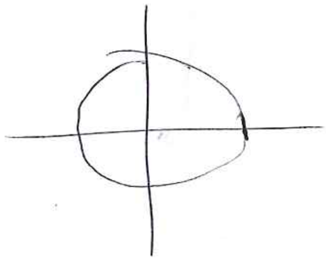
$$\int_0^1\int_0^{2\pi}\sqrt{1-\rho^2}\,ab\rho\,d\theta\,d\rho = ab\,2\pi\int_0^1\sqrt{1-\rho^2}\,\rho\,d\rho =$$

$$=2\pi ab\left[\frac{-1}{\cancel{2}}(1-\rho^2)^{3/2}\cdot\frac{\cancel{2}}{3}\right]_0^1=\frac{-2\pi ab}{3}(0-1)=$$

$$=\boxed{\frac{2\pi}{3}ab}$$

v) $\iint_D \sin(\sqrt{x^2+y^2})\,dx\,dy$ $\qquad D\equiv$ zirkulo unitarioa

ALD - ALD [POLARRAK]



$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta\end{cases} \quad \begin{matrix}\rho\in[0,1] \\ \theta\in[0,2\pi]\end{matrix} \quad J=\rho$$

$$\int_0^1\int_0^{2\pi}\sin\rho\cdot\rho\,d\theta\,d\rho = 2\pi\int_0^1\rho\sin\rho\,d\rho =$$

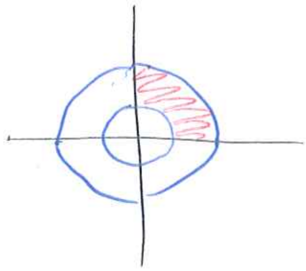
$u=\rho \qquad du=d\rho$

$dv=\sin\rho\,d\rho \qquad v=-\cos\rho$

$$=2\pi\cdot\left[-\rho\cos\rho-\int_0^1-\cos\rho\,d\rho\right]_0^1=2\pi\cdot\left[-\rho\cos\rho+\sin\rho\right]_0^1=\boxed{2\pi(\sin 1-\cos 1)}$$

vi) $\iint_A (x+y)\,dx\,dy \quad A=\{(x,y)\in\mathbb{R}^2 : 1\le x^2+y^2\le 4,\ x\ge 0,\ y\ge 0\}$

$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \quad \rho\in[1,2] \quad \theta\in[0,\frac{\pi}{2}] \quad J=\rho$



$$\int_1^2\int_0^{\pi/2} \rho(\cos\theta+\sin\theta)\rho\,d\theta\,d\rho =$$

$$=\int_1^2 \rho^2\left[\sin\theta-\cos\theta\right]_0^{\pi/2} d\rho = \int_1^2 \rho^2\left(1-0-0-(-1)\right)d\rho =$$

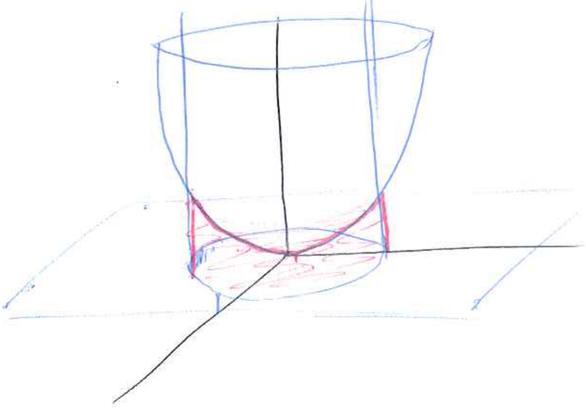
$$= 2\cdot\left[\frac{1}{3}\rho^3\right]_1^2 = \frac{2}{3}(8-1) = \boxed{\frac{14}{3}}$$

8. ARIKETA

iv) $z=x^2+y^2$

$x^2+y^2=1$

$z\ge 0$

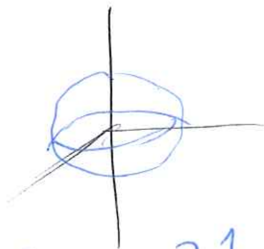


$$\iint_D (\text{paraboloid} - \text{planoa})\,dx\,dy =$$

$$= \iint x^2+y^2-0\ dx\,dy \overset{\text{ALD [POLAR]}}{=} \int_0^1\int_0^{2\pi} \rho^2\cdot\rho\,d\theta\,d\rho =$$

$$= 2\pi\cdot\int_0^1 \rho^3 d\rho = 2\pi\,\frac{1}{4}\left[\rho^4\right]_0^1 = \boxed{\frac{\pi}{2}}$$

v) $x^2+y^2+z^2=1 \Rightarrow z=1-(x^2+y^2)$



ALD-ALD [POLARRAK]

$$B = 2\cdot\int_0^1\int_0^{2\pi} \sqrt{1-\rho^2}\,\rho\,d\rho =$$

$$= 2\pi\cdot 2\cdot\int_0^1 \sqrt{1-\rho^2}\,\rho\,d\rho = 4\pi\cdot\left[-\frac{1}{2}\cdot(1-\rho)^{3/2}\cdot\frac{2}{3}\right]_0^1 =$$

$$= 4\cdot\pi\cdot\frac{-1}{3}\left[0-1\right] = \boxed{\frac{4\pi}{3}}$$

9. ARIKETA

$0 \leq x \leq 2\pi, \quad 0 \leq y \leq \pi$

$\rho(x,y) = y^2 \sin^2(4x) + 2 \quad g/cm^2 \qquad cost: 7 \$/g$

$$m(D) = \iint_D \rho(x,y)\,dx\,dy =$$

$$= \int_0^{2\pi}\int_0^{\pi} y^2 \sin^2(4x) + 2 \, dy\,dx =$$

$$= \int_0^{2\pi} \left[\frac{1}{3} y^3 \sin^2(4x) + 2y \right]_0^{\pi} dx =$$

$$= \int_0^{2\pi} \left(\frac{\pi^3}{3} \sin^2(4x) + 2\pi \right) dx =$$

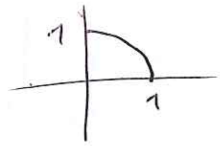
$$= \int_0^{2\pi} \frac{\pi^3}{3} \, \frac{1-\cos 8x}{2} + 2\pi \, dx =$$

$$= \left[\frac{\pi^3}{3} \cdot \left(\frac{x}{2} - \cancel{\frac{\sin 8x}{16}}^{\,0} \right) + 2\pi x \right]_0^{2\pi} =$$

$$= \frac{\pi^3}{3} \cdot \pi + 4\pi^2 = \left(\frac{\pi^4}{3} + 4\pi^2 \right) g$$

$$7 \, \$/g \cdot \left(\frac{\pi^4}{3} + 4\pi^2 \right) g = \frac{7\pi^4}{3} + 28\pi^2 \, \$$$

11. ARIKETA



$\rho(x,y) = \sqrt{1 - (x^2 + y^2)}$

$$m(D) = \iint \rho(x,y)\,dx\,dy = \int_0^1 \int_0^{\pi/2} \sqrt{1-\rho^2}\,\rho \, d\theta \, d\rho =$$

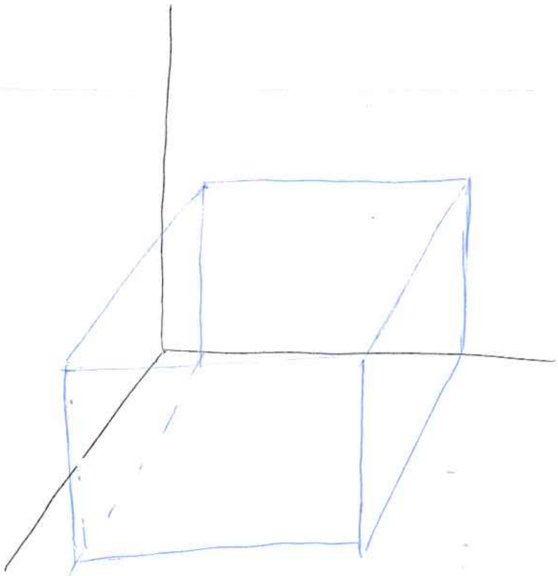
$$= 2\pi \left[\frac{-1}{2} (1-\rho^2)^{3/2} \frac{2}{3} \right]_0^1 = 2\pi \cdot \frac{1}{3} \Rightarrow m(D) = \frac{2\pi}{3}$$

4. GAIA: INTEGRAL HIRUKOITZA

4.1. INTEGRAL HIRUKOITZA PARALELEPIPEDO BATEN GAINEAN.

- DEFINIZIOA (Zuek 77. orrian edo 4. gaian)

$B = [a,b] \times [c,d] \times [e,g] \subset \mathbb{R}^3$ paralelepipedo bat



$$\iiint_B f \, dV = \iiint_B f(x,y,z) \, dx \, dy \, dz$$

TEOREMA 4.1:

Izan bitez $B = [a,b] \times [c,d] \times [e,g] \subset \mathbb{R}^3$ prllppd eta $f: B \longrightarrow \mathbb{R}$. f jarraitua bada edo f-ren etenguneen multzoa bi aldagaiko funtzio jarraituen bildura finitua bada $\Rightarrow$ f integragarria da B eremuan.

TEOREMA 4.2: INTEGRAL ITERATUAK

Izan bitez $B = [a,b] \times [c,d] \times [e,g] \subset \mathbb{R}^3$ eta $f: B \longrightarrow \mathbb{R}$ integragarria

$\Rightarrow$ Edozein integral iteratu existitzen bada, integral hirukoitzaren berdina da:

$$\iiint f(x,y,z) \, dV = \int_a^b \int_c^d \int_e^g f \, dz \, dy \, dx = \int_a^b \int_e^g \int_c^d f \, dy \, dz \, dx =$$

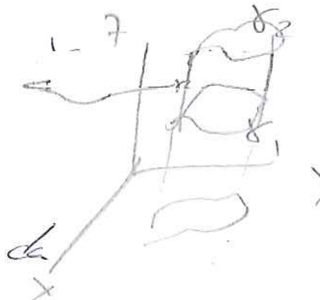
$$= \int_e^g \int_c^d \int_a^b f \, dx \, dy \, dz \quad \Rightarrow \text{INTEGRAL ITERATUAK}$$

4.2. INTEGRAL HIRUKOITZA ESKUALDE OROKORRAGOETAN

DEFINIZIOA: ESKUALDE ELEMENTALAK

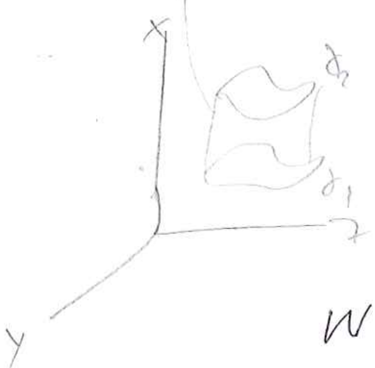
Izan bedi: $W \subset \mathbb{R}^3$ bornatua

i) W <u>I motatako eskualde elementala</u> da, $\exists D \subset \mathbb{R}^2$ eremu elementala eta $\gamma_1, \gamma_2 : D \to \mathbb{R}$ jarraitua non $W = \{(x, y, z) \in \mathbb{R}^2 : (x, y) \in D,$ $\gamma_1(x, y) \leq z \leq \gamma_2(x, y)\}$



ii) W <u>II motako eskualde elementala</u> da $\exists D \subset \mathbb{R}^2$ eremu elementala eta $\gamma_1, \gamma_2 : D \to \mathbb{R}$ jarraituak non

$W = \{(x, y, z) \in \mathbb{R}^2 : (y, z) \in D, \gamma_1(y, z) \leq x \leq \gamma_2(y, z)\}$



iii) W <u>III motako eskualde elementala</u> da $\exists D \subset \mathbb{R}^2$ eremu elementala eta $\gamma_1, \gamma_2 : D \to \mathbb{R}$ jarraituak

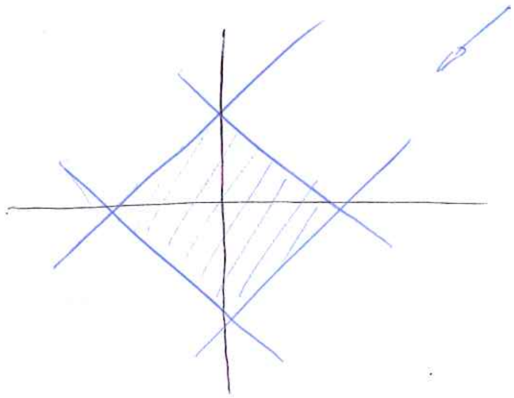
$W = \{(x, y, z) \in \mathbb{R}^3 \mid (x, z) \in D$ non $\gamma_1(x, z) \leq y \leq \gamma_2(x, z)\}$

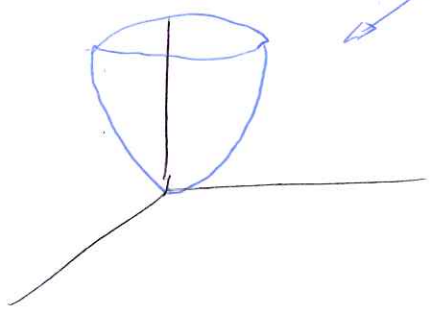
iv) W <u>IV motako eskualde elementala</u> da I, $II \wedge III$ motakoa bada.

ADIBIDEA: Kalkulatu $\iiint_W z \, dV$ non → 1. GOIAN → PLANOA

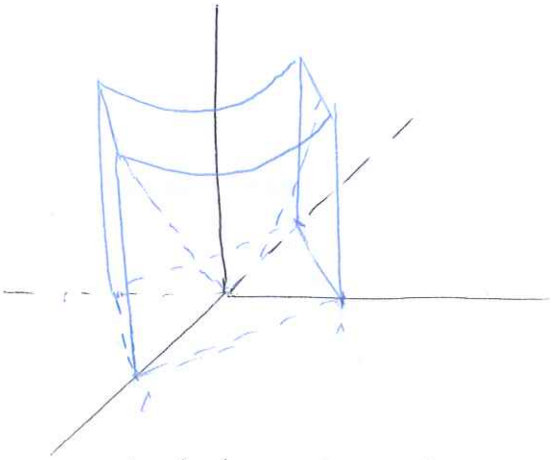
$W = \{(x, y, z) \in \mathbb{R}^3 \mid |x| + |y| \leq 1, \ 0 \leq z \leq x^2 + y^2\}$

PARABOLOIDE





2



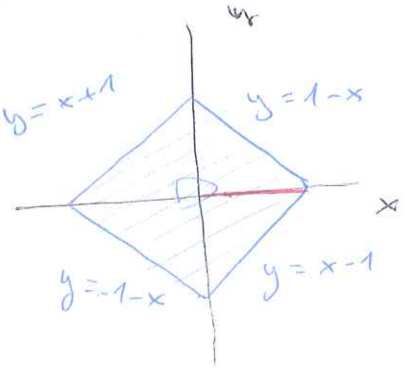
W => I motako E.E

$W = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D :$

$0 \le z \le x^2 + y^2\}$

$\gamma_1(x,y)$; $\gamma_2(x,y)$

Proiektatu W OXY planoan



$$\iiint_D \int_0^{x^2+y^2} z\,dz\,dx\,dy =$$

$$= \iint_D \left.\frac{z^2}{2}\right]_0^{x^2+y^2} = \iint_D \frac{(x^2+y^2)^2}{2}\,dx\,dy =$$

$$= 4 \cdot \iint_D \frac{(x^2+y^2)^2}{2}\,dx\,dy = \frac{4}{2}\int_0^1\int_0^{1-x} (x^2+y^2)^2\,dy\,dx =$$

$$= 2 \cdot \int_0^1\int_0^{1-x} x^4 + 2x^2y^2 + y^4\,dy\,dx =$$

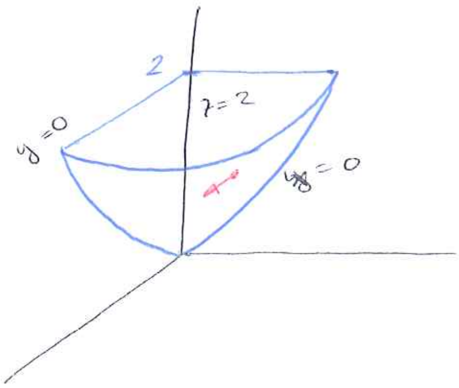
$$= 2 \cdot \int_0^1 \left[x^4 y + 2x^2\frac{y^3}{3} + \frac{y^5}{5}\right]_0^{1-x} dx = [\dots] = \boxed{\frac{7}{45}}$$

ADIBIDEA $\iiint_W x\,dx\,dy\,dz$

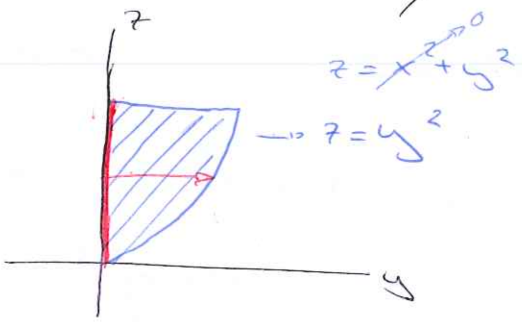
$W = \left\{ \begin{array}{l} x=0,\ y=0 \ \wedge\ z=2 \text{ planoek eta} \\ z = x^2+y^2 \text{ gainazalek mugatutako} \\ \text{gorputza } x \ge 0 \ \wedge\ y \ge 0 \quad \text{izanik} \end{array} \right.$

IV motako E.E

(Konkretuki II motakoa)



Proiektatu OYZ planoan



$$\iiint_W x\,dx\,dy\,dz = \iint_D \int_0^{\sqrt{z-y^2}} x\,dx\,dy\,dz =$$

Z PROI

$$= \iint_D \left[\frac{x^2}{2}\right]_0^{\sqrt{z-y^2}} dy\,dz = \frac{1}{2}\iint_D z - y^2\,dy\,dz \overset{\downarrow}{=} \frac{1}{2}\int_0^2 \int_0^{\sqrt{z}} (z-y^2)\,dy\,dz =$$

$$= \frac{1}{2}\int_0^2 \left[zy - \frac{y^3}{3}\right]_0^{\sqrt{z}} dz = \frac{1}{2}\int_0^2 z^{3/2} - \frac{z^{3/2}}{3}\,dz =$$

$$= \frac{1}{\cancel{2}}\left[z^{5/2}\cdot\frac{\cancel{2}}{5} - \frac{z^{5/2}}{3}\cdot\frac{2}{5}\right]_0^2 = \frac{2}{3}\cdot\left.\frac{z^{5/2}}{5}\right]_0^2 = [\ldots] = \frac{8}{15}\sqrt{2}$$

PROPOSIZIOA 4.3: INTEGRAL HIRUKOITZAREN OINARRIZKO PROP.

Izan bitez $W \subset \mathbb{R}^3$ eskualde elementala eta $f, g : W \to \mathbb{R}$ W-n integragarria

i) $f+g$ integragarria $|x|$-n $\wedge$ $\iiint_W f+g\,dV = \iiint_W f\,dV + \iiint_W g\,dV$

ii) λf integragarria $\forall \lambda \in \mathbb{R}$ $\wedge$ $\iiint_W \lambda f\,dV = \lambda \iiint_W f\,dV$

iii) $f \leq g$ bada $\forall (x,y,z) \in W$

$\Rightarrow \iiint f\,dV \leq \iiint g\,dV$

iv) $W = \bigcup_{i=1}^m W_i$ non W_i e.e diren $\forall i = 1, \ldots, m$

eta $W_i \cap W_j = \emptyset$ $\forall i \neq j$ denean

$$\iiint_W f\,dV = \sum_{i=1}^m \iiint_{W_i} f\,dV$$

v) $|f|$ integragarria W-n eta $\left|\iiint_W f\,dV\right| \leq \iiint_W |f|\,dV$

TEOREMA 4.4: BATAZBESTEKO BALIOAREN TEOREMA

Izan bitez $W \subset \mathbb{R}^3$ E.E finituen bildura eta

$f: W \longrightarrow \mathbb{R}$ jarraitua

$\Rightarrow \exists (x_0, y_0, z_0) \in W$ non

$$\iiint_W f\, dV = f(x_0, y_0, z_0) \cdot \underbrace{\iiint_W 1\, dV}_{\text{BOLUMENA}}$$

4.3. ALDAGAI-ALDAKETA INTEGRAL HIRUKOITZETAN

$f: W \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ integragarria eta

$\iiint_W f\, dV$ Kalkulatu nahi dugu

DEFINIZIOA: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ C^1 klasekoa

Transformazioa $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$

T-ren DETERMINANTE JACOBIARRA hau da

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

TEOREMA 4.5: ALD-ALD INTEGRAL HIRUKOITZETAN

Izan bitez W eta $W^* \subset \mathbb{R}^3$ E.E.

$T: W^* \longrightarrow W$ C^1 motako transformazio

injektiboa eta $f: W \longrightarrow \mathbb{R}$ integragarria.

$$\iiint_W f(x,y,z)\, dV = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot \underbrace{|J|}_{\text{BALIO ABS}}\, du\, dv\, dw$$

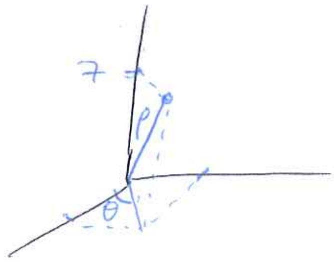
DEFINIZIOA: $(x,y,z) \in \mathbb{R}^3$ puntu bat emanik

(ρ, θ, z) KOORDENATU ZILINDRIKOAK dira non

$T(\rho, \theta, z) = (x_0 + \rho\cos\theta,\ y_0 + \rho\sin\theta,\ z_0 + z)$

$(x_0, y_0, z_0) \Rightarrow$ ZENTRUA

$\theta \in [0, 2\pi]$ $\qquad \rho \geq 0$

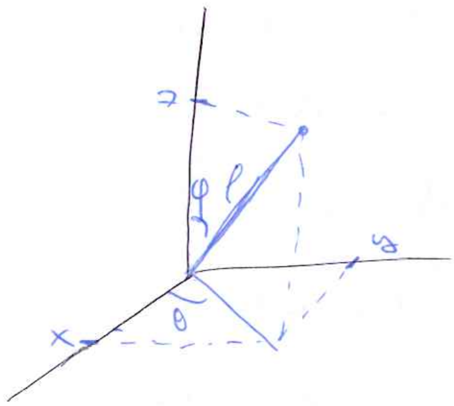


$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = [\ldots] = \boxed{\rho}$$

DEFINIZIOA: $(x,y,z) \in \mathbb{R}^3$ puntu bat emanik (ρ,θ,φ) KOORDENATU ESFERIKOAK dira non

$$T(\rho,\theta,\varphi) = (x_0 + \rho\cos\theta\sin\varphi,\ y_0 + \rho\sin\theta\sin\varphi,\ z_0 + \rho\cos\varphi)$$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = [\ldots] = \rho^2\sin\varphi$$

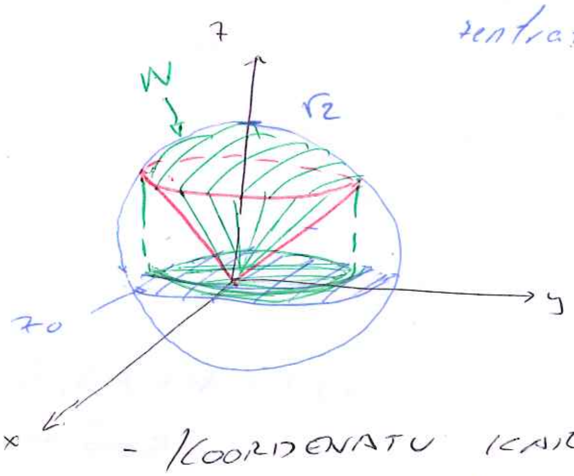


ADIBIDEAK

1) Kalkulatu $\iiint z\,dx\,dy\,dz$

$$W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 2,\ x^2+y^2 \le z^2,\ z \ge 0\}$$

$\sqrt{2}$ erradioko $(0,0,0)$-n zentratutako esfera; $(0,0,0)$-n zentratutako konoa; planoa



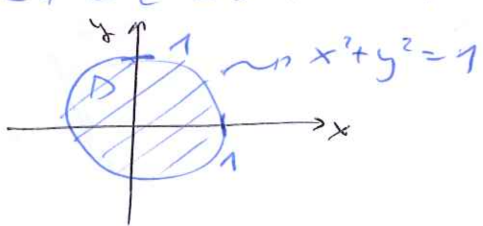
- KOORDENATU KARTESIARRETAN

I motako azalera elementua OXY planoan proiektatuz

$$\begin{cases} x^2+y^2+z^2 = 2 \\ x^2+y^2-z^2 = 0 \end{cases} \Rightarrow \boxed{z=1} \quad x^2+y^2 = 1^2$$

$$2z^2 = 2 \Rightarrow z = \pm 1 \qquad x^2+y^2 = 1$$

$\Rightarrow$



$$\iiint_W z\,dx\,dy = \iiint_D^{esfera}_{konoa} z\,dz\,dx\,dy =$$

$$= \iiint_D \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z\,dz\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z\,dz\,dy\,dx =$$

Bakandu z

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\frac{z^2}{2}\right]_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dy\,dx =$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}\left[2-x^2-y^2-x^2-y^2\right] dy\,dx =$$

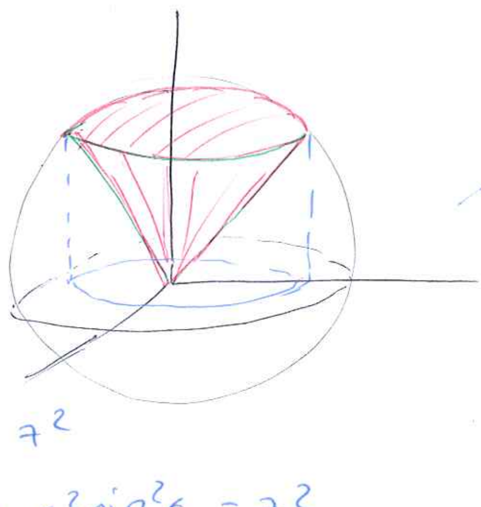
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1-x^2-y^2\,dy\,dx = \int_{-1}^{1}\left[y-x^2y-\frac{1}{3}y^3\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx =$$

$$= \int_{-1}^{1} \sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2} + \sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2}\,dx$$

$$= [\ldots] = \frac{\pi}{2} \quad [Zaila]$$

KOORDENATU ZILINDRIKOAK

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \Rightarrow J = \rho$$



ALDAGAI BERRIAK

$\theta \in [0, 2\pi]$

$\rho \in [0, 1]$

$z \in [konoa, esfera]$

$z \in [\rho, \sqrt{2-\rho^2}]$

- $x^2 + y^2 = z^2$

$\rho^2\cos^2\theta + \rho^2\sin^2\theta = z^2$

$\rho^2 = z^2 \Rightarrow \rho = z$

• $x^2+y^2+z^2=2$

$\rho^2+z^2=2$

$z=\sqrt{2-\rho^2} \geq 0$

$$\iiint_W z\,dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_\rho^{\sqrt{2-\rho^2}} z\cdot\rho\,dz\,d\rho\,d\theta =$$

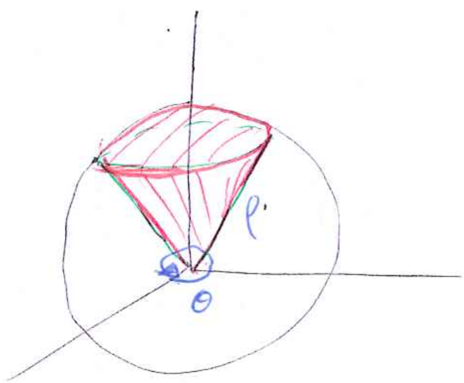
$$= \int_0^{2\pi}\int_0^1 \rho\left[\frac{z^2}{2}\right]_\rho^{\sqrt{2-\rho^2}} d\rho\,d\theta = \int_0^{2\pi}\int_0^1 \rho\left(\frac{2-\rho^2-\rho^2}{2}\right) d\rho\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \rho-\rho^3\,d\rho\,d\theta = \int_0^{2\pi}\left[\frac{1}{2}\rho^2-\frac{1}{4}\rho^4\right]_0^1 d\theta =$$

$$= \int_0^{2\pi}\frac{1}{2}-\frac{1}{4}\,d\theta = \left[\frac{\theta}{2}-\frac{\theta}{4}\right]_0^{2\pi} = \pi-\frac{\pi}{2} = \frac{\pi}{2}$$

KOORDENATU ESFERIKOAK

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \Rightarrow J = -\rho^2\sin\varphi$$



ALD-ALD

$\theta \in [0, 2\pi]$

$\rho \in [0, \sqrt{2}]$

$\varphi \in [\text{Konoa}, 0]$

Konoa $\Rightarrow$ $x^2+y^2=z^2$

$\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi = \rho^2\cos^2\varphi$

[...] $\sin^2\varphi = \cos^2\varphi$

$\sin\varphi = \pm\cos\varphi \Rightarrow \varphi = \frac{\pi}{4}$

$\varphi \in [0, \frac{\pi}{4}]$

$$\iiint_W z\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho\cos\varphi \cdot \overbrace{\rho^2\sin\varphi}^{|J|}\, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\cos\varphi\sin\varphi \underbrace{\left[\frac{\rho^4}{4}\right]_0^{\sqrt{2}}}_{1} d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/4} d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{1}{4}\left[\theta\right]_0^{2\pi} = \boxed{\frac{\pi}{2}}$$

2. ADIBIDEA

$$\iiint_\Omega z\,dx\,dy\,dz$$

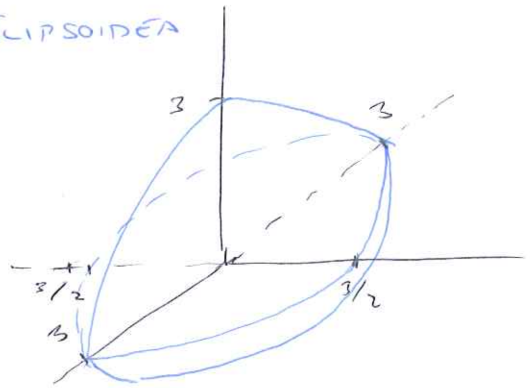
elipsoidea

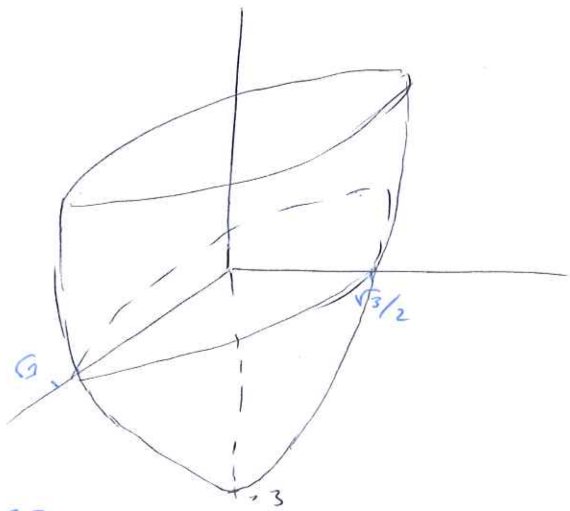
$$\Omega = \{(x,y,z)\in\mathbb{R}^3 :\ z+3 \le x^2+4y^2,\ x^2+4y^2+z^2\le 9\}$$

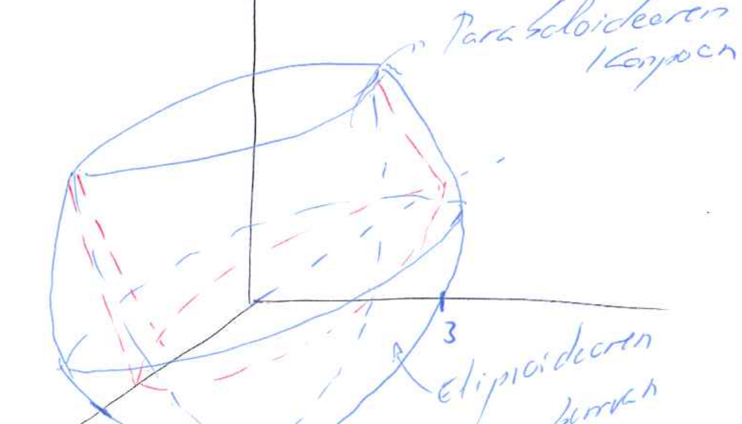
Paraboloide eliptikoa

$$\frac{x^2}{3^2}+\frac{y^2}{(\frac{3}{2})^2}+\frac{z^2}{3^2}=1$$

ELIPSOIDEA







EBAKIDURA

$$\begin{cases} z+3 = x^2+4y^2 \\ x^2+4y^2+z^2 = 9 \end{cases} \Rightarrow z+3+z^2 = 9 \Rightarrow z = 2, -3$$

OXY planoan proiektatu

$$z = 2 \xrightarrow{(1)} z+3 = x^2+4y^2$$

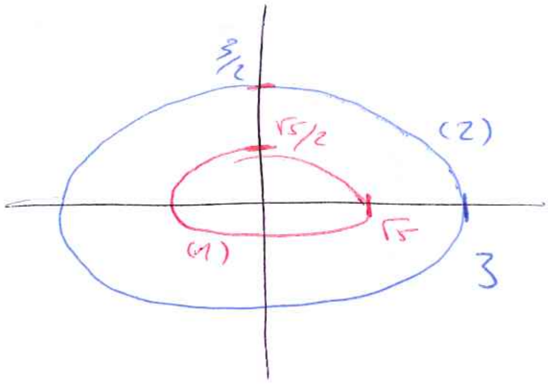
$$5 = x^2+4y^2$$

$$1 = \frac{x^2}{(\sqrt{5})^2} + \frac{y^2}{(\frac{\sqrt{5}}{2})^2}$$

$$z = 0 \xrightarrow{(2)} x^2+4y^2+z^2 = 9$$

$$x^2+4y^2 = 9$$

$$\frac{x^2}{3^2} + \frac{y^2}{(\frac{3}{2})^2} = 1$$



(1) D_1 $\begin{cases} \text{Goitik paraboloidea} \\ \text{Behetik elipsoidea} \end{cases}$

(2) D_2 $\begin{cases} \text{Goitik eta behetik} \\ \text{elipsoidea} \end{cases}$

Aukeratuko dugu aldagai aldaketa (zilindrikoak)

$$\begin{cases} x = \rho\cos\theta \\ y = \frac{1}{2}\rho\sin\theta \\ z = z \end{cases} \qquad |J| = \frac{1}{2}\rho$$

$\boxed{D_1}$

$\theta \in [0, 2\pi]$

$\rho \in [0, \underset{(1)}{(1)\text{ elipsea}}]$

$z \in [\underset{(2)}{\text{elipsoide}}, \underset{(3)}{\text{paraboloide}}]$

$\boxed{D_2}$

$\theta \in [0, 2\pi]$

$\rho \in [\underset{(4)}{(1)\text{ elipse}}, \underset{(5)}{(2)\text{ elipse}}]$

$z \in [\underset{(6)}{\text{elipsoide}}, \underset{(7)}{\text{elipsoide}}]$

4) l $\mathbb{R}^3$-ko zuzena

$\delta(x,y,z) \to (x,y,z) \in W$ puntutik l zuzenerako distantzia. W-ren l-rekiko MOMENTUA:

$$I_l = \iiint_W (\delta(x,y,z))^2 \cdot \rho(x,y,z)\, dx\, dy\, dz$$

$$I_x = \iiint_W (y^2+z^2)\rho(x,y,z)\, dx\, dy\, dz$$

$$I_y = \iiint_W (x^2+z^2)\rho(x,y,z)\, dx\, dy\, dz$$

$$I_z = \iiint_W (x^2+y^2)\rho(x,y,z)\, dx\, dy\, dz$$

I_x, I_y, I_z ardatzekiko INERTZIA MOMENTUAK

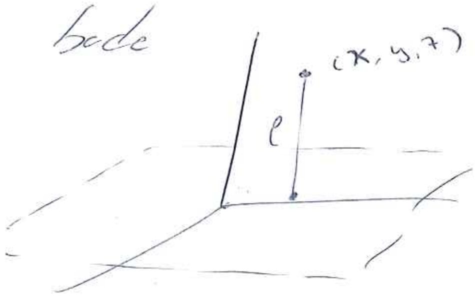
5) PLANOEKIKO INERTZIA MOMENTUAK

[Zuek: 91.orr.]

6) $W \subset \mathbb{R}^3$ eremu elementala eta $f: W \to \mathbb{R}$ jarraitua

$$[f]_m = \frac{\iiint_W f\, dV}{B(W)} \Rightarrow$$ f-ren batazbesteko balioa W eremuan

ADIBIDEA: Kalkulatu $z^2 = x^2+y^2$ konoak eta $2z+y=3$ planoak, $z \geq 0$ espazioerdian mugatzen duten solidoaren masa, $\rho(x,y,z)$ dentsitatea (x,y,z) puntutik OXY planoarekiko distantzia bada



$\Rightarrow \rho(x,y,z) = |z| = z$

(1), (4) $x^2+4y^2=5$

$\rho^2=5 \Rightarrow \rho=\sqrt{5}$

(5) $x^2+4y^2+z^2=9$

$\rho^2+z^2=9 \Rightarrow \rho=\sqrt{9-z^2}$

(6), (2) $z=-\sqrt{9-\rho^2}$

(7) $z=\sqrt{9-\rho^2}$

(3) $z+3=x^2+4y^2$

$\Downarrow$

$z=\rho^2-3$

$$\iiint_\Omega z\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\sqrt{5}}\int_{-\sqrt{9-\rho^2}}^{\rho^2-3} z\cdot\frac{\rho}{2}\,dz\,d\rho\,d\theta +$$

$$+\underbrace{\int_0^{2\pi}\int_0^{\sqrt{9-z^2}}\int_{-\sqrt{9-\rho^2}}^{\sqrt{9-\rho^2}} z\,\frac{\rho}{2}\,dz\,d\rho\,d\theta}_{=0} = [\ldots] = \boxed{-\frac{125}{24}\pi}$$

4.4. INTEGRAL HIRUKOITZAREN APLIKAZIOAK

$W\subset\mathbb{R}^3$

1) $B(W)=\iiint_W 1\,dx\,dy\,dz \longrightarrow$ W-ren bolumena

2) $m(W)=\iiint_W \rho(x,y,z)\,dx\,dy\,dz \longrightarrow$ W-ren MASA

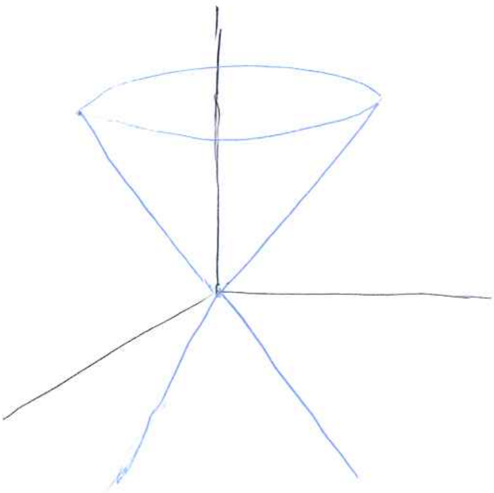
3) $(\bar{x},\bar{y},\bar{z})$ W-ren MASA ZENTRUA

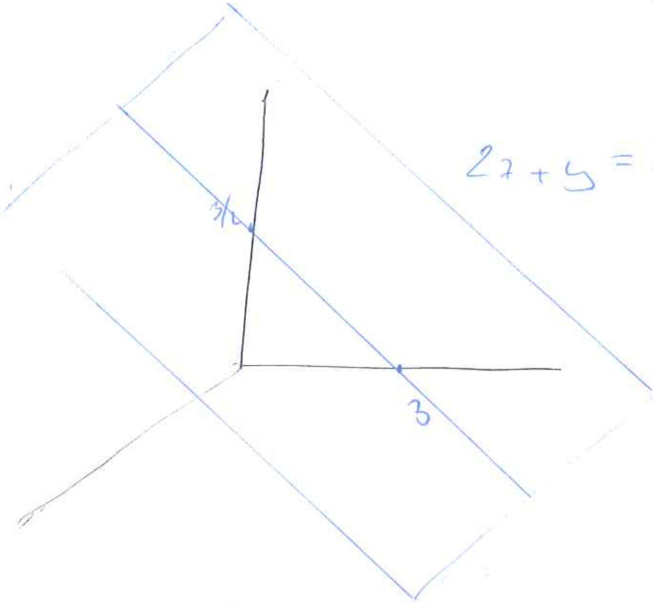
$$\bar{x}=\frac{\iiint_W x\,\rho(x,y,z)\,dx\,dy\,dz}{m(W)}$$

$$\bar{y}=\frac{\iiint_W y\,\rho(x,y,z)\,dx\,dy\,dz}{m(W)}$$

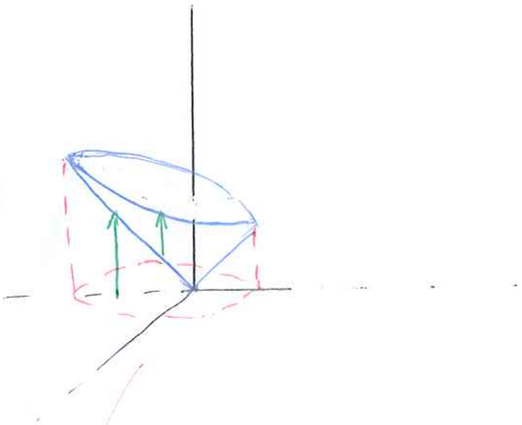
$\rho(x,y,z)$ = DENTSITATEA

$$\bar{z}=\frac{\iiint_W z\,\rho(x,y,z)\,dx\,dy\,dz}{m(W)}$$





$$2z + y = 3$$

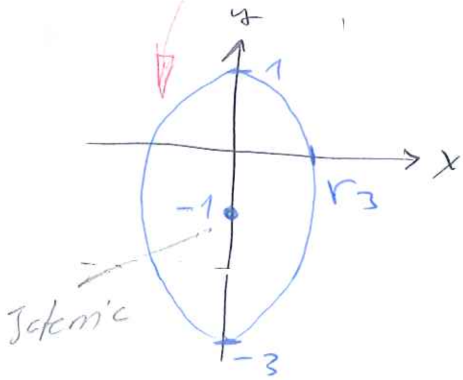


Proiektatu OXY

$$\begin{cases} 2z + y = 3 \rightarrow z = \dfrac{3-y}{2} \\ z^2 = x^2 + y^2 \longrightarrow \left(\dfrac{3-y}{2}\right)^2 = x^2 + y^2 \end{cases}$$

$$[\ldots]$$

$$\frac{x^2}{(\sqrt{3})^2} + \frac{(y+1)^2}{2^2} = 1$$



ALDAGAI - ALDAKETA

$$\begin{cases} x = \sqrt{3}\rho\cos\theta \\ y = 2\rho\sin\theta - 1 \Rightarrow |J| = \sqrt{3}\cdot 2\rho \\ z = z \end{cases}$$

$$\begin{cases} \theta \in [0, 2\pi) \\ \rho \in [0, \text{elipsea}] = [0, 1] \\ z \in [\text{Konoa}, \text{planoa}] \end{cases}$$

$$\frac{x^2}{(\sqrt{3})^2} + \frac{y^2}{2^2} = 1 \xrightarrow{\text{ALD ALD}} \rho = 1$$

KONOA: $z^2 = x^2 + y^2 \overset{\text{ALD}}{\Rightarrow} z = \sqrt{3\rho^2\cos^2\theta + 4\rho^2\sin^2\theta - 4\rho\sin\theta + 1}$

PLANOA: $z = \dfrac{3-y}{2} \overset{\text{ALD}}{\Longrightarrow} z = \dfrac{3 - (2\rho\sin\theta - 1)}{2} = 2 - \rho\sin\theta$

$$m(W) = \iiint_W z\,dV = \int_0^{2\pi}\int_0^1\int_{\sqrt{3\rho^2\cos^2\theta + 4\rho^2\sin^2\theta - 4\rho\sin\theta + 1}}^{2-\rho\sin\theta} z\cdot\sqrt{3}\cdot 2\rho\, dz\, d\rho\, d\theta = [\ldots] = \boxed{\frac{3\sqrt{3}\pi}{2}}$$

ANALISI BEKTORIALA ETA KONPLEXUA

4. Gaia: INTEGRAL HIRUKOITZA

Ariketak

1. Kalkula itzazu ondoko solidoen bolumenak:

+ (i) $x^2+y^2=z$ eta $x^2+y^2+z^2=2$ gainazalek $z\geq 0$ espazioerdian mugatzen dutena.
+ (ii) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ eta $x=0$, $y=0$, $z=0$ planoek lehenengo oktantean mugatzen duten solidoarena, a, b, c positiboak izanik.
+ (iii) $az=a^2-x^2-y^2$ gainazalak eta $z=a-x-y$, $x=0$, $y=0$, $z=0$ planoek lehenengo oktantean mugatutako solidoaren bolumena, a positiboa izanik.
+ (iv) $x^2+y^2=2ax$, $z=0$, $x^2+y^2=z^2$ gainazalek $z\geq 0$ espazioerdian definitzen duten solidoa.
+ (v) $x^2+y^2+z^2=R^2$ eta $(x^2+y^2)^2=R^2(x^2-y^2)$ gainazalek $x\geq 0$ espazioerdian mugatzen duten bolumena, zilindroaren barrukoa dena.
+ (vi) $A=\left\{(x,y,z)\in\mathbf{R}^3:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 2,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq\frac{z^2}{c^2},\ z\geq 0\right\}$ eskualdearen bolumena.
+ (vii) $z=x^2+y^2$, $x^2+y^2=x$, $x^2+y^2=2x$ gainazalek $z\geq 0$ espazioerdian mugatzen duten eskualdearen bolumena.
+ (viii) $x^2+y^2-az=0$, $(x^2+y^2)^2=a^2(x^2-y^2)$, $z=0$ gainazalek mugatzen dutena.
+ (ix) $x^2+y^2+z^2=a^2$, $x^2+y^2+z^2=b^2$, $x^2+y^2=z^2$ gainazalek, $z\geq 0$ espazioerdian mugatzen duten solidoa, $0<a<b$ izanik.
+ (x) $z=x^2+y^2$ paraboloideak eta $z=x$ planoak definitzen duten solidoaren bolumena.
+ (xi) $\Omega=\{(x,y,z)\in\mathbf{R}^3:\ \frac{x^2}{25}+\frac{y^2}{25}+\frac{z^2}{9}\leq 1,\ x^2+y^2\geq 4\}$ solidoaren bolumena.
+ (xii) $z=4-x^2-y^2$ paraboloideak eta $z=2+y^2$ zilindro parabolikoak mugatzen duten solidoaren bolumena.
+ (xiii) $\frac{x^2}{9}+\frac{y^2}{4}\leq 1$, $\frac{y^2}{4}+\frac{z^2}{9}\leq 1$, $x\geq 0$, $y\geq 0$ eta $z\geq 0$ baldintzek definitzen duten solidoaren bolumena.
+ (xiv) $z=1-x^2-y^2$ eta $x+z=1$ gainazalek mugatzen duten solidoaren bolumena.

Em.:: (i) $2\pi\left(\frac{2\sqrt{2}-1}{3}-\frac{1}{4}\right)$; (ii) $\frac{abc}{6}$; (iii) $\left(\frac{\pi}{8}-\frac{1}{6}\right)a^3$; (iv) $\frac{32a^3}{9}$; (v) $\frac{3\pi+20-16\sqrt{2}}{9}R^3$; (vi) $\frac{4(\sqrt{2}-1)\pi abc}{3}$; (vii) $\frac{45\pi}{32}$; (viii) $\frac{a^3\pi}{8}$; (ix) $\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)(b^3-a^3)$; (x) $\frac{\pi}{32}$; (xi) $\frac{84\pi\sqrt{21}}{5}$; (xii) $\sqrt{2}\pi$; (xiii) 12; (xiv) $\pi/32$

$\pi/4$

$(-24)^{3/2}-(-3)^{3/2}$

$24\cdot\sqrt{-24}$

2. Izan bitez W OXY planoarekiko simetrikoa den I motako eskualde elementala eta W^+ $z \geq 0$ espazioerdian geratzen den W-ren zatia.

(i) Froga ezazu f z aldagaian bikoitia denean, hots $f(x,y,-z) = f(x,y,z)$ denean $(x,y,z) \in W^+$ guztietarako, orduan

$$\iiint_W f(x,y,z)\,dxdydz = 2\iint_{W^+} f(x,y,z)\,dxdydz$$

dela.

(ii) Froga ezazu f z aldagaian bakoitia denean, hots $f(x,y,-z) = -f(x,y,z)$ denean $(x,y,z) \in W^+$ guztietarako, orduan

$$\iiint_W f(x,y,z)\,dxdydz = 0$$

dela.

3. Kalkulatu, $u = x+y+z$, $uv = y+z$, $uvw = z$ aldagai-aldaketaren bidez, ondoko integral hirukoitza:

$$\iiint_E xyz(1-x-y-z)\,dV,$$

non E $x \geq 0$, $y \geq 0$, $z \geq 0$, $x+y+z \leq 1$ desberdintzek definitutako tetraedroa den.

Em.: $\dfrac{1}{5040}$.

4. Ondorengo integralak kalkula itzazu:

+ (i) $\iiint_V xyz\,dxdydz$, non V $x=0$, $y=0$, $z=0$ planoz eta $x^2+y^2+z^2=1$ esferaz mugatutako lehen oktanteko eskualdea den.

+ (ii) $\iiint_S \dfrac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$, S solidoa $x^2+y^2+z^2=a^2$ eta $x^2+y^2+z^2=b^2$ esferez bornatutakoa bada, $0<a<b$ izanik.

+ (iii) $\iiint_A zy\sqrt{x^2+y^2}\,dxdydz$, $A = \{(x,y,z) \in R^3 \,:\, 0 \leq z \leq x^2+y^2, 0 \leq y \leq \sqrt{2x-x^2}\}$.

+ (iv) $\iiint_A ze^{-(x^2+y^2)}\,dxdydz$, $A = \{(x,y,z) \in R^3 \,:\, 2(x^2+y^2) \leq z^2 \leq x^2+y^2+1,\ z \geq 0\}$.

+ (v) $\iiint_W z^2\,dV$, non $W = \{(x,y,z) \in \mathbf{R}^3 \,:\, x^2+y^2+z^2 \leq R^2,\ x^2+y^2+(z-R)^2 \leq R^2\}$ den.

(vi) $\iiint_V (x^2+y^2+z^2)\,dxdydz$, baldin eta V multzoa $x \geq 0$, $y \geq 0$ eta $-\sqrt{4-x^2-y^2} \leq z \leq \dfrac{1}{2}\sqrt{4-x^2-y^2}$ desberdintzak betetzen dituzten puntuek osatzen dutena bada.

− + (vii) $\iiint_A x^2\,dxdydz$, non $A = \{(x,y,z) \in \mathbf{R}^3 \,:\, x^2+y^2+(z-1)^2 \leq 1,\ 3z^2 \geq x^2+y^2\}$ den.

+ (viii) $\iiint_W z\,dxdydz$, non $W = \{(x,y,z) \in \mathbf{R}^3 \,:\, x^2+y^2+z^2 \leq 9,\ z \leq x^2+y^2-7\}$ den.

Em.: (i) $\dfrac{1}{48}$; (ii) $4\pi \ln \dfrac{b}{a}$; (iii) $\dfrac{16}{9}$; (iv) $\dfrac{\pi}{2e}$; (v) $\dfrac{59}{480}\pi R^5$; (vi) $\dfrac{22\pi}{5}$; (vii) $\dfrac{81\pi}{320}$; (viii) $-\dfrac{9\pi}{4}$.

5. (i) Kalkula ezazu $x^2 + y^2 + z^2 = 2$, $x^2 + y^2 = z^2$, $z > 0$ solidoaren OZ ardatzarekiko inertzia-momentua (dentsitatea konstantea da).

(ii) Froga ezazu $z = y^2 + 2$, $z = 3y^2$, $x = 0$, $x = 3$ kurbek mugatzen duten eskualdearen masa-zentrua $(3/2, 0, 7/5)$ puntua dela (dentsitatea konstantea da).

(iii) Kalkula ezazu $x^2 = 2az$, $x = 0$ eta $z^2 + y^2 = az$ $(a > 0)$ gainazalek mugatutako solidoaren masa, dentsitatea puntu bakoitzean $\rho(x, y, z) = x$ izanik.

(iv) Kalkula itzazu $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$ elipsoidearen ardatz koordenatuekiko inertzia-momentuak, a, b eta c positiboak izanik eta dentsitatea, d, konstantea.

(v) Izan bedi V $x^2 + y^2 = 2y$ gainazal zilindrikoak eta $z = 0$, $y + z = 2$ planoek mugatutako solidoa. Kalkulatu V-ren masa dentsitatea $\rho(x, y, z) = z$ bada.

Em.: (i) $2\pi d\left(\dfrac{8\sqrt{2}}{15} - \dfrac{2}{3}\right)$, d dentsitatea izanik; (iii) $\dfrac{a^4\pi}{8}$;

(iv) $I_x = \dfrac{4}{15}(b^2 + c^2)abcd\pi$, $I_y = \dfrac{4}{15}(a^2 + c^2)abcd\pi$, $I_z = \dfrac{4}{15}(a^2 + b^2)abcd\pi$; (v) $5\pi/4$.

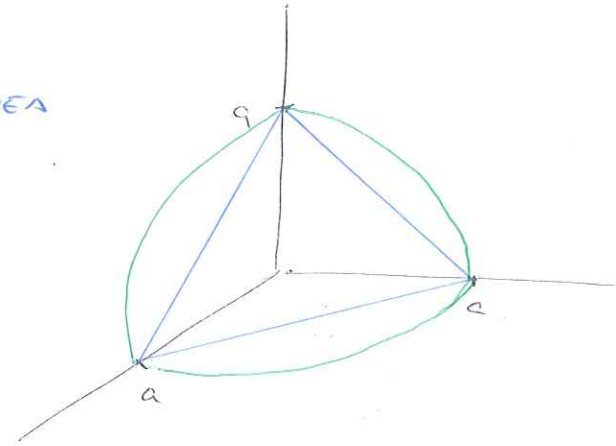
4. INTEGRAL HIRUKOITZA
ARIKETAK

1. ARIKETA

iii) $az = a^2 - x^2 - y^2 \to$ PARABOLOIDEA

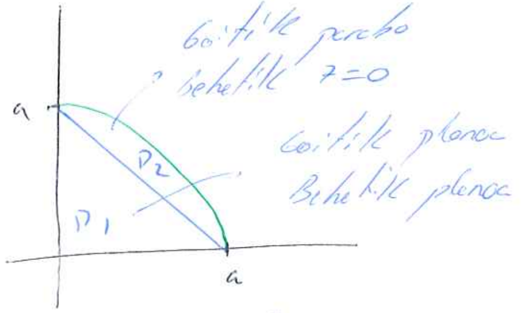
Planoak $\left\{ \begin{array}{l} z = a - x - y \\ x = 0, \; y = 0, \; z = 0 \end{array} \right.$

Goitik paraboloidea
Behetik planoa $\quad (W)$



$$B(W) = \iiint 1 \, dx\, dy\, dz$$

PROIEKTATU OXY:



$$B(W) = \iint_{D_1} \int_{a-x-y}^{\frac{a^2-x^2-y^2}{a}} 1 \, dz\, dx\, dy + \iint_{D_2} \int_0^{\frac{a^2-x^2-y^2}{a}} 1 \, dz\, dx\, dy =$$

$$= \int_0^a \int_0^{a-x} \int_{a-x-y}^{\frac{a^2-x^2-y^2}{a}} 1 \, dz\, dy\, dx + \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} \int_0^{\frac{a^2-x^2-y^2}{a}} 1 \, dz\, dx\, dy =$$

$$= [\ldots] = \left(\frac{\pi}{8} - \frac{1}{6} \right) a^3$$

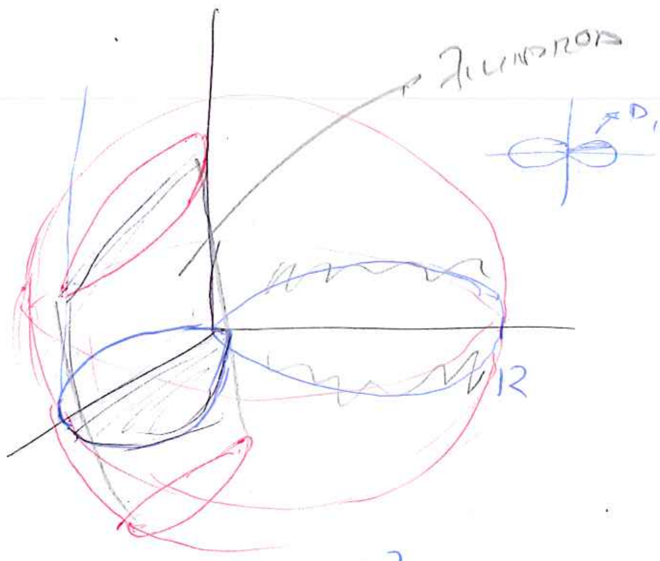
- $\int \sqrt{a^2 - x^2} \, dx = \ldots \quad \left\{ \begin{array}{l} x = a \sin\theta \\ dx = a\cos\theta \, d\theta \end{array} \right.$

- $\int x^2 \sqrt{a^2 - x^2} \, dx = \ldots$

v) $x^2+y^2+z^2=R^2 \to$ Esfera

Lemniskata $(x^2+y^2)^2=R^2(x^2-y^2)$

$x \geq 0$



$$B(W)=4\iiint_0^{\sqrt{R^2-x^2-y^2}} 1\,dz\,dx\,dy =$$

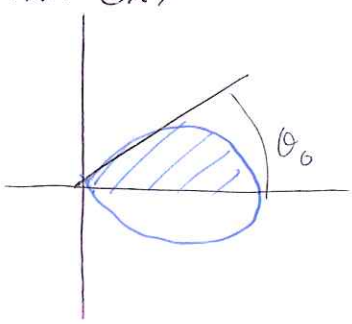
ALD-ALD [ZILINDRIKOA]

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z \end{cases} \qquad |J|=\rho$$

$\theta \in [0, \theta_0]$

$\rho \in [0, \rho_0]$

PROIEKTATU OXY



$\theta_0 \quad (x^2+y^2)^2=R^2(x^2-y^2)$

$\rho^4=R^2\rho^2\cos 2\theta$

$R^4-R^2\rho^2\cos 2\theta=0$

$\rho^2=0 \Rightarrow \rho=0$

$\rho_0^2=R^2\cos 2\theta \to \rho=R\sqrt{\cos 2\theta} \to \cos 2\theta \geq 0$

$2\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \qquad \theta \in [-\frac{\pi}{4}, \underset{\theta_0}{\frac{\pi}{4}}]$

$\theta \in [0, \pi/4] \qquad \rho \in [0, R\sqrt{\cos 2\theta}]$

$z \in [0, \text{esfera}] = [0, \sqrt{R^2-\rho^2}]$

$x^2+y^2+z^2=R^2$

$z^2=R^2-\rho^2 \to z=\sqrt{R^2-\rho^2}$

$$B=4\cdot\int_0^{\pi/4}\int_0^{R\sqrt{\cos 2\theta}}\int_0^{\sqrt{R^2-\rho^2}} 1\cdot\rho\cdot dz\,d\rho\,d\theta=$$

$$=4\int_0^{\pi/4}\int_0^{R\sqrt{\cos 2\theta}}[z]_0^{\sqrt{R^2-\rho^2}}\rho\,d\rho\,d\theta=4\int_0^{\pi/4}\int_0^{R\sqrt{\cos 2\theta}}\sqrt{R^2-\rho^2}\,\rho\,d\rho\,d\theta=$$

$$B=\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{\rho^2} 1\cdot\rho\,dz\,d\rho\,d\theta=\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\rho\,[z]_0^{\rho^2}\,d\rho\,d\theta=$$

$$=\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\rho^3\,d\rho\,d\theta=\int_{-\pi/2}^{\pi/2}\frac{1}{4}\left[\rho^4\right]_{\cos\theta}^{2\cos\theta}d\theta=$$

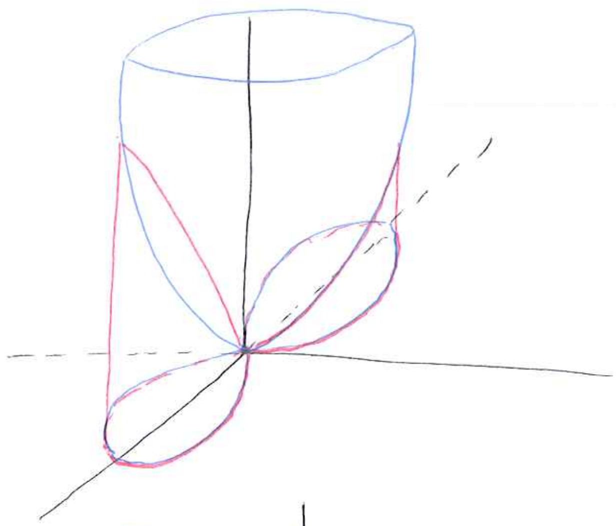
$$=\int_{-\pi/2}^{\pi/2}4\cos^4\theta-\frac{1}{4}\cos^4\theta\,d\theta=\frac{15}{4}\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2d\theta=$$

$$=\frac{15}{4}\cdot\frac{1}{4}\int_{-\pi/2}^{\pi/2}1+2\cos 2\theta+\cos^2 2\theta\,d\theta=\dots=\frac{45\pi}{32}$$

viii) $x^2+y^2-az=0\rightarrow$ PARABOLOIDEA

$(x^2+y^2)=a^2(x^2-y^2)$

$z=0$



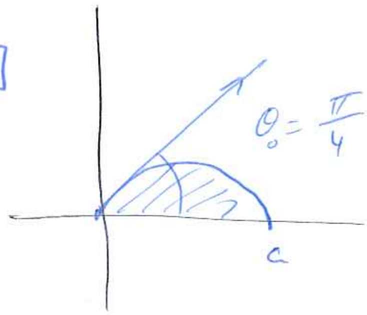
PROIEKTATU OXY

ALDAGAI - ALDAKETA

- ZILINDRIKOA -

$$\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\\ z=z\end{cases}\qquad |J|=\rho$$

$\theta\in[0,\pi/4]$



Aurrekoan egind

$\rho\in[0,\text{lemniskata}]=[0,a\sqrt{\cos 2\theta}]$

$z\in[0,\text{paraboloidea}]=[0,\rho^2/a]$

$x^2+y^2-az=0\Rightarrow z=\frac{\rho^2}{a}$

$$B=4\cdot\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}}\int_0^{\rho^2/a}1\,\rho\,dz\,d\rho\,d\theta=[\dots]=\frac{a^3\pi}{8}$$

$[\dots]\Rightarrow \cos^2 2\theta=\frac{1+\cos 4\theta}{2}$

$$= 4\int_0^{\pi/4} \left.\frac{(R^2-\rho^2)^{3/2}}{-2\,\frac{3}{2}}\right]_0^{R\sqrt{\cos 2\theta}} d\theta = \frac{-4}{3}R^3\int_0^{\pi/4}(\underbrace{1}_{\cos^2+\sin^2\theta}-\cos 2\theta)^{3/2}-1\,d\theta =$$

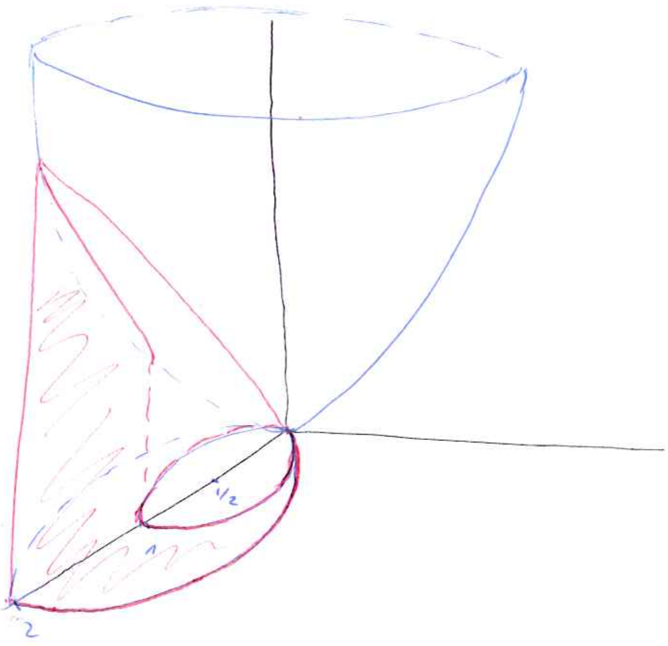
$$= -\frac{4}{3}R^3\int_0^{\pi/4}(2\sin^2\theta)^{3/2}\,d\theta = -\frac{4}{3}R^3\,2^{3/2}\int_0^{\pi/4}\underbrace{\sin^3\theta}_{(1-\cos^2\theta)\sin\theta}-1\,d\theta =$$

$$= [\dots] = R^3\left[\frac{20-16\sqrt{2}+3\pi}{9}\right]$$

vii) $z = x^2+y^2 \rightarrow$ PARABOLOIDEA

Zilindroak $\begin{cases} x = x^2+y^2 \rightarrow (x-\frac{1}{2})^2+y^2=\frac{1}{4} \\ 2x = x^2+y^2 \rightarrow (x-1)^2+y^2=1 \end{cases}$

$z \geq 0$

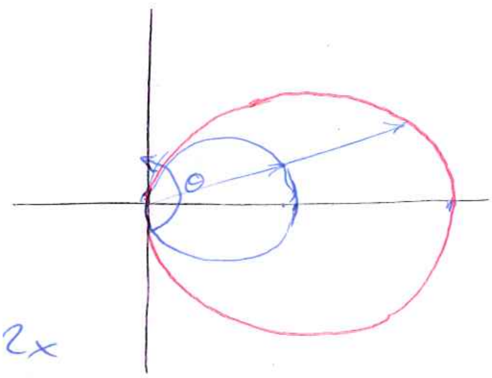


ALDAGAI - ALDAKETA

- ZILINDRIKOAK -

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \qquad |J| = \rho$$

PROIEKTATU OXY



$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\rho \in [\text{txikia}, \text{handia}] = [\cos\theta, 2\cos\theta]$

Tx: $x^2+y^2 = x$

$\rho^2 = \rho\cos\theta$

$\rho(\rho-\cos\theta) = 0$

$\rho = 0$

$\rho = \cos\theta$

HN: $x^2+y^2 = 2x$

$\rho^2 = 2\rho\cos\theta$

$\rho = 2\cos\theta$

$z \in [0, \text{paraboloide}] = [0, \rho^2]$

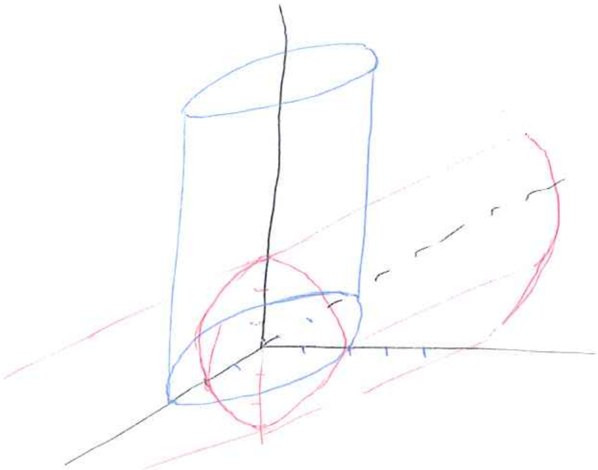
$z = x^2+y^2$

$z = \rho^2$

xiii) $\frac{x^2}{9}+\frac{y^2}{4}\leq 1$

$\frac{y^2}{4}+\frac{z^2}{9}\leq 1$

ZILINDRO ELIPTIKOAK



$x\geq 0,\ y\geq 0,\ z\geq 0$

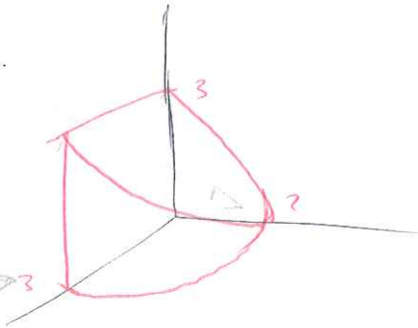
ALDAGAI - ALDAKETA

- ZILINDRIKOAK -

$x = 3\rho\cos\theta$

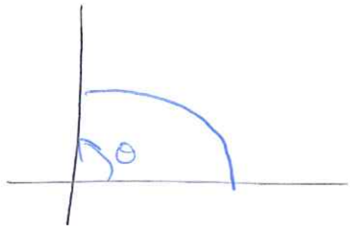
$y = 2\rho\sin\theta$ $\quad |J| = 3\cdot 2\cdot\rho = 6\rho$

$z = z$



$\theta\in[0,\pi/2]$

$\rho\in[0,1]$



$\frac{x^2}{9}+\frac{y^2}{4}=1 \overset{ALD}{\Rightarrow} \rho = 1$

$z\in[0, \text{zilindroa}] = [0, 3\sqrt{1-\rho^2\sin^2\theta}\,]$

$\frac{y^2}{4}+\frac{z^2}{9}=1 \overset{ALD}{\Rightarrow} z = 3\sqrt{1-\rho^2\sin^2\theta}$

$$B = \int_0^{\pi/2}\int_0^1\int_0^{3\sqrt{1-\rho^2\sin^2\theta}} 1\cdot\underbrace{6\rho}_{J}\, dz\,d\rho\, d\theta = [\dots] = \cancel{6}$$

$$= 6\int_0^{\pi/2}\int_0^1 3\rho\sqrt{1-\rho^2\sin^2\theta}\, d\rho\, d\theta =$$

$$= 18\int_0^{\pi/2} -\frac{1}{2\sin^2\theta}\,\frac{\left[(1-\rho^2\sin^2\theta)^{3/2}\right]_0^1}{3/2} = [\dots] =$$

$$= 6\int_0^{\pi/2}\frac{1-\cos^3\theta}{\sin^2\theta}\,d\theta = 6\int_0^{\pi/2}\frac{\cancel{(1-\cos\theta)}(1+\cos\theta)}{\cancel{\sin^2\theta}(1+\cos\theta)}\,d\theta =$$

$(1-\sin^2\theta)\cos\theta$

$t=\tan(\frac{\theta}{2})$

$= [\dots] = 12$

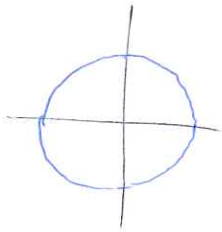
4. ARIKETA

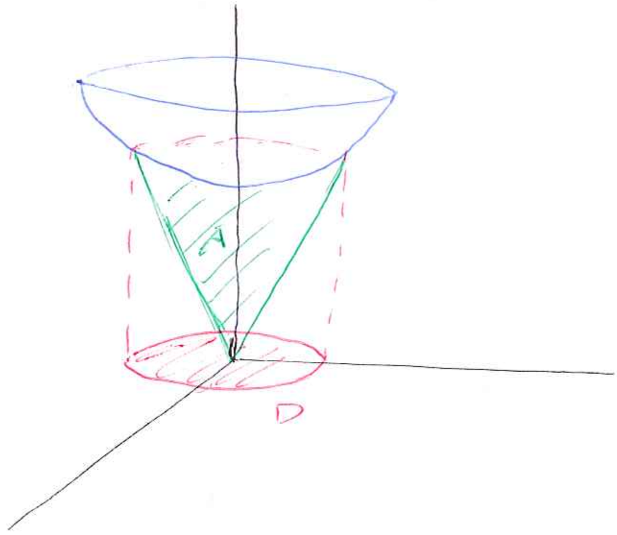
iv) $$\iiint_A z \cdot e^{-(x^2+y^2)}\, dx\, dy\, dz$$

$A = \{(x,y,z) \in \mathbb{R}^3 :$ ~~$0 \le z \le x^2+y^2,\ 0 \le y \le \sqrt{2x-x^2}$~~

EBAKIDURA

$$\begin{cases} 2(x^2+y^2) = z^2 \\ x^2+y^2+1 = z^2 \end{cases}$$





ALD - ALD

- ZILINDRIKOAK -

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \qquad \theta \in [0, 2\pi] \qquad \rho \in [0,1]$$

$z \in [\text{konoa},\ \text{bi orrieko hiperboloid}] = [\sqrt{2}\rho, \sqrt{\rho^2+1}]$

$2(x^2+y^2) = z^2$

$2\rho^2 = z^2 \Rightarrow z = \sqrt{2}\rho$

$x^2+y^2+1 = z^2$

$\rho^2+1 = z^2$

$z = \sqrt{\rho^2+1}$

$$\iiint_A z e^{-(x^2+y^2)}\, dx\, dy\, dz = \int_0^{2\pi}\int_0^1\int_{\sqrt{2}\rho}^{\sqrt{\rho^2+1}} z e^{-\rho^2} \underset{\overset{\shortparallel}{J}}{\rho}\, dz\, d\rho\, d\theta =$$

$$= [\ldots] = \frac{\pi}{2e}$$

$$\int (1-\rho^2)\rho\, e^{-\rho} d\rho = \ldots = \rho^2 \frac{e^{-\rho^2}}{2}$$

$dv = \rho e^{-\rho^2} d\rho$

$u = 1-\rho^2$

v) $\iiint_W z^2 dV$ non $W =$

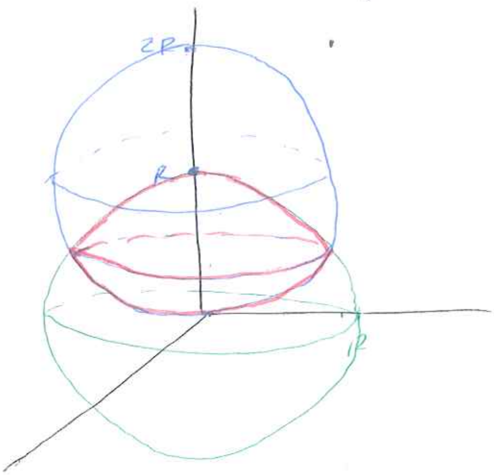
$W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2, \; x^2+y^2+(z-R)^2 \leq R^2$

$x^2+y^2+z^2=R^2$ $\qquad$ $x^2+y^2+(z-R)^2=R^2$

ZENTRUA $(0,0,0)$ $\quad r=R$ $\qquad$ ZENTRUA $(0,0,R)$ $\quad r=R$

ALDAGAI-ALDAKETA

— ESFERIKOAK —

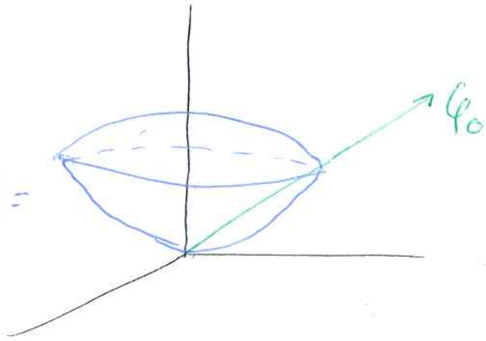


$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho \end{cases}$$

$|J| = \rho^2\sin\varphi$

$\theta \in [0, 2\pi]$

$\varphi \in [0, \varphi_0] \cup [\underset{\frac{\pi}{3}}{\varphi_0}, \pi/2]$



$\rho \in [0, \text{esf}_{(x^2+y^2+z^2=R^2)}] \cup [0, \text{esf}_{\text{beste}}] =$

$= [0, R] \cup [0, 2R\cos\varphi]$

EBAKIDURA

$$\begin{cases} x^2+y^2+z^2=R^2 \\ x^2+y^2+(z-R)^2=R^2 \end{cases} \Rightarrow z = \frac{1}{2}R$$

$\underset{R}{\rho}\cos\varphi = \frac{1}{2}R \Rightarrow \cos\varphi = \frac{1}{2} \Rightarrow \varphi_0 = \frac{\pi}{3}$

$$\iiint_W z^2 dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^R (\rho\cos\varphi)^2\rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta +$$

$$+ \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2R\cos\varphi} (\rho\cos\varphi)^2\rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta =$$

$$= [\ldots] = \frac{59R^5\pi}{480}$$

vii) $\iiint x^2 dx\, dy\, dz$

$A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+(z-1)^2 \leq 1,\ 3z^2 \geq x^2+y^2\}$

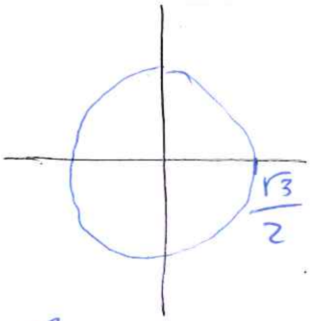
ESFERA $x^2+y^2+(z-1)^2 \leq 1$

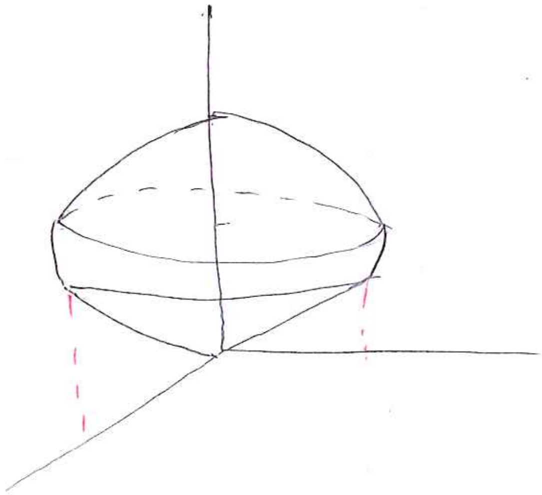
· ZENTRUA = $(0,0,1)$ $r=1$

$3z^2 = x^2+y^2$
KONOA

EBAKIDURA

$$\begin{cases} x^2+y^2+(z-1)^2=1 \\ 3z^2 = x^2+y^2 \end{cases} \quad \begin{matrix} \nearrow z=0 \\ \searrow z=\frac{1}{2} \\ \downarrow \\ x^2+y^2=\frac{3}{4} \end{matrix}$$





ALD - ALD — ESFERIKOAK —

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \qquad |J| = \rho^2\sin\varphi$$

$\theta \in [0, 2\pi]$

$\varphi \in [0, \underset{\text{KONOA}}{\varphi_0}] \Rightarrow [0, \pi/3]$

$3z^2 = x^2+y^2 \Rightarrow \sqrt{3} = \tan\varphi \Rightarrow \varphi = \frac{\pi}{3}$

$\rho \in [0, \text{esf}] = [0, 2\cos\varphi]$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{2\cos\varphi} (\rho\cos\theta\sin\varphi)^2 \cdot \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = [\dots] = \frac{81\pi}{320}$$

$$\int \sin^3\varphi\cos\varphi(1-\sin^2\varphi)^2 d\varphi \underset{t=\sin\varphi}{=} \int t^3(1-t^2)^2 dt$$

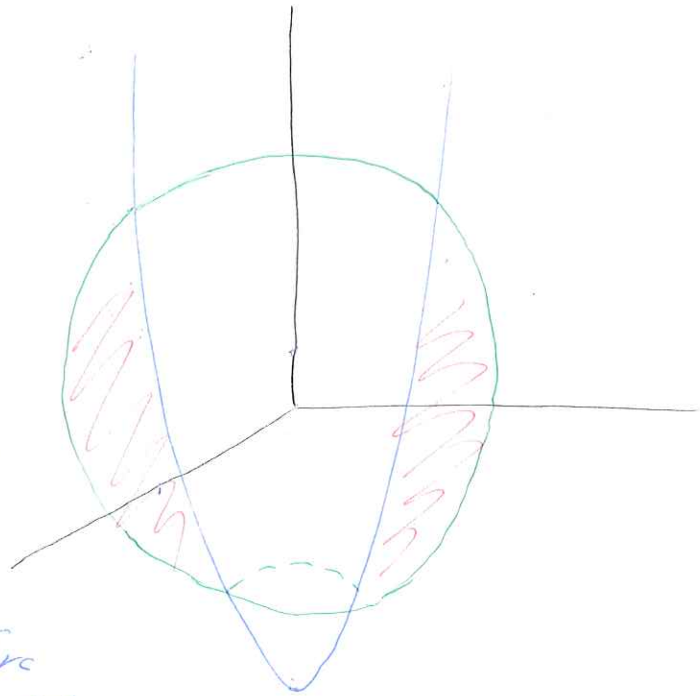
8

viii) $\iiint_W z\,dx\,dy\,dz$ non

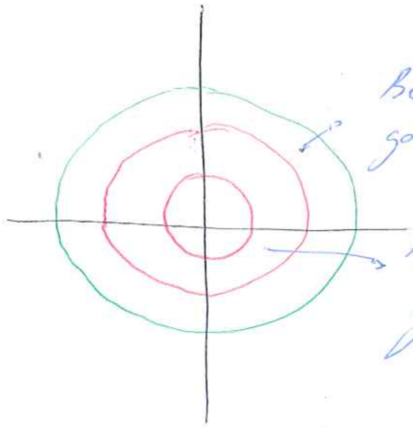
$W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 9, \ z \le x^2+y^2-7\}$

ALDAGAI - ALDAKETA

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \qquad J = \rho$$



PROIEKTATU OXY



$\theta \in [0, 2\pi]$

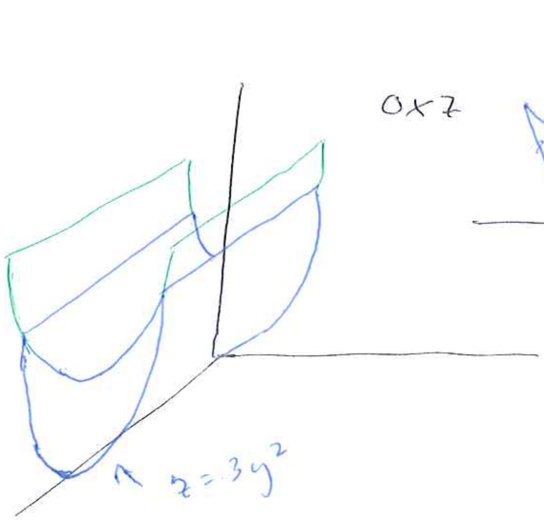
$\rho \in [\sqrt{5}, \sqrt{8}] \cup [\sqrt{8}, 3]$

$z \in [-\sqrt{9-\rho^2}, \rho^2-7] \cup [-\sqrt{9-\rho^2}, \sqrt{9-\rho^2}]$

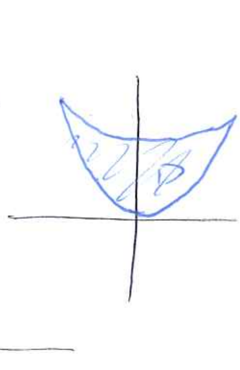
$$\int_0^{2\pi}\int_{\sqrt{5}}^{\sqrt{8}}\int_{-\sqrt{9-\rho^2}}^{\rho^2-7} z\cdot\rho\,dz\,d\rho\,d\theta + \int_0^{2\pi}\int_{\sqrt{8}}^{3}\int_{-\sqrt{9-\rho^2}}^{\sqrt{9-\rho^2}} z\,\rho\,dz\,d\rho\,d\theta = \dots = \pi\,\frac{-9}{4}$$

5. ARIKETA

iv) W $\begin{cases} z = y^2+2 \\ z = 3y^2 \\ x = 0 \\ x = 3 \end{cases}$ zilindro paraboliko



OXZ



$(\bar{x}, \bar{y}, \bar{z}) = (\frac{3}{2}, 0, \frac{7}{5})$

9

$$m(W) = \iiint_W k\,dx\,dy\,dz = \int_{-1}^{1}\int_{3y^2}^{y^2+2}\int_0^3 dx\,dz\,dy = [\dots] = 8k$$

$$\bar{x} = \frac{\iiint_W kx\,dx\,dy\,dz}{m(W)} = \frac{k\int_{-1}^{1}\int_{3y^2}^{y^2+2}\int_0^3 x\,dx\,dz\,dy}{8k} = \dots = \frac{3}{2}$$

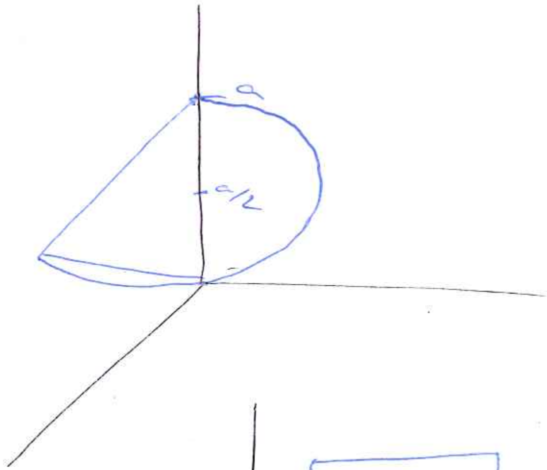
$$\bar{y} = \frac{\iiint_W ky\,dx\,dy\,dz}{m(W)} = \frac{k\int_{-1}^{1}\int_{3y^2}^{y^2+2}\int_0^3 y\,dx\,dz\,dy}{8k} = \dots = 0$$

viii)

$$W \equiv \begin{cases} x^2 = 2az \rightarrow \text{zilindro parabolikoa} \\ x = 0 \\ z^2+y^2 = az \rightarrow \text{zilindroa} \end{cases}$$

$y^2 + (z - \frac{a}{2})^2 = \frac{a^2}{4}$

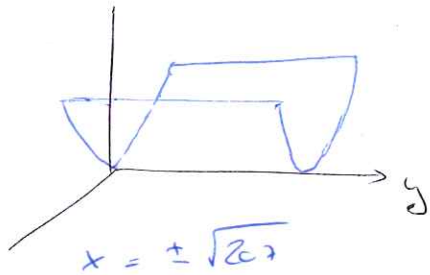
$[a>0]$



$\rho(x,y,z) = z$

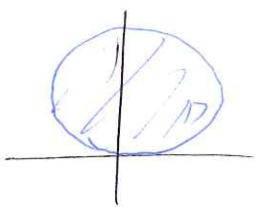
$$\begin{cases} x = x \\ y = \rho\cos\theta \\ z = \rho\sin\theta \end{cases}$$ ZILINDRIKOAK

PROIEKTATU OYZ



$x = \pm\sqrt{2az}$

$x = \sqrt{2az}$



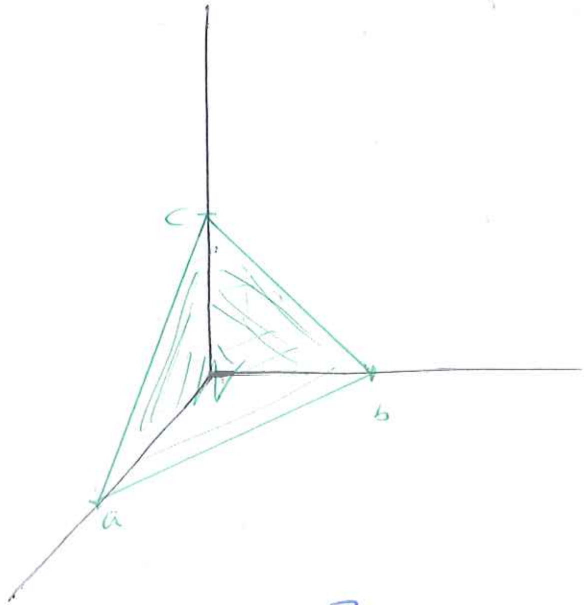
$\theta \in [0, \pi]$

$\rho \in [0, \text{zirkunf}]$
$\uparrow$
$\rho = a\sin\theta$

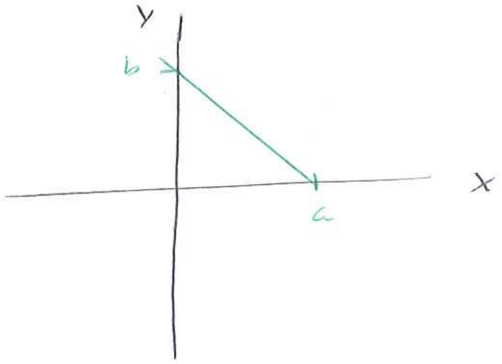
$$m(W) = \int_0^{\pi}\int_0^{a\sin\theta}\int_0^{\sqrt{2a\rho\sin\theta}} x\,\rho\,dx\,d\rho\,d\theta = \frac{a^4\pi}{8}$$

1. ARIKETA

ii) $\begin{cases} \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \\ x=0 \quad z=0 \\ y=0 \end{cases}$



OXY PLANOAN PROIEKTATU



$z \in [0, plano] =$
$= [0, c\cdot(1-\frac{y}{b}-\frac{x}{a})]$
$y \in [0, b\cdot(1-\frac{x}{a})]$
$x \in [0, a]$

$B(W) = \iiint_W dx\,dy\,dz$

$B(W) = \int_0^a \int_0^{b\cdot(1-\frac{x}{a})} \int_0^{c(1-\frac{y}{b}-\frac{x}{a})} 1\, dz\,dy\,dx =$

$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\cdot(1-\frac{y}{b}-\frac{x}{a})dy\,dx = c\cdot \int_0^a \left[y-\frac{y^2}{2b}-\frac{xy}{a}\right]_0^{b(1-\frac{x}{a})} dx =$

$= cb\cdot \int_0^a 1-\frac{x}{a}-\frac{b}{2b}+\frac{2b\cancel{x}}{\cancel{2}ba}-\frac{x^2b}{2ba^2}-\frac{\cancel{x}}{\cancel{a}}+\frac{x^2}{a^2}dx =$

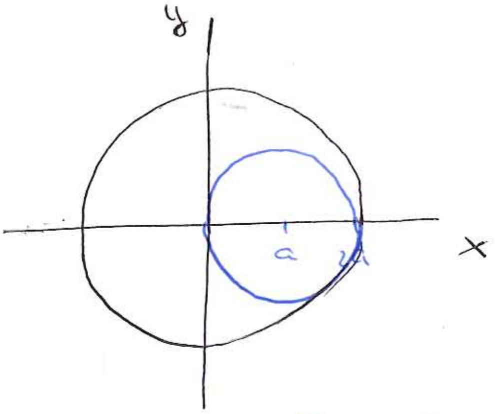
$= cb\cdot\left[x-\frac{x^2}{2a}-\frac{x}{2}-\frac{x^3}{6a^2}+\frac{x^3}{3a^2}\right]_0^a =$

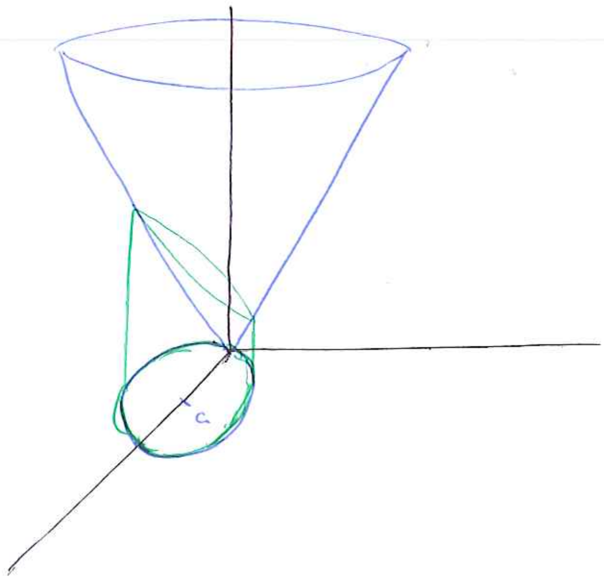
$= cb\cdot\left[\cancel{\frac{6a}{6}-\frac{3a}{6}}-\frac{3a}{6}-\frac{a}{6}+\frac{2a}{6}\right] =$

$= \boxed{\frac{abc}{6}}$

iv) $\begin{cases} x^2+y^2=2ax \Rightarrow (x-a)^2+y^2=a^2 \\ x^2+y^2=z^2 \to \text{KONOA} \\ z=0,\ z\geq 0 \end{cases}$

OXY PLANOAN PROIEKTATU





$z \in [0, \text{Konoa}] = [0, \rho]$

$x^2+y^2=z^2 \Rightarrow \rho^2=z^2\ [z\geq 0]$

$\rho \in [0, \text{zirkunf}] = [0, 2a\cos\theta]$

$\rho^2 = 2a\rho\cos\theta$

$\theta \in [-\pi/2, \pi/2]$

$\rho^2 = 2a\rho\cos\theta \hookrightarrow \cos\theta = 0\ [\rho=0]$

ALD-ALD: ZILINDRIKOAK

$x=\rho\cos\theta$

$y=\rho\sin\theta$

$z=z$

$|J|=\rho$

$$B(W) = \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta}\int_0^{\rho} \rho\, dz\, d\rho\, d\theta =$$

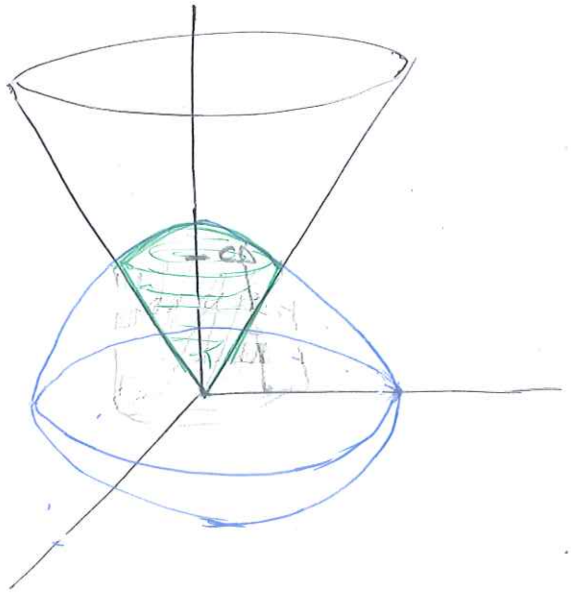
$$= \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} \rho^2\, d\rho\, d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{1}{3}\rho^3\right]_0^{2a\cos\theta} d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8a^3}{3}\cos^3\theta\, d\theta = \frac{8a^3}{3}\int_{-\pi/2}^{\pi/2}(1-\sin^2\theta)\cos\theta\, d\theta =$$

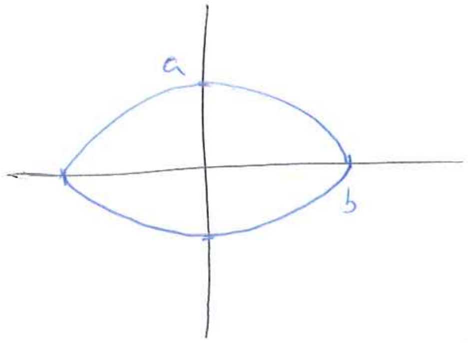
$$= \frac{8a^3}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = \frac{8a^3}{3}\cdot\left[\sin\frac{\pi}{2} - \frac{\sin^3\frac{\pi}{2}}{3} - \sin\frac{-\pi}{2} + \frac{\sin^3\frac{-\pi}{2}}{3}\right]$$

$$= \frac{8a^3}{3}\cdot\left(\frac{3}{3}-\frac{1}{3}+\frac{3}{3}-\frac{1}{3}\right) = \boxed{\frac{32a^3}{9}}$$

vi) $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 2 & \to \text{ELIPSOIDE} \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq \dfrac{z^2}{c^2} & \to \text{Kono} \\ z \geq 0 \end{cases}$



OXY PLANOAN PROIEKTATU



ALD-ALD: ZILINDRIKOAK

$x = a\rho\cos\theta$

$y = b\rho\sin\theta$

$z = z$

$J = ab\rho$

$z \in [0, \text{elipsoide}] = [0, c\sqrt{2-\rho^2}]$

$\rho^2 + \dfrac{z^2}{c^2} = 2 \Rightarrow z^2 = c^2(2-\rho^2)$

$\rho \in [0, \text{Kono}] = [0, 1]$

$\theta \in [0, 2\pi]$

$\rho^2 = \dfrac{z^2}{c^2} \Rightarrow z = c\rho$

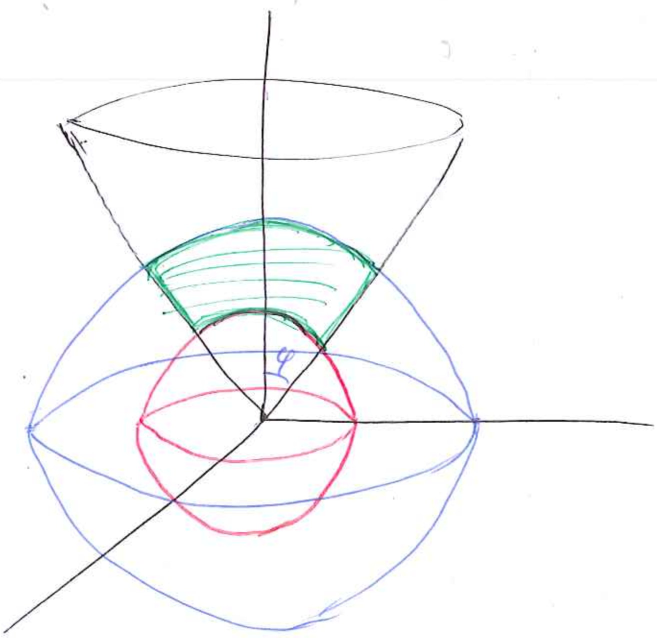
$$B(W) = \int_0^{2\pi}\int_0^1\int_{c\rho}^{c\sqrt{2-\rho^2}} ab\rho \, dz \, d\rho \, d\theta =$$

$$= \int_0^{2\pi}\int_0^1 abc\left[\rho\sqrt{2-\rho^2} - c\rho^2\right] d\rho d\theta = \int_0^{2\pi}\left[abc\left(\frac{-1}{3}(2-\rho^2)^{3/2} - \frac{1}{3}\rho^3\right)\right]_0^1$$

$$= \int_0^{2\pi} abc\frac{1}{3}\left[-(2-1)^{3/2} + 2^{3/2} - 1 = \int_0^{2\pi}\frac{abc}{3}\left[2\sqrt{2} - 2\right] d\theta =\right.$$

$$\boxed{= \frac{abc}{3}4\pi\left[\sqrt{2} - 1\right]}$$

ix) $\begin{cases} x^2+y^2+z^2=a^2 \\ x^2+y^2+z^2=b^2 \Rightarrow \text{eferak} \\ x^2+y^2=z^2 \to \text{konoa} \\ z\geq 0 \\ 0<a<b \end{cases}$



ALD-ALD: ESFERIKOAK

$x=\rho\cos\theta\sin\varphi$

$y=\rho\sin\theta\sin\varphi$

$z=\rho\cos\varphi$

$|J|=\rho^2\sin\theta$

$\rho\in[a,b]$

$\theta\in[0,2\pi]$

$\varphi\in[0,\text{kono}]=[0,\pi/4]$

$\rho^2\cos^2\theta\sin^2\varphi+\rho^2\sin^2\theta\sin^2\varphi=\rho^2\cos^2\varphi$

$\rho^2\sin^2\varphi=\rho^2\cos^2\varphi$

$\sin^2\varphi=\cos^2\varphi$

$\rho=\frac{\pi}{4}$

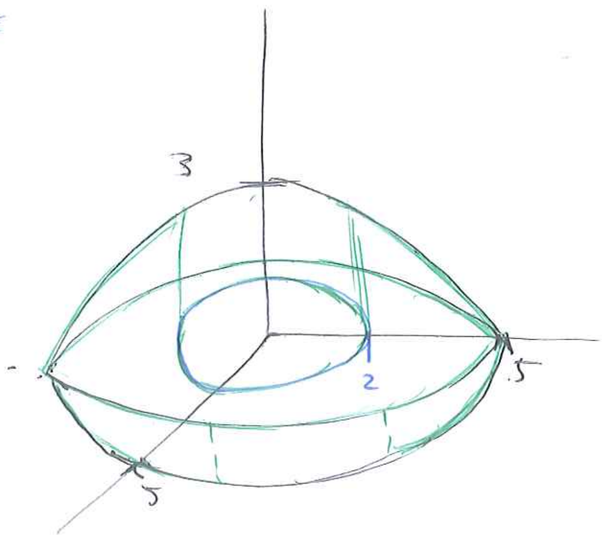
$$B(W)=\int_0^{\pi/4}\int_0^{2\pi}\int_a^b\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi=$$

$$=\int_0^{\pi/4}\int_0^{2\pi}\left[\frac{\rho^3}{3}\sin\varphi\right]_a^b d\theta\,d\varphi=\int_0^{\pi/4}\int_0^{2\pi}\sin\varphi\,\frac{b^3-a^3}{3}\,d\theta\,d\varphi=$$

$$=\int_0^{\pi/4}\sin\varphi\,\frac{b^3-a^3}{3}\,2\pi\,d\varphi=\frac{a^3-b^3}{3}2\pi\left[\cos\varphi\right]_0^{\pi/4}=$$

$$\boxed{=\frac{a^3-b^3}{3}2\pi\cdot\left[\frac{\sqrt{2}}{2}-1\right]}$$

xi) $\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{25}+\dfrac{z^2}{9}\le 1 \to \text{ELIPSOIDE} \\ x^2+y^2\ge 4 \to \text{ZILINDRO} \end{cases}$



ALD-ALD: ZILINDRIKOAK

$\rho \in [2, 5]$

$\theta \in [0, 2\pi]$

$z \in [\text{elipsoid}, \text{elipsoid}] =$

$= \left[-\sqrt{1-\rho^2}\cdot 3,\ 3\sqrt{1-\rho^2}\right]$

$x = \rho\cos\theta$
$y = \rho\sin\theta$
$z = z$
$|J| = \rho$

$$B(W) = \int_0^{2\pi}\int_2^5\int_{-\sqrt{1-\frac{\rho^2}{25}}\cdot 3}^{3\sqrt{1-\frac{\rho^2}{25}}} \rho\, dz\, d\rho\, d\theta =$$

$$= \int_0^{2\pi}\int_2^5 3\cdot 2\sqrt{1-\frac{\rho^2}{25}}\,\rho\, d\rho\, d\theta =$$

$$= \int_0^{2\pi}\left[25\cdot 2\cdot\left(1-\frac{\rho^2}{25}\right)^{3/2}\right]_2^5 d\theta =$$

$$= \int_0^{2\pi} -25\cdot 2\cdot\left[\cancel{\left(1-\frac{25}{25}\right)^{3/2}}^{\,0} - \left(1-\frac{4}{25}\right)^{3/2}\right] d\theta =$$

$$= \left[\cancel{25}\cdot 2\cdot\frac{21^{3/2}}{25^{\cancel{3/2}}}\,\theta\right]_0^{2\pi} = \frac{2\cdot 2\cdot 21\cdot\pi\cdot\sqrt{21}}{5} =$$

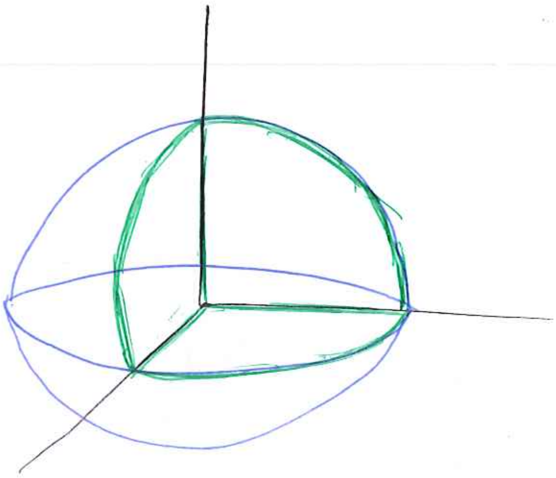
$$= \boxed{\frac{84\pi\sqrt{21}}{5}}$$

15

4. Ariketa

i) $\iiint_V xyz \, dx\,dy\,dz$

$$V \equiv \begin{cases} x=0 \\ y=0 \\ z=0 \\ x^2+y^2+z^2=1 \end{cases}$$



Ald.-ald: Esferikoak

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$|J| = \rho^2\sin\varphi$

$\rho \in [0,1]$

$\varphi \in [0, \pi/2]$

$\theta \in [0, \pi/2]$

$$\iiint_V xyz\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^5 \cos\theta\sin\theta\sin^3\varphi\cos\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{6}\cos\theta\sin\theta\sin^3\varphi\cos\varphi\, d\varphi\, d\theta =$$

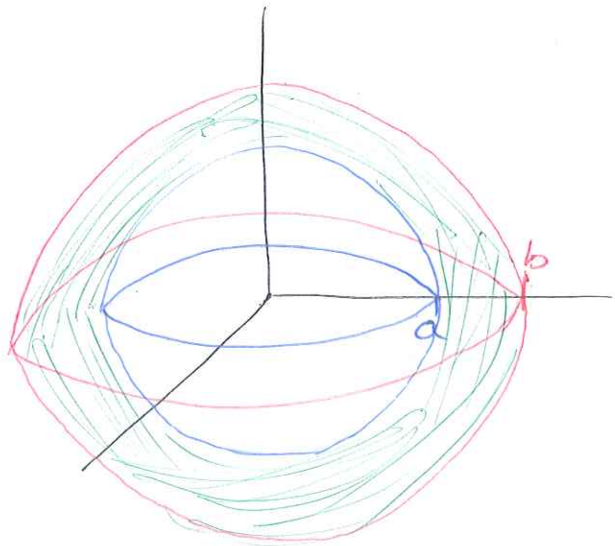
$$= \int_0^{\pi/2} \left[\frac{1}{6}\cos\theta\sin\theta\frac{1}{4}\sin^4\varphi\right]_0^{\pi/2} d\theta =$$

$$= \int_0^{\pi/2} \frac{1}{24}\cos\theta\sin\theta\, d\theta = \int_0^{\pi/2}\frac{1}{48}\sin 2\theta\, d\theta =$$

$$= \left[\frac{-1}{48\cdot 2}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{48\cdot 2}\left(1-(-1)\right) = \boxed{\frac{1}{48}}$$

iii) $\displaystyle\iiint_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$

$$S \equiv \begin{cases} x^2+y^2+z^2=a^2 \\ x^2+y^2+z^2=b^2 \\ 0<a<b \end{cases}$$



ALD-ALD: ESFERIKOAK

$x=\rho\cos\theta\sin\varphi$

$y=\rho\sin\theta\sin\varphi$

$z=\rho\cos\varphi$

$|J|=\rho^2\sin\varphi$

$\rho\in[a,b]$

$\theta\in[0,2\pi]$

$\varphi\in[0,\pi]$

$$x^2+y^2+z^2 \overset{\text{ALD-ALD}}{=} \rho^2\sin^2\varphi+\rho^2\cos^2\varphi=\rho^2$$

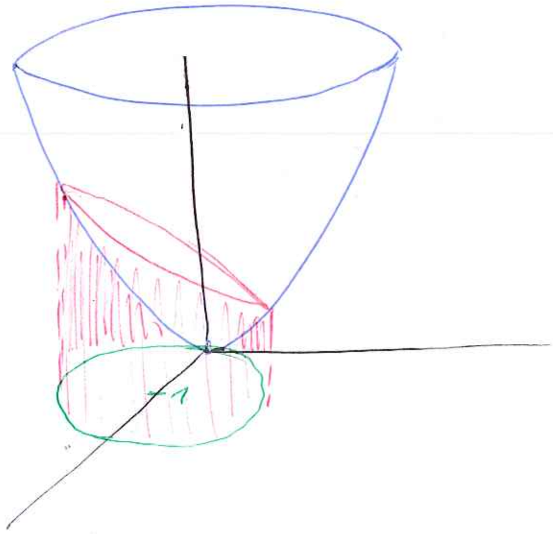
$$\iiint_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}} = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{\rho^2\sin\varphi}{\rho^3}\,d\rho\,d\varphi\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi}\Big[\ln\rho\ \sin\varphi\Big]_a^b d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\ln\frac{b}{a}\sin\varphi\,d\varphi\,d\theta =$$

$$= \int_0^{2\pi}\Big[\ln\frac{b}{a}(-\cos\varphi)\Big]_0^{\pi}d\theta = \int_0^{2\pi}\ln\frac{b}{a}(\cos 0-\cos\pi)\,d\theta =$$

$$= \int_0^{2\pi}2\ln\frac{b}{a}\,d\theta = \boxed{4\pi\ln\frac{b}{a}}$$

iii) $\iiint_A zy\sqrt{x^2+y^2}\,dx\,dy\,dz$



$$A \equiv \begin{cases} 0 \le z \\ z \le x^2+y^2 \rightarrow \text{Paraboloide} \\ 0 \le y \\ y \le \sqrt{2x-x^2} \end{cases}$$

$\downarrow$

$(x-1)^2+y^2=1$

ALD-ALD: ZILINDRIKOAK

$x=\rho\cos\theta$

$y=\rho\sin\theta$

$z=z$

$|J|=\rho$

$\rho\in[0,\rho_0]=[0,2\cos\theta]$

$\theta\in[0, \quad]$

$z\in[0,\text{paraboloid}]=$

$=[0,\rho^2]$

$\rho_0 \Rightarrow$

$(\rho\cos\theta-1)^2+\rho^2\sin^2\theta=1$

$\rho^2\cos^2\theta-2\rho\cos\theta+\cancel{1}+\rho^2\sin^2\theta=\cancel{1}\Rightarrow\rho=2\cos\theta$

$$\iiint_A zy\sqrt{x^2+y^2}\,dx\,dy\,dz=\int_0\int_0^{2\cos\theta}\int_0^{\rho^2} z\rho^3\sin\theta\,dz\,d\rho\,d\theta=$$

$$=\int_0\int_0^{2\cos\theta}\frac{\rho^7}{2}\sin\theta\,d\rho\,d\theta=\int_0\frac{1}{16}\sin\theta\left[\rho^8\right]_0^{2\cos\theta}d\theta=$$

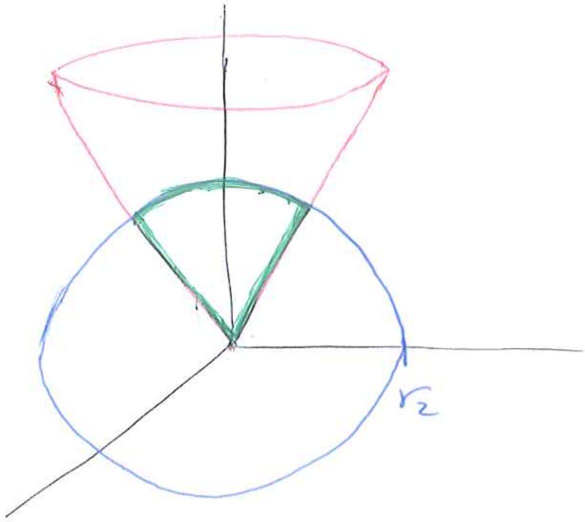
$$=\int_0 -\frac{2^8}{2^4}\sin\theta\cos^8\theta\,d\theta=\frac{-2^4}{9}\left[\cos 9\theta\right]_0^{\pi/2}=\boxed{\frac{16}{9}}$$

5. ARIKETA

i) $\begin{cases} x^2+y^2+z^2=2 \\ x^2+y^2=z^2 \end{cases}$

$z>0$

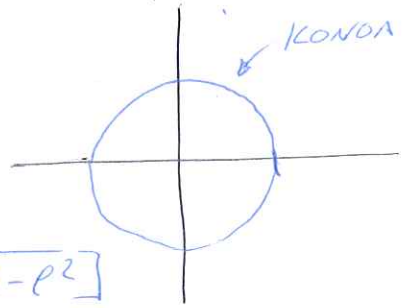
$\rho = d$



$$I_z = \iiint (x^2+y^2) d \, dx\,dy\,dz$$

ALD-ALD: ZILINDRIKOAK

OXY-n PROIEKTATU



$x = \rho\cos\theta$

$y = \rho\sin\theta \qquad |J| = \rho$

$z = z$

$\theta \in [0, 2\pi) \quad z \in [kon, esf] = [0, \sqrt{2-\rho^2}]$

$\rho \in [0, 1]$

$$I_z = \int_0^{2\pi}\int_0^1\int_\rho^{\sqrt{2-\rho^2}} \rho^3 d\,dz\,d\rho\,d\theta = \int_0^{2\pi}\int_0^1 \rho^3 d\sqrt{2-\rho^2} - \rho^4 d\,d\rho\,d\theta$$

$u = 2-\rho^2 \qquad du = -2\rho\,d\rho \Rightarrow d\rho = -\frac{1}{2\rho}du$

$\rho = \sqrt{2-u} \qquad u$

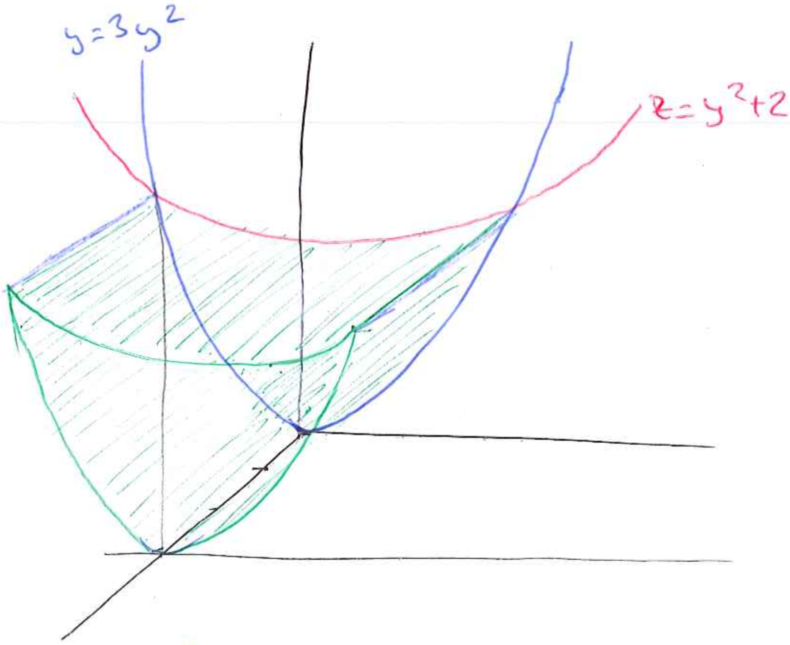
$$I = \int_0^{2\pi}\int (2-u)^{3/2} \cdot \frac{-\sqrt{u}}{2\sqrt{2-u}} d\,du\,d\theta = \int_0^{2\pi}\int \frac{d}{2}(u^{3/2}-2\sqrt{u})\,du\,d\theta =$$

$$= \int_0^{2\pi} \frac{d}{2}\left[\frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3}\right] d\theta = \int_0^{2\pi} d \cdot \left[\frac{(2-\rho^2)^{5/2}}{5} - \frac{2(2-\rho^2)^{3/2}}{3} - \frac{1}{5}\rho^5\right]_0^1 d\theta$$

$$= \int_0^{2\pi} d \cdot \left[\cancel{\frac{1}{5}} - \frac{2}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3} - \cancel{\frac{1}{5}}\right]$$

$$= 2\pi d \cdot \left(\frac{8\sqrt{2}}{15} - \frac{2}{3}\right)$$

ii) $\begin{cases} z = y^2 + 2 \\ z = 3y^2 \\ x = 0,\ x = 3 \end{cases}$ $\rho = d$

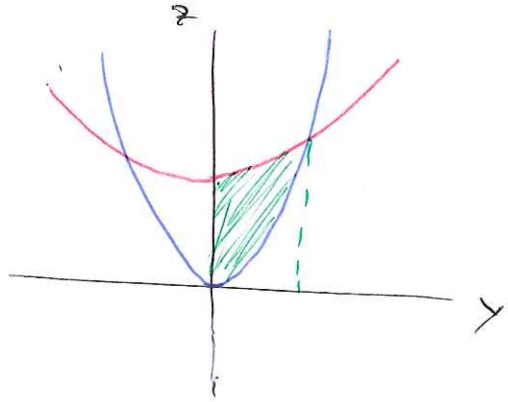


$$m(W) = \iiint_W \rho(x,y,z)\,dx\,dy\,dz$$

$x \in [0, 3]$

$$\begin{cases} z = y^2 + 2 \\ z = 3y^2 \end{cases} \Rightarrow y^2 + 2 = 3y^2$$

$2y^2 = 2 \to y = \pm 1$ OZY PLANOAN PROIEKTATU



$$2 \times \begin{cases} y \in [0, 1] \\ z \in [3y^2, y^2 + 2] \end{cases}$$

$$m(W) = \int_0^3 \int_0^1 \int_{3y^2}^{y^2+2} 2 \cdot d\,dz\,dy\,dx =$$

$$= \int_0^3 \int_0^1 2 \cdot d\,[y^2 + 2 - 3y^2]\,dy\,dx = 2d \int_0^3 \int_0^1 2 - 2y^2\,dy\,dx$$

$$= 4d \int_0^3 \left[y - \frac{1}{3}y^3\right]_0^1 dx = 4d \int_0^3 1 - \frac{1}{3}\,dx =$$

$$= 4d\,\frac{2}{3}\,x \Big|_0^3 = 8d$$

$$\bar{x} = \frac{\iiint_W x\,\rho(x,y,z)\,dx \cdot dy\,dz}{m(W)} =$$

$$= \frac{1}{8d} \cdot \int_0^3 \int_0^1 \int_{3y^2}^{y^2+2} 2 \cdot d\,x\,dz\,dy\,dx = \frac{1}{4} \int_0^3 \int_0^1 x \cdot (y^2 + 2 - 3y^2)\,dy\,dx =$$

$$= \frac{1}{4} \int_0^3 x \cdot 2 \cdot \left[y - \frac{1}{3}y^3\right]_0^1 dx = \frac{1}{2} \cdot \int_0^3 \frac{2x}{3}\,dx = \frac{1}{2}\,\frac{x^2}{3}\Big|_0^3 = \frac{3}{2}$$ ✓

$$\bar{y} = \frac{\iiint\limits_W y\rho(x,y,z)\,dx\,dy\,dz}{m(W)} =$$

$$= \frac{1}{8d} \cdot \int_0^3 \int_{-1}^1 \int_{3y^2}^{y^2+2} d \cdot y \, dz\,dy\,dx =$$

?

$$= \frac{1}{4} \int_0^3 \int_{-1}^1 y \cdot (y^2+2-3y^2)\,dy\,dx = \frac{1}{4} \int_0^3 \int_{-1}^1 2y - 2y^3\,dy\,dx =$$

$$= \frac{1}{2} \int_0^3 \left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_{-1}^1 dx = \frac{1}{2} \int_0^3 \cancel{\frac{1}{2} - \frac{1}{4} - \frac{1}{2} + \frac{1}{4}}\,dx = 0 \checkmark$$

$$\bar{z} = \frac{\iiint\limits_W z\rho(x,y,z)\,dx\,dy\,dz}{m(W)} =$$

$$= \frac{1}{8d} \int_0^3 \int_0^1 \int_{3y^2}^{y^2+2} 2 \cdot d \cdot z\,dz\,dy\,dx =$$

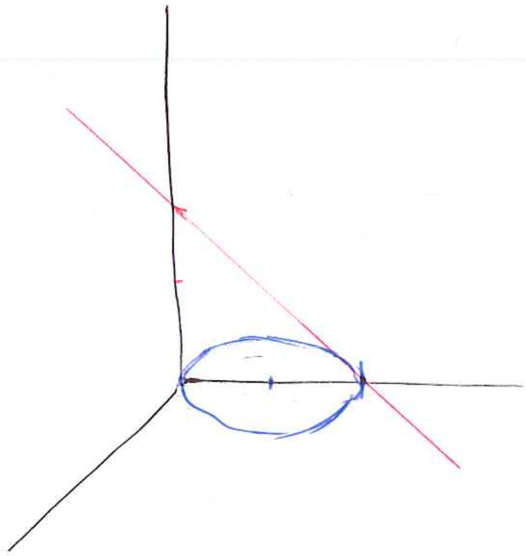
$$= \frac{1}{4} \int_0^3 \int_0^1 \frac{1}{2} z^2 \Big|_{3y^2}^{y^2+2} dy\,dx = \frac{1}{8} \int_0^3 \int_0^1 (y^2+2)^2 - 9 \cdot y^4\,dy\,dx =$$

$$= \frac{1}{8} \int_0^3 \int_0^1 y^4 + 4y^2 + 4 - 9y^4\,dy\,dx =$$

$$= \frac{1}{8} \int_0^3 \int_0^1 -8y^4 + 4y^2 + 4\,dy\,dx = \frac{1}{8} \int_0^3 \left[-\frac{8}{5}y^5 + \frac{4}{3}y^3 + 4y\right]_0^1 dx =$$

$$= \frac{1}{8} \int_0^3 \frac{56}{15}\,dx = \frac{7 \cdot \cancel{8} \cdot \cancel{3}}{\cancel{8} \cdot 5 \cdot \cancel{3}} = \frac{7}{5} \checkmark$$

v) $\begin{cases} x^2+y^2=2y \rightarrow x^2+(y-1)^2=1 \\ y+z=2 \rightarrow z=2-y \\ z=0 \end{cases}$



$\rho(x,y,z)=z$

ALD-ALD: ZILINDRIKOAK

$x=\rho\cos\theta$

$y=1+\rho\sin\theta \qquad |J|=\rho$

$z=z$

$\rho\in[0,1]$

$\theta\in[0,2\pi]$

$z\in[0,\text{zuzena}]=[0,2-y]=[0,1-\rho\sin\theta]$

$$m(W)=\iiint_W \rho(x,y,z)\,dx\,dy\,dz=$$

$$=\int_0^{2\pi}\int_0^1\int_0^{1-\rho\sin\theta} z\rho\,dz\,d\rho\,d\theta=\int_0^{2\pi}\int_0^1 \frac{1}{2}\rho\cdot[1-\rho\sin\theta]^2\,d\rho\,d\theta=$$

$$=\int_0^{2\pi}\int_0^1 \frac{1}{2}\rho\cdot(1-2\rho\sin\theta+\rho^2\sin^2\theta)\,d\rho\,d\theta=$$

$$=\int_0^{2\pi}\frac{1}{2}\left[\frac{1}{2}\rho^2-\frac{2}{3}\rho^3\sin\theta+\frac{1}{4}\rho^4\sin^2\theta\right]_0^1 d\theta=$$

$$=\int_0^{2\pi}\frac{1}{2}\cdot\left(\frac{1}{2}-\frac{2}{3}\sin\theta+\frac{1}{4}\sin^2\theta\right)d\theta=$$

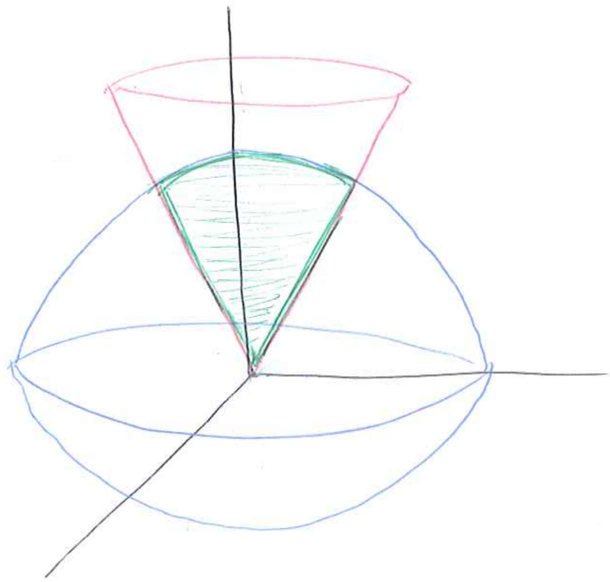
$$=\int_0^{2\pi}\frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}\sin\theta+\frac{1}{4}\cdot\frac{1-\cos 2\theta}{2}\right)d\theta=$$

$$=\frac{1}{2}\cdot\left(\frac{\theta}{2}+\cancel{\frac{2}{3}\cos\theta}^{\,0}+\frac{\theta}{8}-\cancel{\frac{\sin 2\theta}{8\cdot 2}}^{\,0}\right)\Bigg|_0^{2\pi}=$$

$$=\frac{\pi}{2}\cdot\left(1+\frac{1}{4}\right)=\boxed{\frac{5\pi}{8}}$$

1. Ariketa

i) $\begin{cases} x^2+y^2=z \\ x^2+y^2+z^2=2 \\ z\geq 0 \end{cases}$



$\begin{cases} x^2+y^2=z \\ x^2+y^2+z^2=2 \end{cases}$

$z^2+z-2=0 \quad z=1, \quad \cancel{z=-2} \; z\geq 0$

$x^2+y^2=1$

ALD-ALD: ZILINDRIKOAK

$x=\rho\cos\theta$

$y=\rho\sin\theta \qquad |J|=\rho$

$z=z$

$x^2+y^2=1 \Rightarrow \rho^2=1 \Rightarrow \rho=1$

$\rho\in[0,1] \qquad z\in[\rho^2,\sqrt{2-\rho^2}]$

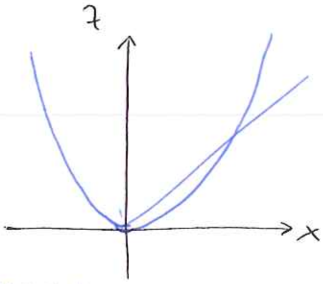
$\theta\in[0,2\pi]$

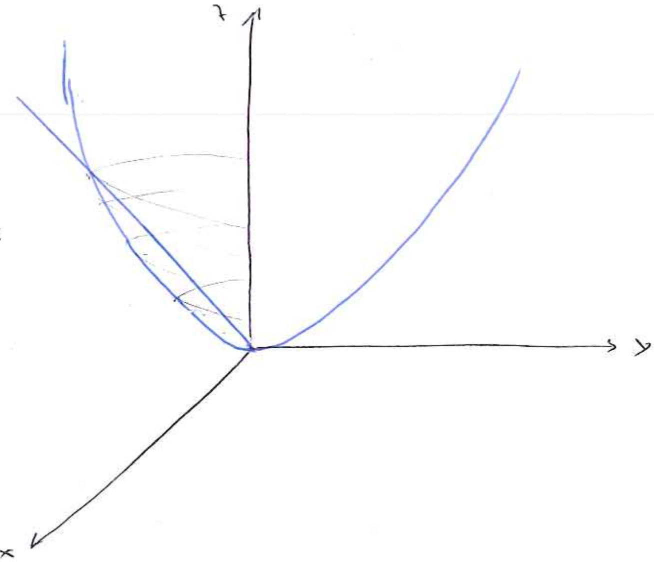
$$B(W)=\int_0^{2\pi}\int_0^1\int_{\rho^2}^{\sqrt{2-\rho^2}}\rho\, dz\, d\rho\, d\theta=\int_0^{2\pi}\int_0^1\rho\left(\sqrt{2-\rho^2}-\rho^2\right)d\rho\, d\theta=$$

$$=\int_0^{2\pi}\left[\frac{-1}{3}(2-\rho^2)^{3/2}-\frac{1}{4}\rho^4\right]_0^1 d\theta=2\pi\cdot\left(\frac{-1}{3}-\frac{1}{4}+\frac{2\sqrt{2}}{3}\right)=$$

$$=2\pi\cdot\left(\frac{2\sqrt{2}-1}{3}-\frac{1}{4}\right)$$

x) $\begin{cases} z = x^2 + y^2 \\ z = x \end{cases}$





ALD-ALD: ZILINDRIKOAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

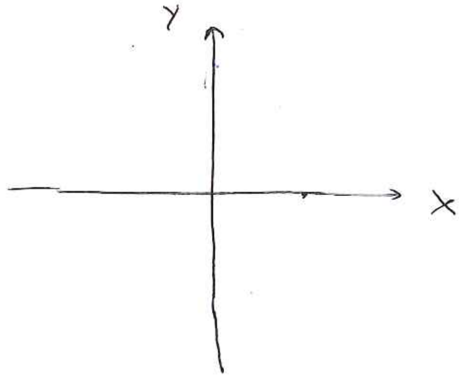
$|J| = \rho$

$\theta \in [0, \pi/2]$

$\rho \in [0, \rho_0]$

$z \in$ [parabo, plano]

OXY-n PROIEKTATU



$\rho_0 \Rightarrow \begin{cases} z = \rho^2 \\ z = \rho\cos\theta \end{cases} \Rightarrow \rho = \cos\theta$

$z \in [\rho^2, \rho\cos\theta]$

$$B(W) = 2\int_0^{\pi/2}\int_0^{\cos\theta}\int_{\rho^2}^{\rho\cos\theta} \rho \, dz\, d\rho\, d\theta =$$

$$= 2\int_0^{\pi/2}\int_0^{\cos\theta} \rho\left[\rho\cos\theta - \rho^2\right] d\rho\, d\theta = 2\int_0^{\pi/2}\left[\frac{1}{3}\rho^3\cos\theta - \frac{1}{4}\rho^4\right]_0^{\cos\theta} d\theta =$$

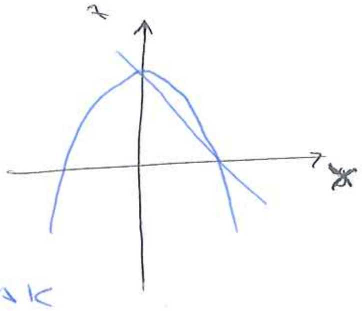
$$= 2\int_0^{\pi/2} \frac{1}{3}\cos^4\theta - \frac{1}{4}\cos^4\theta\, d\theta = 2\int_0^{\pi/2}\frac{1}{12}\cos^4\theta\, d\theta =$$

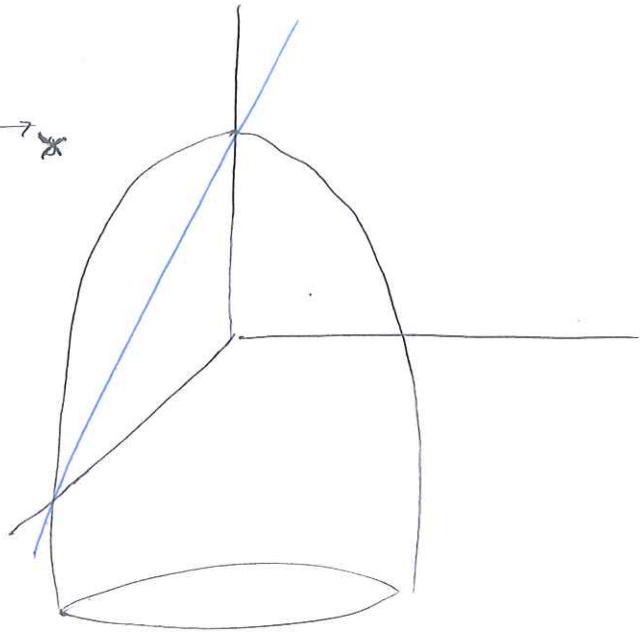
$$= \frac{1}{6}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{1}{3\cdot 2^3}\int_0^{\pi/2} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\, d\theta =$$

$$= \frac{1}{3\cdot 2^3}\left[\theta + \cancel{\sin 2\theta}^{\,0} + \frac{\theta}{2} + \frac{1}{8}\cancel{\sin 4\theta}^{\,0}\right]_0^{\pi/2} = \frac{1}{3\cdot 2^3}\cdot\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = \boxed{\frac{\pi}{32}}$$

xiv)

$$\begin{cases} z = 1 - x^2 - y^2 \\ x + z = 1 \end{cases}$$





ALD-ALD: ZILINDRIKOAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

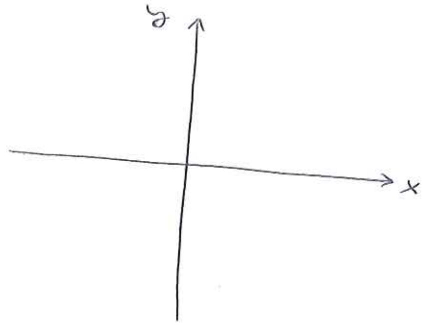
$|J| = \rho$

$\rho \in [0, \rho_0]$

$\theta \in [0, \pi/2]$

$z \in [\text{plano}, \text{parabo}]$

OXY-n PROIEKTATU



$\rho_0 \Rightarrow \quad z = 1 - \rho^2$

$\quad z = 1 - \rho\cos\theta \Rightarrow \rho_0 = \cos\theta$

$z \in [1 - \rho\cos\theta, 1 - \rho^2]$

$$B(W) = 2\int_0^{\pi/2}\int_0^{\cos\theta}\int_{1-\rho\cos\theta}^{1-\rho^2} \rho \, dz \, d\rho \, d\theta =$$

$$= 2\int_0^{\pi/2}\int_0^{\cos\theta} \rho\left[-\rho^2 + \rho\cos\theta\right] d\rho \, d\theta = 2\int_0^{\pi/2}\left[-\frac{1}{4}\rho^4 + \frac{1}{3}\rho^3\cos\theta\right]_0^{\cos\theta} d\theta =$$

$$= 2\int_0^{\pi/2} \frac{1}{3}\cos^4\theta - \frac{1}{4}\cos^4\theta \, d\theta = 2\int_0^{\pi/2} \frac{1}{12}\cos^4\theta \, d\theta =$$

$$= \frac{1}{6}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{3\cdot 2^3}\int_0^{\pi/2} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \, d\theta =$$

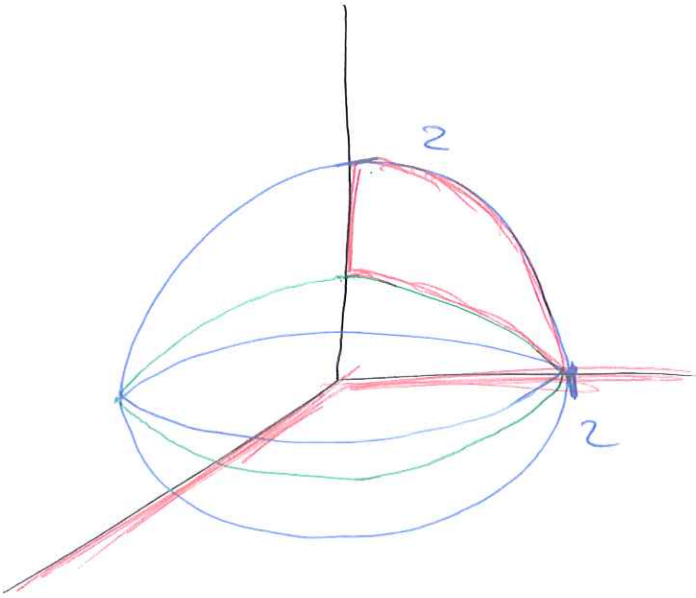
$$= \frac{1}{3\cdot 2^2}\left[\theta + \sin 2\theta + \frac{\theta}{2} + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{3\cdot 2^3}\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = \boxed{\frac{\pi}{32}}$$

vi) $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ -\sqrt{4-x^2-y^2} \leq z \\ z \leq \frac{1}{2}\sqrt{4-x^2-y^2} \end{cases}$$

$z^2 = 4-x^2-y^2$

$4z^2 = 4-x^2-y^2$



OXY PROIEKTATU

1- vi)

Zmen madina

$z \in [\text{konosa}, \text{elipsoidosa}] = [\rho c, c\sqrt{2-\rho^2}]$

$\rho^2 = \frac{z^2}{c^2}$ $\qquad$ $\rho^2 + \frac{z^2}{c^2} = 2$

$z = \rho c$ $\qquad$ $z = c\sqrt{2-\rho^2}$

Nik

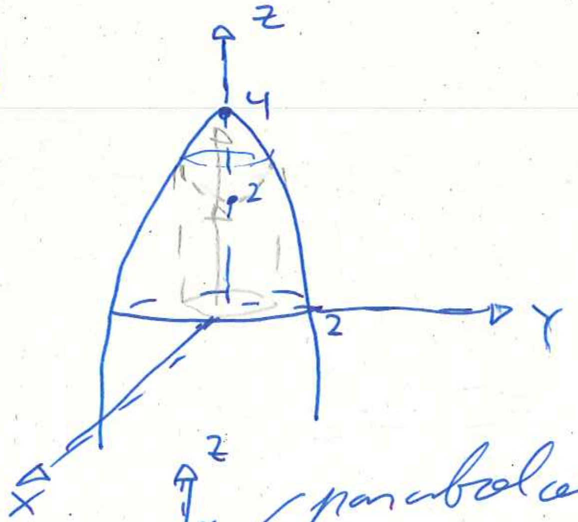
$$\begin{cases} x = a\rho\cos\varphi \\ y = a\rho\sin\varphi \\ z = cz \end{cases}$$

$\varphi \in [0, 2\pi]$

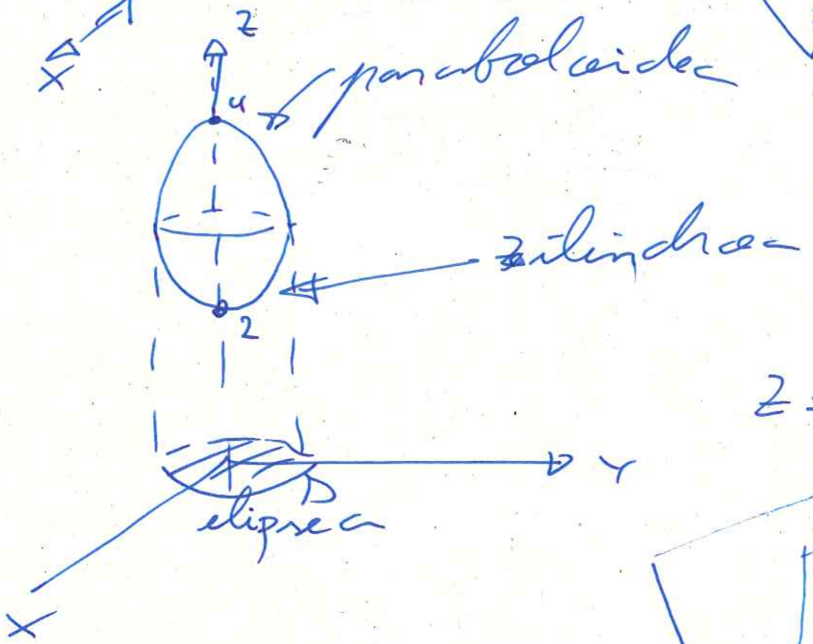
$J = abc\rho$ $\qquad$ $\rho \in [0, 1]$

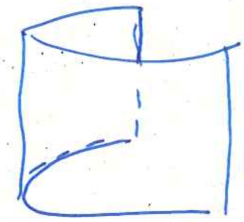
$z \in [\rho, \sqrt{2-\rho^2}]$

xii)

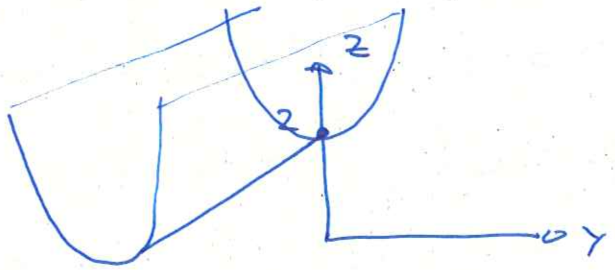


$$\left\{\begin{array}{l} z=4-x^2-y^2 \\ \boxed{x=0} \longrightarrow z=4-y^2 \quad OYZ \\ \boxed{y=0} \longrightarrow z=4-x^2 \\ z=0 \longrightarrow x^2+y^2=4 \end{array}\right.$$





$z=2+y^2 \quad OYZ$



$$\left\{\begin{array}{l} z=4-x^2-y^2 \\ z=2+y^2 \end{array}\right.$$

$$\frac{x^2}{2}+\frac{y^2}{1^2}=1$$

$$\left\{\begin{array}{l} x=\sqrt{2}\rho\cos\theta \\ y=1\rho\sin\theta \\ z=z \end{array}\right.$$

$$|J|=\sqrt{2}\rho$$

$$z\in[zil, parab]$$

$$z=4-x^2-y^2$$

$$z=4-2\rho^2\cos^2\theta-\rho^2\sin^2\theta=$$

$$=4-\rho^2\cos^2\theta-\rho^2$$

$$z=2+y^2=2+\rho^2\sin^2\theta$$

5. GAIA: LERRO INTEGRALAK

5.1. IBILBIDEAK: ARKU-LUZERA

DEFINIZIOA:

$$\sigma : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$t \longrightarrow \sigma(t) = (x_1(t), \dots, x_n(t))$$

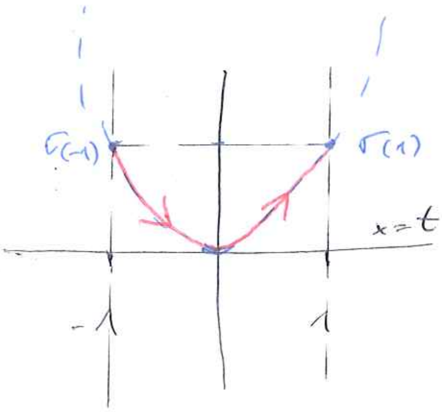
σ $\mathbb{R}^n$ espazioko IBILBIDEA

- σ-ren IRUDIA ($t \in [a,b]$ denean) kurba bat da
- $\sigma(a) \wedge \sigma(b) \rightarrow$ σ-ren MUTURRAK

ADIBIDEAK

1) $\sigma : [-1,1] \longrightarrow \mathbb{R}^2$

$\sigma(t) = (\underset{x(t)}{t}, \underset{y(t)}{t^2}) \longrightarrow \begin{cases} x = t \\ y = t^2 = x^2 \end{cases}$



$\left.\begin{array}{l} \sigma(-1) = (-1,1) \\ \sigma(1) = (1,1) \end{array}\right\} \Rightarrow$ MUTURRAK

2) $\sigma : [0, 2\pi] \longrightarrow \mathbb{R}^3$

$\sigma(t) = (\underset{x(t)}{\cos t}, \underset{y(t)}{\sin t}, \underset{z(t)}{t})$

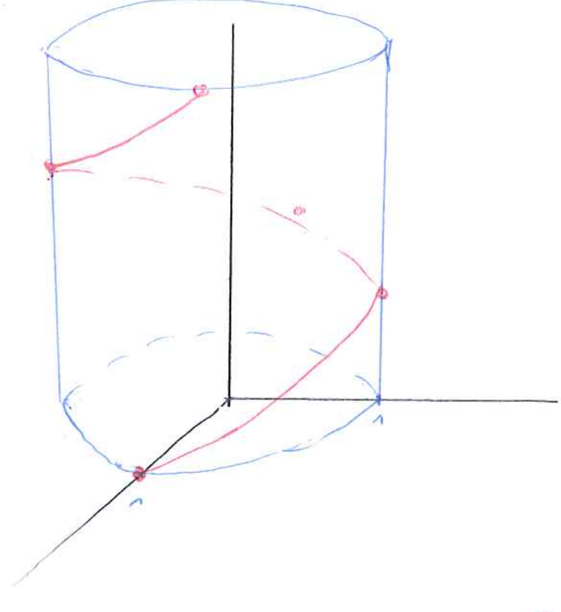
$\begin{cases} x = \cos t \\ y = \sin t \Rightarrow x^2 + y^2 = 1 \text{ betetzen da} \\ z = t \end{cases}$

$\left.\begin{array}{l} \sigma(0) = (1,0,0) \\ \sigma(2\pi) = (1,0,2\pi) \end{array}\right\}$ MUTURRAK

$\sigma(\frac{\pi}{2}) = (0, 1, \frac{\pi}{2})$

$\sigma(\pi) = (-1, 0, \pi)$

$\sigma(\frac{3\pi}{2}) = (0, -1, \frac{3\pi}{2})$.



DEFINIZIOA

$\sigma: I \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ C^1 klasekoa

i) $\sigma'(t) = (x_1'(t), \dots, x_n'(t)) \Rightarrow$ σ-ren ABIADURA BEKTOREA

ii) $\|\sigma'(t)\| = \sqrt{\sum_{i=1}^{n} (x_i'(t))^2} \Rightarrow$ σ-ren LASTERTASUNA

DEFINIZIOA

$\sigma: [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ C^1 klasekoa

$\sigma(t) = (x(t), y(t), z(t))$

$\ell(\sigma) = \int_a^b \|\sigma'(t)\| dt \longrightarrow$ σ-ren ARKU LUZERA

OHARRA: σ ez du zertan C^1 klasekoa izan behar $[a,b]$ tarte osoan, nahikoa da zatika C^1 klasekoa bada.

ADIBIDEA

$\sigma: [-1, 1] \longrightarrow \mathbb{R}^3$

$\sigma(t) = (\underbrace{|t|}_{x(t)}, \underbrace{|t - \frac{1}{2}|}_{y(t)}, \underbrace{0}_{z(t)})$ $\rightarrow$ OXY planoan dago

$|t| = \begin{cases} t, & t \geq 0 \\ -t, & t \leq 0 \end{cases}$

$|t - \frac{1}{2}| = \begin{cases} t - \frac{1}{2}, & t \geq \frac{1}{2} \\ \frac{1}{2} - t, & t < \frac{1}{2} \end{cases}$

σ bai da zatika C^1:

$\sigma_1(t): [-1, 0] \longrightarrow \mathbb{R}^3$ $\sigma_2(t): [0, \frac{1}{2}] \longrightarrow \mathbb{R}^3$

$\sigma_1(t) = (-t, \frac{1}{2} - t, 0)$ $\sigma_2(t) = (t, \frac{1}{2} - t, 0)$

$\sigma_3(t): [\frac{1}{2}, 1] \longrightarrow \mathbb{R}^3$

$\sigma_3(t) = (t, t - \frac{1}{2}, 0)$

$$l(\sigma) = \int_{-1}^{1} \|\sigma'(t)\| dt = \cancel{\int_{-1}^{0} \|(1,-1,0)} =$$

$$= \int_{-1}^{1} \|(1,-1,0)\| dt + \int_{0}^{1/2} \|(1,-1,0)\| dt + \int_{1/2}^{1} \|(1,1,0)\| dt =$$

$$= \int_{-1}^{0} \sqrt{2} dt + \int_{0}^{1/2} \sqrt{2} dt + \int_{1/2}^{1} \sqrt{2} dt$$

S. 2. LEHEN ETA BIGARREN MAILAKO LERRO-INTEGRALAK

$f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ funtzio ESKALARRA

$\vec{F} : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ funtzio BEKTORIALA

DEFINIZIOA

$\sigma : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ C^1 klaseko ibilbidea

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ funtzio eskalar jarraitua

$$\int_\sigma f ds = \int_a^b f(\sigma(t)) \cdot \|\sigma'(t)\| dt \Rightarrow$$

LEHEN MAILAKO LERRO INTEGRALA (V IBILBIDE INTEGRALA)

ADIBIDEA

$\sigma : [0, 2\pi] \longrightarrow \mathbb{R}^3$

$\sigma(t) = (\cos t, \sin t, t)$ $C^1([0,2\pi])$

$f(x,y,z) = x^2 + y^2 + z^2$ jarraitua

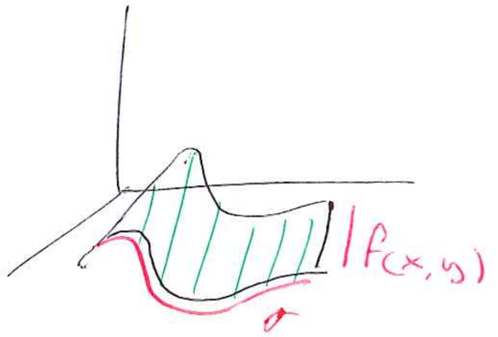
$$\int_\sigma f ds = \int_0^{2\pi} f(\sigma(t)) \cdot \|\sigma'(t)\| dt$$

$$\int_\sigma f ds = \int_0^{2\pi} (\underbrace{\cos^2 t + \sin^2 t}_{1} + t^2) \cdot \sqrt{\underbrace{(-\sin t^2) + (\cos t)^2}_{1} + 1^2} dt =$$

$$= \int_0^{2\pi} (1 + t^2) \cdot \sqrt{2} dt = \sqrt{2} \left[t + \frac{1}{3} t^3 \right]_0^{2\pi} = \boxed{\sqrt{2} \cdot 2\pi + \frac{8\pi^3}{3}}$$

OHARRA: $f(x,y) \geq 0$

Esanahi geometrikoa



$$\int_\sigma f\,ds = \text{horma honen azalera}$$

$[f(x,y)$ altuera$]$

PROPOSIZIOA: 5.1:

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ funtzio eskalarra

$\rho = \rho(\theta) \longrightarrow \sigma$ ibilbidearen ekuazio polarra

$$\theta \in [\theta_1, \theta_2] \Rightarrow \int_{\theta_1}^{\theta_2} f(\underbrace{\rho(\theta)\cos\theta}_{x(\theta)}, \underbrace{\rho(\theta)\sin\theta}_{y(\theta)}) \sqrt{\rho^2 + (\rho')^2}\, d\theta$$

$\Rightarrow$ LEHEN MAILAKO LERRO INTEGRALA POLARRETAN

DEFINIZIOA

$\sigma: [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ C^1 klasekoa ibilbidea

$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ funtzio bektorial jarraitua

$$\int_\sigma \vec{F}\,ds = \int_a^b \vec{F}(\sigma(t)) \underset{\text{ESKALARRA}}{\cdot} \sigma'(t)\,dt$$

$\Rightarrow \vec{F}$-ren BIGARREN MAILAKO LERRO INTEGRALA

OHARRA:

1) $\vec{F}$-ren bigarren mailako lerro integralaren esanahi fisikoa ~~fisikoa~~: partikula bat $\sigma(a)$-tik $\sigma(b)$-ra mugitzen bada eta F bere gainean aplikatzen indar bat bada

$\int_\sigma \vec{F}\,ds$ sortzen den LANA da.

2) $\vec{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

$\sigma(t) = (x(t), y(t), z(t))$

$\sigma'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$

$$\int_\sigma \vec{F} ds = \int_a^b \vec{F}(\sigma(t))\,\sigma'(t)\,dt = \int_\sigma F_1\,dx + F_2\,dy + F_3\,dz$$

ADIBIDEA

$\sigma : [0, 2\pi] \longrightarrow \mathbb{R}^3$

$\sigma(t) = (\sin t, \cos t, t)$

$\vec{F}(x,y,z) = (x, y, z)$

Kalkulatu $\int_\sigma \vec{F} ds$

$$\int_\sigma \vec{F}\cdot ds = \int_0^{2\pi} \vec{F}(\sigma(t))\cdot\sigma'(t)\,dt =$$

$$= \int_0^{2\pi} (\sin t, \cos t, t) \overset{\text{EBK}}{\cdot} (\cos t, -\sin t, 1)\,dt =$$

$$= \int_0^{2\pi} \cancel{\sin t \cos t - \cos t \sin t} + t\,dt =$$

$$= \left[\frac{1}{2}t^2\right]_0^{2\pi} = 2\pi^2$$

$$\int_\sigma \vec{F}\cdot ds \overset{\text{OHARRA}_2}{=} \int_\sigma \underset{\sin t}{\underset{\shortparallel}{x}}\, \overset{(\sin t)' \ldots}{\overset{\shortparallel}{dx}} + y\,dy + z\,dz =$$

$$= \int_0^{2\pi} (\sin t \cos t + \cos t(-\sin t) + t)\,dt = \int_0^{2\pi} t\,dt = 2\pi^2$$

TEOREMA S.R:

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ C^1 eta $\sigma : [a,b] \longrightarrow \mathbb{R}^3$

C^1 ibilbidea.

Orduan $\underbrace{\int_\sigma \overset{*}{\nabla f}\, ds}_{\text{Bigarren mailako lerro integrala}} = f(\sigma(b)) - f(\sigma(a))$

$*$ $\vec{F}$ emanik $\exists f$ existitzen non $\nabla f = \vec{F}$ orduan f, $\vec{F}$-ren POTENTZIALA da

ADIBIDEA: ...

$\sigma(t) = (\overbrace{\frac{t^4}{4}}^{x(t)}, \sin^3(\frac{t\pi}{2}), 0) \quad t \in [0,1]$

Kalkulatu $\int y\,dx + x\,dy$

$\int_\sigma y\,dx + x\,dy = \int_\sigma y\,dx + x\,dy + 0\,dz = \dots$

$\vec{F} = (F_1, F_2, F_3) = (y, x, 0)$

$\int_\sigma \vec{F}\,ds = \int_\sigma \nabla f\,ds \overset{\text{TEOR 5.2}}{=} f(\sigma(1)) - f(\sigma(0)) \overset{*}{=}$

Aurkitu ahal badugu f non $\nabla f = \vec{F}$

Bilatuko dugu f non $\nabla f = \vec{F}$

$\nabla f = \vec{F}$

$\frac{\partial f}{\partial x} = y \longrightarrow \frac{\partial f}{\partial x} = y \Rightarrow f(x,y,z) = yx + h(y,z)$ — A

$\frac{\partial f}{\partial y} = x \longrightarrow \frac{\partial f}{\partial y} = x \Rightarrow f(x,y,z) = xy + h(x,z)$

$\frac{\partial f}{\partial z} = 0 \longrightarrow \frac{\partial f}{\partial z} = 0 \Rightarrow f(x,y,z) = h(x,y)$

BERE MODUA $\Rightarrow$ $\frac{\partial f}{\partial y} = x \Rightarrow x + \overset{0}{\frac{\partial h}{\partial y}} = x \Rightarrow h(y,z) = k(z)$

Aukeratu

$\frac{\partial f}{\partial z} = 0 \Rightarrow 0 + \frac{\partial k}{\partial z} = 0 \Rightarrow k(z) = C = 0$

$\overset{*}{=} f(\frac{1}{4}, 1, 0) - f(0,0,0) = \frac{1}{4} - 0 = \boxed{\frac{1}{4}}$

$f = yx$

5.3. BIRPARAMETRIZAZIOA

$\sigma \wedge \rho$ bi IBILBIDE desberdin IRUDI berdinarekin,

$$\int_\sigma f\,ds \underset{?}{=} \int_\rho f\,ds \qquad \int_\sigma F\,ds \underset{?}{=} \int_\rho F\,ds$$

DEFINIZIOA

$h : [\alpha, \beta] \longrightarrow [a, b]$ C^1 funtzio bijektiboa izan bitez $\sigma : [a, b] \longrightarrow \mathbb{R}^3 \wedge \rho : [\alpha, \beta] \longrightarrow \mathbb{R}^3$ non $\rho = \sigma \circ h$. ρ σ-ren birparametrizazioa da.

ADIBIDEA:

$\sigma : [0, 2\pi] \longrightarrow \mathbb{R}^3$

$\sigma(t) = (\cos t, \sin t, t)$ helize zirkularra

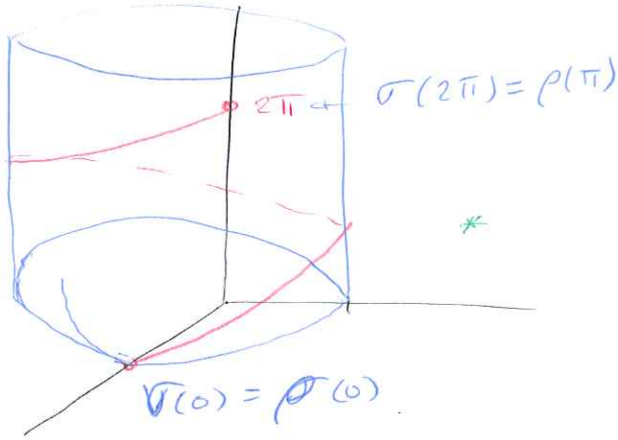
$\rho : [0, \pi] \longrightarrow \mathbb{R}^3$

$\rho(t) = (\cos 2t, \sin 2t, 2t)$

$h(t) = 2t$ hartzen badugu $h : [0, \pi] \longrightarrow [0, 2\pi]$

$\rho = \sigma \circ h$

(ρ-k denbora erdia behar izan du σ-ren ibilbide bera egiteko)



DEFINIZIOA: $\sigma : [a, b] \longrightarrow \mathbb{R}^3$ ibilbidea

$h : [\alpha, \beta] \longrightarrow [a, b]$ C^1 funtzio bijektiboa eta $\rho : [\alpha, \beta] \longrightarrow \mathbb{R}^3$ ibilbidea $\rho = \sigma \circ h$

i) $\begin{cases} \rho(\alpha) = \sigma(a) \\ \rho(\beta) = \sigma(b) \end{cases} \Longrightarrow \rho$-k ORIENTAZIOA MANTENTZEN du. * ads

ii) $\begin{cases} \rho(\alpha) = \sigma(b) \\ \rho(\beta) = \sigma(a) \end{cases} \Rightarrow \rho$-k ORIENTAZIOA ALDATZEN DU

ADIBIDEA:

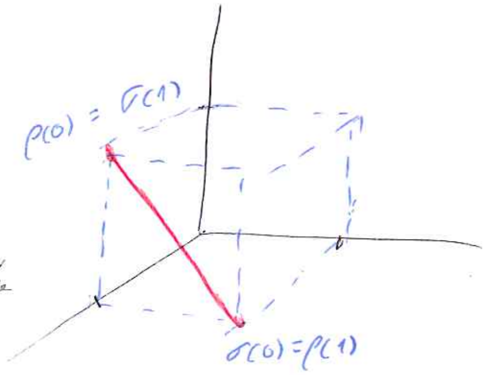
$\sigma : [0,1] \longrightarrow \mathbb{R}^3$ $\qquad$ $\rho : [0,1] \longrightarrow \mathbb{R}^3$

$\sigma(t) = (1, 1-t, t)$ $\qquad$ $\rho(t) = (1, t, 1-t)$

$\sigma(0) = (1,1,0)$ $\qquad$ $\rho(0) = (1,0,1)$

$\sigma(1) = (1,0,1)$ $\qquad$ $\rho(1) = (1,1,0)$

ρ-k orientazioa aldatzen du



TEOREMA 5.3:

$\sigma : [a,b] \to \mathbb{R}^3$ C^1 ibilbidea

ρ, σ-ren birparametrizazioa

eta $f : \mathbb{R}^3 \to \mathbb{R}$ funtzio eskalar jarraitua

$$\Rightarrow \int_\sigma f\, dl = \int_\rho f\, ds$$

TEOREMA 5.4:

$\sigma : [a,b] \to \mathbb{R}^3$ C^1 ibilbidea

$\rho : [\alpha,\beta] \to \mathbb{R}^3$ σ-ren birparametrizazioa

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ funtzio bektorial jarraitua

$$\int_\sigma \vec{F}\, ds = \int_\rho \vec{F}\, dS \quad \text{eta} \quad \int_\sigma \vec{F}\, ds = -\int_\rho \vec{F}\, dS$$

Orientazioa mantendu $\qquad$ Orientazioa aldatu

5.4. LERRO-INTEGRALAK KURBA GEOMETRIKOEN GAINEAN

DEFINIZIOA: $\sigma : [a,b] \to \mathbb{R}^3$ zatik C^1 injektiboa

σ-ren irudia Γ kurba sinpl bat da eta σ Γ-ren parametrizazio bat da

$\sigma(a), \sigma(b) \rightarrow \Gamma$-ren MUTURRAK

Γ Kurbak bi ORIENTAZIO ditu

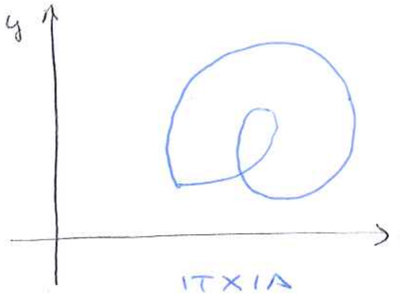
$\sigma(a)$-tik $\sigma(b)$-ra $\wedge$ $\sigma(b)$-tik $\sigma(a)$-ra

- Γ kurba sinple bat orientazio batekin, KURBA SINPLE NORABIDETUA da.

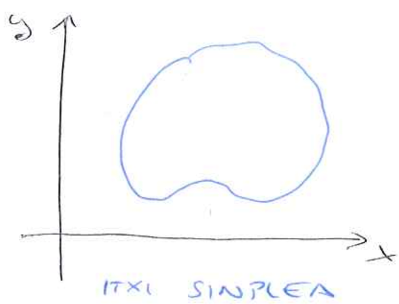
DEFINIZIOA:

- $\left\{ \begin{array}{l} \sigma: [a,b] \rightarrow \mathbb{R}^3 \text{ zatika } C^1 \\ \sigma(a) = \sigma(b) \wedge \Gamma \text{ } \sigma\text{-ren irudia} \end{array} \right. \Rightarrow \Gamma$ Kurba ITXIA

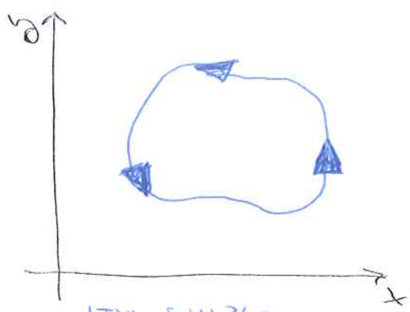
- $\left\{ \begin{array}{l} \sigma: [a,b] \rightarrow \mathbb{R}^3 \text{ zatika } C^1 \\ \sigma(a) = \sigma(b) \quad \text{INJEKTIBOA} \\ \Gamma, \sigma\text{-ren irudia} \end{array} \right. \Rightarrow \Gamma$ kurba ITXI SINPLEA



ITXIA



ITXI SINPLEA



ITXI, SINPLE, NORABIDETUA

DEFINIZIOA:

Γ Kurba sinple norabidetua σ orientazioa mantentzen duen Γ-ren parametrizazioa

$f: \mathbb{R}^3 \rightarrow \mathbb{R} \wedge F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ jarraituak

$$\Rightarrow \left\{ \begin{array}{l} \int_\Gamma f\,ds = \int_\sigma f\,ds = \int_a^b f(\sigma(t)) \cdot \|\sigma'(t)\| dt \\ \int_\Gamma \vec{F}\,ds = \int_\sigma \vec{F}\,ds = \int_a^b \vec{F}(\sigma(t)) \cdot \sigma'(t)\, dt \end{array} \right.$$

ADIBIDEA:

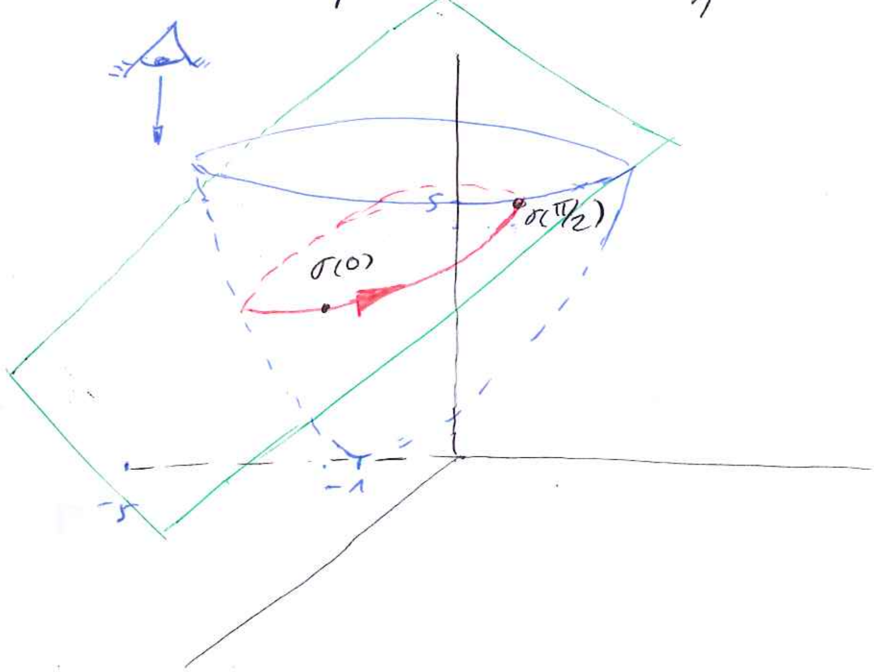
$$\Gamma: \left\{ \begin{array}{l} z = 5 + y \\ 2z = x^2 + (y+1)^2 \end{array} \right.$$

$$\Gamma : \begin{cases} z = 5 + y & \rightarrow \text{PLANOA} \\ 2z = x^2 + (y+1)^2 & \rightarrow \text{PARBOLOIDEA} \end{cases}$$

Gainazalen arteko ebakidura (gainetik begiratuta erlojuaren orratzen kontrako orientazioa)

$$\left.\begin{aligned} f(x,y,z) &= \sqrt{5+x^2} \\ \vec{F}(x,y,z) &= (yz, -xz, xy) \end{aligned}\right\} \text{ jarraituak}$$

Kalkulatu $\int_\Gamma f\,ds \ \wedge \ \int_\Gamma \vec{F}\,ds$



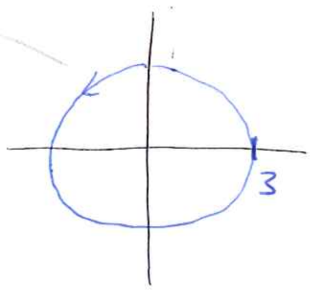
Bilatu Γ-ren parametrizazioa:

σ-k $z = 5 + y \ \wedge \ 2z = x^2 + (y+1)^2$ ekuazioak bete behar ditu.

EBAKIDURA: $\begin{cases} z = 5 + y \\ 2z = x^2 + (y+1)^2 \end{cases} \Rightarrow x^2 + y^2 = 9$

$\sigma(t) = (3\cos t, \ 3\sin t, \ 5 + 3\sin t)$

$t \in [0, 2\pi]$



$\sigma(0) = (3, 0, 5)$

$\sigma(\frac{\pi}{2}) = (0, 3, 8)$ $\Rightarrow$ ORIENTAZIOA MANTENDU

$$\int_{\Gamma} f\,ds = \int_{\sigma} f\,ds = \int_0^{2\pi} f(r(t)) \cdot \|\sigma'(t)\|\,dt =$$

$$= \int_0^{2\pi} \sqrt{9 + (3\cos t)^2} \cdot \sqrt{(-3\sin t)^3 + (3\cos t)^2 + (3\cos t)^2}\,dt =$$

$$= \int_0^{2\pi} 9 + 9\cos^2 t\,dt = 9 \cdot \int_0^{2} 1 + \frac{1+\cos 2t}{2}\,dt = \dots = \boxed{27\pi}$$

$$\int_{\Gamma} \vec{F}\,ds = \int_{\sigma} \overset{\overset{(yz,-xz,xy)}{\shortparallel}}{\vec{F}} \cdot ds \overset{\substack{\text{DEF} \\ \text{OHARRA} \\ 2}}{=} \int_0^{2\pi} yz\,dx - xz\,dy + xy\,dz =$$

$$= \int_0^{2\pi} (3\sin t)(5+3\sin t) \cdot (-3\sin t) - 3\cos t(3\sin t+5) 3\cos t +$$

$$+ 3\cos t \cdot 3\sin t\,(3\cos t)\,dt = [\dots] = \boxed{-90\pi}$$

TEOREMA 5.5:

Γ Kurba sinple norabidetua eta Γ^- kurba bera kontrako noranzkoan hartuta.

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ jarraitua $\Rightarrow \int_{\Gamma} \vec{F} \cdot ds = -\int_{\Gamma^-} \vec{F} \cdot ds$

OHARRA:

$f: \mathbb{R}^3 \to \mathbb{R}$ jarraitua $\Rightarrow \int_{\Gamma} f\,ds = \int_{\Gamma^-} f\,ds$

TEOREMA 5.6:

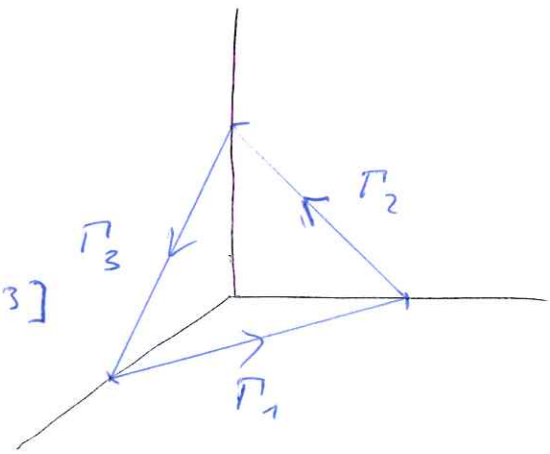
Γ_i, $i = 1, \dots, m$ kurba sinple norabidetuak eta demagun Γ_i-ren bukaerako puntua Γ_{i+1}-en hasierako puntuaren berdina dela. Orduan $\Gamma = \Gamma_1 + \Gamma_1 + \dots + \Gamma_m$ eta $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ jarraitua $\Rightarrow \int_{\Gamma} \vec{F}\,ds = \sum_{i=1}^{m} \int_{\Gamma_i} \vec{F}\,ds$

ADIBIDEA:

Kalkulatu $\int_\Gamma \overset{F_1}{y}\,dx + \overset{F_2}{x}\,dy + \overset{F_3}{xz}\,dz$ non

Γ kurba $x+y+z=3$ planoa eta plano koordenatuek ebakitzean lortzen den kurba itxia den goitik begiratuta erlojuen orratzen kontrako orientazioarekin

$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$



Γ_1-en parametrizazioa:

$z=0 \wedge x+y=3$

$\sigma_1(t) = (3-t, t, 0) : t\in[0,3]$

$\left.\begin{array}{l}\sigma_1(0) = (3,0,0) \\ \sigma_1(3) = (0,3,0)\end{array}\right\} \Rightarrow$ ORIENTAZIOA MANTENDU

Γ_2-ren parametrizazioa

$x=0 \wedge y+z=3$

$\sigma_2(t) = (0, 3-t, t) : t\in[0,3]$

$\left.\begin{array}{l}\sigma_2(0) = (0,3,0) \\ \sigma_2(3) = (0,0,3)\end{array}\right\} \Rightarrow$ ORIENTAZIOA MANTENDU

Γ_3-ren parametrizazioa

$y=0 \wedge x+z=3$

$\sigma_3(t) = (3-t, 0, t) \quad t\in[0,3]$

$\left.\begin{array}{l}\sigma_3(0) = (3,0,0) \\ \sigma_3(3) = (0,0,3)\end{array}\right\} \Rightarrow$ ORIENTAZIOA ALDATU

$$\int_\Gamma y\,dx + x\,dy + xz\,dz = \int_{\Gamma_1} \vec{F}\,ds + \int_{\Gamma_2} \vec{F}\,ds + \int_{\Gamma_3} \vec{F}\,ds$$

$$= \int_{\Gamma_1} \vec{F} ds + \int_{\Gamma_2} \vec{F} ds - \int_{\Gamma_3} \vec{F} ds =$$

$$= \int_0^3 t(-1) + (3-t) + (3-t) \cdot 0 \, dt + \int_0^3 0 \, dt - \int_0^3 0 + (3-t) \cdot 0 + (3-t) t \cdot 1 dt =$$

$$= [\dots] = -\frac{9}{2}$$

ANALISI BEKTORIALA ETA KONPLEXUA

5. Gaia: LERRO-INTEGRALAK

Ariketak

1. Kalkula itzazu ondoko ibilbideen arku-luzerak emandako tarteetan:

   (i) $\sigma(t) = (2t, t^2, \log t)$ ibilbidearen $(2,1,0)$ eta $(4,4,\ln 2)$ puntuen arteko arkuaren luzera.

   (ii) $\sigma(t) = (1, 3t^2, t^3)$ ibilbidearen $[0,1]$ tarteko arku-luzera.

   (iii) $\sigma(t) = (a\cos t, a\sin t, bt)$ ibilbidearen arku-luzera, $0 \leq t \leq 2\pi$ izanik eta $a > 0$, $b > 0$.

   (iv) $\sigma(t) = (t, t, \frac{2}{3}t^{3/2})$ ibilbidearen arku-luzera, $t \in [t_0, t_1]$ delarik.

   Em.: (i) $3 + \log 2$; (ii) $5\sqrt{5} - 8$; (iii) $2\pi\sqrt{a^2 + b^2}$; (iv) $\frac{2}{3}\left((t_1+2)^{3/2} - (t_0+2)^{3/2}\right)$.

2. Kalkula itzazu ondoko funtzioen lerro-integralak emandako ibilbideen gainean:

   (i) $f(x,y,z) = y, \quad \sigma(t) = (0,0,t),\ 0 \leq t \leq 1.$ *Em.:* 0.

   (ii) $f(x,y,z) = \cos z, \quad \sigma(t) = (\sin t, \cos t, t),\ 0 \leq t \leq \pi.$ *Em.:* 0.

   (iii) $f(x,y,z) = xyz, \quad \sigma(t) = (e^t\cos t, e^t \sin t, 3),\ 0 \leq t \leq 2\pi.$ *Em.:* $\frac{3\sqrt{2}}{13}(1 - e^{6\pi})$.

   (iv) $\vec{F}(x,y,z) = (x,y,z), \quad \sigma(t) = (\cos t, \sin t, 0),\ t \in [0, 2\pi].$ *Em.:* 0.

   (v) $\vec{F}(x,y,z) = (y, 2x, y),\ \sigma(t) = (t, t^2, t^3),\ 0 \leq t \leq 1.$ *Em.:* $\frac{34}{15}$.

3. (i) Izan bitez $\nabla f(x,y,z) = (2xyz, x^2z, x^2y)$ eta $f(1,1,1) = 1$. Kalkulatu $f(1,2,4)$.

   (ii) Izan bitez $\nabla f(x,y,z) = (2xyze^{x^2}, ze^{x^2}, ye^{x^2})$ eta $f(0,0,0) = 5$; $f(1,1,2)$ kalkulatu.

   Em.: (i) 8; (ii) $5 + 2e$.

4. Ondorengo lerro-integralak kalkulatu:

   (i) $\int_{OA} (x+y)ds$, OA ibilbidea $O(0,0)$ eta $A(1,1)$ puntuak lotzen dituen zuzenkia da.

   (ii) $\int_C |y|\, ds$, C kurba $(x^2+y^2)^2 = a^2(x^2 - y^2)$ lemniskata da.

   (iii) $\int_C \sqrt{x^2+y^2}\, ds$, C kurba $x^2 + y^2 = ax$ zirkunferentzia izanik eta $a > 0$.

   (iv) $\int_C \sqrt{2y^2 + z^2}\, ds$, C $x^2+y^2+z^2 = a^2$ esfera eta $x = y$ planoaren arteko ebakidura izanik.

   Em.: (i) $\sqrt{2}$; (ii) $2a^2(2 - \sqrt{2})$; (iii) $2a^2$; (iv) $2a^2\pi$.

5. Hurrengo (bigarren mailako) lerro-integralak kalkulatu:

(i) $\displaystyle\int_{ABC} yzdx + xzdy + xydz$, ABC kurba $A(1,0,0)$, $B(0,1,0)$ eta $C(0,0,1)$ erpinetako lerro poligonala da.

(ii) $\displaystyle\int_{OA} xdy + ydx$, OA ibilbidea $y = 2x^2$ parabolaren zatia da, $O(0,0)$ eta $A(1,2)$ puntuak izanik.

(iii) $\displaystyle\oint_{\sigma} (x+y)dx + (x-y)dy$, σ ibilbidea $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ elipsea da (erlojuen orratzen aurkako norantzaz harturik).

(iv) $\displaystyle\int_{C} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$, C lehen oktanteko plano koordenatuen eta $x^2 + y^2 + z^2 = 1$ esferaren arteko ebakidura-kurba da (esfera gainean eta C kurban zehar 1. oktanteko esfera zatia ezkerraldean geratzen delarik).

(v) $\displaystyle\int_{\sigma} y^2dx + z^2dy + x^2dz$, σ ibilbidea $z \geq 0$ espazioerdian dagoen $x^2 + y^2 + z^2 = a^2$ eta $x^2 + y^2 = ax$ gainazalen arteko ebakidura da (erlojuen orratzen aurkako norantzaz hartuta).

(vi) $\displaystyle\int_{AB} \sin y dx + \sin x dy$, non AB kurba $A(0,\pi)$ eta $B(\pi,0)$ lotzen dituen zuzenkia den.

(vii) $\displaystyle\int_{C} \frac{dx + dy}{|x| + |y|}$, non C erpinak $(1,0)$, $(0,1)$, $(-1,0)$ eta $(0,-1)$ puntuetan dituen laukia den, norantza positiboan.

(viii) $\displaystyle\int_{C} (4xy + y^2)\,dx - (xy + 3x^2)\,dy$, non C $(x+1)^2 + (y-2)^2 = 9$ ekuazioko zirkunferentzia den, norantza positiboan hartuta.

Em.: (i) 0; (ii) 2; (iii) 0; (iv) -4; (v) $-a^3\pi/4$; (vi) 0; (vii) 0; (viii) 36π.

5. Lerro-integralak

Ariketak

1. Ariketak

i) $\sigma(t) = (2t, t^2, \ln t)$

$\sigma(1) = (2,1,0) \wedge (4,4,\ln 2) = \sigma(2)$

$$\ell(\sigma) = \int_c^s \|\sigma'(t)\| dt$$

$$\ell(\sigma) = \int_1^2 \|(2, 2t, \tfrac{1}{t})\| dt =$$

$$= \int_1^2 \sqrt{2^2 + (2t)^2 + (\tfrac{1}{t})^2}\, dt = \int_1^2 \sqrt{4+4t^2+\tfrac{1}{t^2}}\, dt =$$

$$= \int_1^2 \sqrt{\frac{4t^2+4t^4+1}{t^2}}\, dt = \int_1^2 \frac{1}{t}\sqrt{(\underset{\geq 0}{2t^2+1})^2}\, dt = \int_1^2 \frac{2t^2+1}{t} dt =$$

$$= \int_1^2 2t + \frac{1}{t}\, dt = \left[t^2 + \ln t\right]_1^2 = 4 + \ln 2 - 1 - \overset{0}{\ln 1} =$$

$$= 3 + \ln 2$$

iii) $\sigma(t) = (a\cos t, a\sin t, bt)$

$t \in [0, 2\pi] \quad a, b > 0$

$$\ell(\sigma) = \int_0^{2\pi} \|(-a\sin t, a\cos t, b)\| dt =$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}\, dt = \int_0^{2\pi} \sqrt{a^2+b^2}\, dt =$$

$$= 2\pi\sqrt{a^2+b^2}$$

2. Ariketa

iii) $f(x,y,z) = xyz$, $\sigma(t) = (e^t\cos t, e^t\sin t, 3)$ $0 \le t \le 2\pi$

$$\int_\sigma f\, ds = \int_0^{2\pi} f(\sigma(t)) \cdot \|\sigma'(t)\| dt =$$

$$= \int_0^{2\pi} e^{2t} \cos t \sin t \cdot 3 \cdot \|(e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t, 0)\| dt =$$

$$= 3 \int_0^{2\pi} e^{2t} \cos t \sin t \cdot \underbrace{\sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}}_{= \sqrt{2} e^t} dt =$$

$$= \frac{\sqrt{2}}{2} 3 \cdot \int_0^{2\pi} e^{3t} \cdot \sin 2t = \frac{\sqrt{2} \cdot 3}{2} \int_0^{2\pi} e^{3t} \cdot \sin 2t \, dt =$$

$$I = \int_0^{2\pi} e^{3t} \sin 2t \, dt = -\frac{1}{2} e^{3t} \cos t \Big|_0^{2\pi} - \int_0^{2\pi} -\frac{1}{2} \cos 2t \, 3 e^{3t} dt$$

$u = e^{3t}$

$dv = \sin 2t \, dt$

$$= \dots = -\frac{1}{2} e^{3t} \cos 2t + \frac{3}{4} e^{3t} \sin 2t \Big|_0^{2\pi} - \frac{9}{4} I$$

$u = e^{3t}$

$dv = \cos 2t \, dt$

$$= \frac{\sqrt{2} \cdot 3}{2} \left[\frac{-\frac{1}{2} e^{3t} \cos 2t + \frac{3}{4} e^{3t} \sin 2t}{1\frac{3}{4}} \right]_0^{2\pi} = \boxed{\frac{3\sqrt{2}}{13} [1 - e^{6\pi}]}$$

v) $\vec{F}(x, y, z) = (y, 2x, y)$

$\sigma(t) = (t, t^2, t^3) \qquad t \in [0, 1]$

$$\int_\sigma \vec{F} ds = \int_0^1 \vec{F}(\sigma(t)) \cdot \sigma'(t) dt =$$

$$= \int_0^1 (t^2, 2t, t^2) \cdot (1, 2t, 3t^2) \, dt =$$

$$= \int_0^1 t^2 + 4t^2 + 3t^4 dt = [\dots] = \boxed{\frac{34}{15}}$$

3. ARIKETA

ii) $\nabla f(x,y,z) = (2xyz e^{x^2}, z e^{x^2}, y e^{x^2})$

$f(0,0,0) = 5 \Rightarrow f(1,1,2) = ?$

Bi modu:

1) Kalkulatu f, atera $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$f(x,y,z) = \dots + k$

k atera $f(0,0,0)=5$ erabiliz

f deskargu jada eta $f(1,1,2)$ atera

2) TEOR 5.2

Aukeratu σ ibilbide bat (edozein bat eta erreztena) $\sigma(b)=(1,1,2) \wedge \sigma(a)=(0,0,0)$

$\Rightarrow \int_\sigma \nabla f \, ds = f(\sigma(b)) - f(\sigma(a))$

ADIBIDEZ

$\sigma(t) = (t,t,2t) \qquad t \in [0,1]$ hartuz

$\sigma(0) = (0,0,0)$
$\sigma(1) = (1,1,2)$ $\Rightarrow \int_\sigma \nabla f \, ds =$

$\int_\sigma \nabla f \, ds = \int_\sigma 2xyz e^{x^2} dx + z e^{x^2} dy + y e^{x^2} dz =$

$= \int_0^1 4t^3 \cdot e^{t^2} 1 + 2t e^{t^2} \cdot 1 + t e^{t^2} \cdot 2 \, dt =$

$= \int_0^1 e^{t^2}(4t^3 + 4t) dt = 4 \int_0^1 e^{t^2} t (t^2+1) dt =$

$\left| \begin{array}{l} u = t^2+1 \\ dv = e^{t^2} t dt \end{array} \right. \Rightarrow \begin{array}{l} du = 2t dt \\ v = \frac{1}{2} e^{t^2} \end{array}$

$= 2e = f(1,1,2) - \overset{5}{f(0,0,0)} \Rightarrow \boxed{f(1,1,2) = 2e+5}$

3

4. ARIKETA

ii) $\int_C |y|\, ds$, $C \equiv (x^2+y^2)^2 = a^2(x^2-y^2)$

$\cos 2\theta \geq 0$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow \rho^4 = a^2\rho^2\cos 2\theta$

$\theta \in [-\pi/4, \pi/4] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}]$

$\rho(\theta) = a\sqrt{\cos 2\theta}$

$f(x,y) = |y|$

$$\int_\sigma f\,ds = \int_{\theta_0}^{\theta_1} f(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)\sqrt{\rho^2+(\rho')^2}\,d\theta =$$

$$\int_\sigma f\,ds = 4\int_0^{\pi/4} |a\sqrt{\cos 2\theta}\sin\theta| \cdot \sqrt{a^2\cos 2\theta + (a\tfrac{1}{2}(\cos 2\theta)^{-1/2}\,2\sin 2\theta)^2}$$

$$= [\dots] = 4a^2\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta \cdot \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\,d\theta =$$

$\theta \in [0, \pi/4]$

$a\sqrt{\cos 2\theta}\sin\theta \geq 0$

$$= 4a^2\int_0^{\pi/4}\sin\theta \cdot 1\,d\theta = \dots = 2a^2(2-\sqrt{2})$$

iii) $\int_C \sqrt{x^2+y^2}\,ds =$ $\qquad f(x,y) = \sqrt{x^2+y^2}$

1. mailako l.i

$x^2+y^2 = ax$

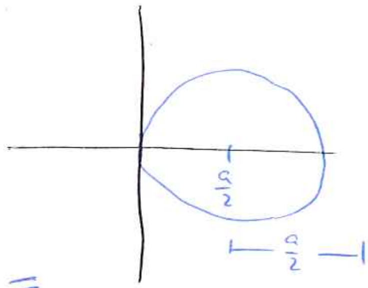
bidetu γ

$$= \int_0^{?} f(\sigma(\theta))\,\|\sigma'(\theta)\|\,d\theta =$$

$(x-\frac{a}{2})^2 + y^2 = \frac{a^2}{4}$

$\sigma(\theta) = (\frac{a}{2} + \frac{a}{2}\cos\theta,\ 0 + \frac{a}{2}\sin\theta)$

C-ren parametrizazioa



$$= \int_0^{2\pi}\sqrt{(\tfrac{a}{2}+\tfrac{a}{2}\cos\theta)^2 + (\tfrac{a}{2}\sin\theta)^2}\cdot\frac{a}{2}\,d\theta =$$

$\sigma'(\theta) = (-\frac{a}{2}\sin\theta, \frac{a}{2}\cos\theta)$

$$\|\sigma'(\theta)\| = \sqrt{\frac{a^2}{4}\sin^2\theta + \frac{a^2}{4}\cos^2\theta} = \sqrt{\frac{a^2}{4}} \underset{a>0}{=} \frac{a}{2}$$

$$= \dots = \frac{a^2}{4}\sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\theta}\,d\theta =$$

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

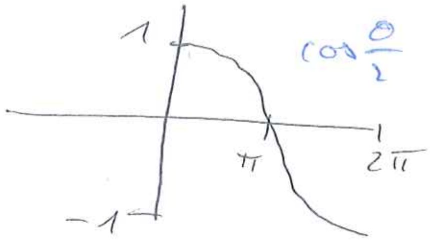
$$\cos u = \sqrt{\frac{1+\cos 2u}{2}}$$

$$|\cos u| = \sqrt{\frac{1+\cos 2u}{2}} \xrightarrow{u=\theta/2} \left|\cos\frac{\theta}{2}\right| = \sqrt{\frac{1+\cos 2\frac{\theta}{2}}{2}}$$

$$\sqrt{2}\left|\cos\frac{\theta}{2}\right| = \sqrt{1+\cos\theta}$$

$$= \frac{a^2}{4}\sqrt{2}\int_0^{2\pi}\sqrt{2}\left|\cos\frac{\theta}{2}\right|d\theta = \frac{a^2}{4}\sqrt{2}\sqrt{2}\cdot 2\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta =$$

$$= \dots = 2a^2$$



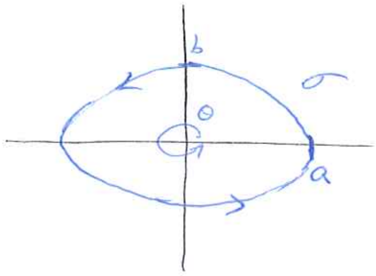
5. ARIKETA

iii) $$\oint_\sigma \overbrace{(x+y)}^{F_1}dx + \overbrace{(x-y)}^{F_2}dy =$$

$\sigma \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ elipsea (counterclockwise)

$\sigma(\theta) = (a\cos\theta, b\sin\theta)$

$\theta \in [0, 2\pi]$



$\sigma(0) = (a,0)$

$\sigma(\pi/2) = (0,b)$ $\Rightarrow$ ORIENTAZIOA MANTENDU $\Rightarrow$ $+$

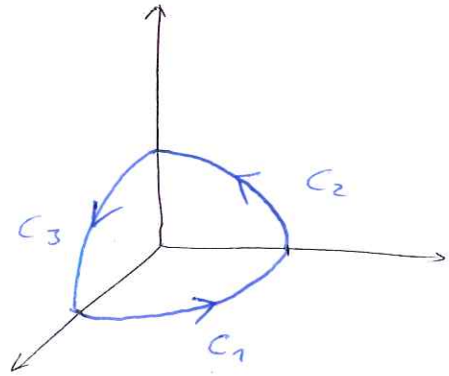
$$= +\int_0^{2\pi}(a\cos\theta+b\sin\theta)(-a\sin\theta)+(a\cos\theta-b\sin\theta)b\cos\theta\,d\theta =$$

$$= \int_0^{2\pi} ab\cos 2\theta - \frac{a^2+b^2}{2}\sin 2\theta\,d\theta = [\dots] = 0$$

iv) $$\int_c (y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz$$

iv) $\int_C (y^2-z^2)\,dx + (z^2-x^2)\,dy + (x^2-y^2)\,dz$

$x^2+y^2+z^2=1 \;\cancel{\;}\; \begin{matrix} x=0 \\ y=0 \\ z=0 \end{matrix} \quad \Rightarrow C$



$C_1 \to \sigma_1(\theta) = (\cos\theta, \sin\theta, 0)$

$\theta \in [0, \pi/2]$

$\sigma_1(0) = (1,0,0)$

$\sigma_1(\pi/2) = (0,1,0)$

$\sigma_1(\theta)$ orientazioa mantendu

$C_2 \to \sigma_2(\theta) = (0, \cos\theta, \sin\theta)$

$\theta \in [0, \pi/2]$

$\sigma_2(0) = (0,1,0)$ $\Rightarrow$ orientazioa mantendu

$\sigma_2(\pi/2) = (0,0,1)$

$C_3 \to \sigma_3(\theta) = (\sin\theta, 0, \cos\theta)$

$\theta \in [0, \pi/2]$

$\sigma_3(0) = (0,0,1)$ $\Rightarrow$ Orientazioa mantendu

$\sigma_3(\pi/2) = (1,0,0)$

$$I = \int_0^{\pi/2} (\sin^2\theta - 0)(-\sin\theta) + (0^2-\cos^2\theta)\cos\theta +$$

$$+ (\cos^2\theta - \sin^2\theta)\,0 \; d\theta +$$

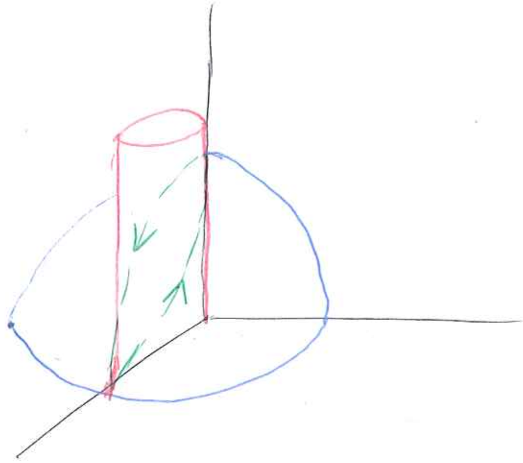
$$+ \int_0^{\pi/2} (\cos^2\theta - \sin^2\theta).0 + (\sin^2\theta - 0^2)(-\sin\theta) + (-\cos^2\theta)\cos\theta\, d\theta +$$

$$+ \int_0^{\pi/2} (0^2-\cos^2\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)0 + \sin^2\theta(-\sin\theta\, d\theta =$$

$$= [\dots] = -3\int_0^{\pi/2} \underbrace{\sin^3\theta}_{(1-\cos^2\theta)\sin\theta} + \underbrace{\cos^3\theta}_{(1-\sin^2\theta)\cos\theta}\, d\theta = [\dots] = -4$$

v) $\int_\sigma y^2dx + z^2dy + x^2dz$

$$\begin{cases} z \geq 0 \\ x^2+y^2+z^2=a^2 \\ x^2+y^2=ax \end{cases}$$



$(x-\frac{a}{2})^2+y^2=\frac{a^2}{2}$

Gure parametrizazioa

$(x-\frac{a}{2})^2+y^2=\frac{a^2}{4}$ bete behar

$\sigma(\theta)=(\frac{a}{2}+\frac{a}{2}\cos\theta, \frac{a}{2}\sin\theta, ?)$

$x^2+y^2+z^2=a^2$ bete behar

$(\frac{a}{2}+\frac{a}{2}\cos\theta)^2+(\frac{a}{2}\sin\theta)^2+z^2=a^2$

$z=\pm a\sqrt{\frac{1-\cos\theta}{2}} \overset{z\geq 0}{\Rightarrow} z=a\sqrt{\frac{1-\cos\theta}{2}}=a|\sin\frac{\theta}{2}|$

$\sigma(\theta)=(\frac{a}{2}+\frac{a}{2}\cos\theta, \frac{a}{2}\sin\theta, a|\sin\frac{\theta}{2}|)$

$\theta\in[0,2\pi]$

$\downarrow \theta\in[0,2\pi]$

$a\sin\frac{\theta}{2}$

$\sigma(0)=(a,0,0)$

$\sigma(\pi/2)=(\frac{a}{2},\frac{a}{2},a\frac{\sqrt{2}}{2})$ $\Rightarrow$ Orientazioa mantentzen du

$\sigma(\theta)=(\frac{a}{2}(1+\cos\theta), \frac{a}{2}\sin\theta, a\sin\frac{\theta}{2})\Rightarrow$

$\sigma(\theta)=a(\cos^2\frac{\theta}{2}, \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$ $\theta\in[0,2\pi]$

$\sigma(\theta)=a(\cos^2\theta, \sin\theta\cos\theta, \sin\theta)$ $\theta\in[0,\pi]$

$\int_\sigma \vec{F}ds=\int_0^\pi a^2\sin^2\theta\cos^2\theta\, 2a\cos\theta(-\sin\theta)+$

$+a^2\sin\theta(a(\cos^2\theta-\sin^2\theta))+a^3\cos^4\theta\cos\theta\, d\theta=$

$=[\dots]=\frac{-a^3\pi}{4}$

1. ARIKETA

iii) $\sigma(t) = (1, 3t^2, t^3)$ $[0,1]$

$$\ell(\sigma) = \int_a^b \|\sigma'(t)\| dt = \int_0^1 \|(0, 6t, 3t^2)\| dt =$$

$$= \int_0^1 \sqrt{6^2 t^2 + 9t^4}\, dt = \int_0^1 3t\sqrt{4+t^2}\, dt =$$

$$= \Big[(4+t^2)^{3/2}\Big]_0^1 = \boxed{5\sqrt{5} - 8}$$

iv) $\sigma(t) = (t, t, \frac{2}{3} t^{3/2})$ $t \in [t_0, t_1]$

$$\ell(\sigma) = \int_a^b \|\sigma'(t)\| dt = \int_{t_0}^{t_1} \|(1, 1, \sqrt{t})\| dt =$$

$$= \int_{t_0}^{t_1} \sqrt{2+t}\, dt = \left[\frac{2}{3}(2+t)^{3/2}\right]_{t_0}^{t_1} = \boxed{\frac{2}{3}\left[(2+t_1)^{3/2} - (2+t_0)^{3/2}\right]}$$

2. ARIKETA

i) $f(x,y,z) = y$, $\sigma(t) = (0,0,t)$ $0 \le t \le 1$

$$\int_\sigma f ds = \int_a^b f(\sigma(t)) \|\sigma'(t)\| dt = \int_a^b 0 \cdot \|\sigma'(t)\| dt \boxed{= 0}$$

iii) $f(x,y,z) = \cos z$, $\sigma(t) = (\sin t, \cos t, t)$ $0 \le t \le \pi$

$$\int_\sigma f ds = \int_a^b f(\sigma(t)) \|\sigma'(t)\| dt =$$

$$\|\sigma'(t)\| = \|(\cos t, -\sin t, 1)\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$$

$$= \int_0^\pi \cos t \cdot \sqrt{2}\, dt = \sqrt{2}\Big[\sin t\Big]_0^\pi \boxed{= 0}$$

iv) $\vec{F}(x,y,z) = (x,y,z)$, $\sigma(t) = (\cos t, \sin t, 0)$ $t \in [0, 2\pi]$

$$\int_\sigma \vec{F} dS = \int_c^b \vec{F}(\sigma(t)) \cdot \sigma'(t) dt =$$

$$= \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt =$$

$$= \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t)\, dt \boxed{= 0}$$

3. ARIKETA

$\nabla f(x,y,z) = (2xyz, x^2 z, x^2 y)$ $\quad f(1,1,1) = 1 \quad f(1,2,4) = ?$

$$\int_\sigma \nabla f = \int_\sigma 2xyz\, dx + x^2 z\, dy + x^2 y\, dz =$$

$f(1,1,1) = 1$

$f(1,2,4) = ?$ $\quad \Rightarrow \quad \sigma(t) = (1, 1+t, 1+3t) \quad t \in [0,1]$

$\sigma'(t) = (0, 1, 3)$

$$= \int_0^1 0 + 1 + 3t + 3(1+t)\, dt =$$

$$= \int_0^1 4 + 6t\, dt = \left[4t + 3t^2\right]_0^1 \boxed{= 7}$$

4. ARIKETA

i) $\int_{OA} (x+y)\, ds \qquad O(0,0) \to A(1,1) \qquad f(x,y) = x+y$

$\sigma(t,t) \qquad t \in [0,1] \qquad \sigma'(t) = (1,1)$

$$\int_\sigma f\, ds = \int_a^b f(\sigma(t)) \|\sigma'(t)\|\, dt =$$

$$= \int_0^1 2t \cdot \sqrt{2}\, dt = \sqrt{2}\left[t^2\right]_0^1 \boxed{= \sqrt{2}}$$

5. ARIKETA

ii) $\oint_\sigma x\, dy + y\, dx \qquad y = 2x^2 \qquad O(0,0) \to A(1,2)$

$\sigma(t) = (t, 2t^2)$

$$\int_0^1 t\, 4t + 2t^2\, dt = \int_0^1 6t^2\, dt = \left[2t^3\right]_0^1 \boxed{= 2}$$

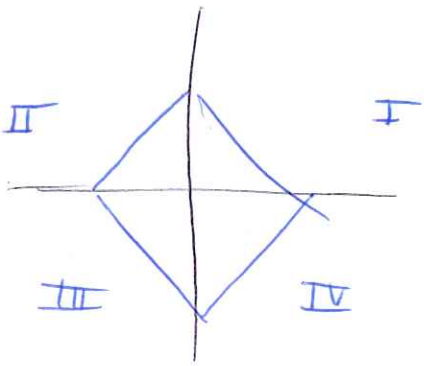
9

vi) $\int_{AB} \sin y\,dx + \sin x\,dy \qquad A(0,\pi) \to B(\pi,0)$

$$\sigma(t) = (t, \pi - t)$$

$$\int_0^\pi \underbrace{\sin(\pi-t)}_{\sin t} - \sin t\,dt = 0$$

vii) $\int_C \frac{dx+dy}{|x|+|y|} \qquad C \equiv (1,0), (0,1)(-1,0), (0,-1)$



I: $y,x>0 \Rightarrow \frac{1}{x+y} \wedge \sigma_1(t)=(t,1-t)$

II: $x<0, y>0 \Rightarrow \frac{1}{-x+y} \wedge \sigma_2(t)=(t-1,t)$

III: $x,y\leq 0 \Rightarrow \frac{1}{-x-y} \wedge \sigma_3(t)=(t-1,-t)$

IV: $x>0, y<0 \Rightarrow \frac{1}{x-y} \wedge \sigma_4(t)=(t,t-1)$

$$I = \int_0^1 \cancel{1-1}\,dt + \cancel{\int_0^1 -2\,dt} + \int_0^1 \cancel{1-1}\,dt + \cancel{\int_0^1 2\,dt} = 0$$

4. ARIKETA

iv) $\int_C \sqrt{2y^2+z^2}\,ds \qquad C \equiv x^2+y^2+z^2=c^2 \wedge x=y$

$$x^2+y^2+z^2=a^2 \Rightarrow (a\cos\theta\sin\varphi,\ a\sin\theta\sin\varphi,\ a\cos\varphi)$$

$$x=y \Rightarrow a\cos\theta\sin\varphi = a\sin\theta\sin\varphi \Rightarrow \theta=\frac{\pi}{4}$$

$$\sigma(\varphi) = \frac{\sqrt2}{2}a\sin\varphi,\ \frac{\sqrt2}{2}a\sin\varphi,\ a\cos\varphi$$

$$\|\sigma'(\varphi)\| = \sqrt{\left(\frac{\sqrt2}{2}a\cos\varphi\right)^2+\left(\frac{\sqrt2}{2}a\cos\varphi\right)^2 + a^2\sin^2\varphi} = a$$

$$\int_C f(\sigma(\varphi))\cdot\|\sigma'(\varphi)\| = \int_0^{2\pi}\sqrt{2\frac{1}{2}a^2\sin^2\varphi + a^2\cos^2\varphi}\cdot a\,d\varphi = \int_0^{2\pi} a^2\,d\varphi = \boxed{2\pi a^2}$$

GOGORATU

3. GAIA: $D \subset \mathbb{R}^2 \longrightarrow \iint_D f\,dA$ integral bikoitza

4. GAIA: $W \subset \mathbb{R}^3 \longrightarrow \iiint_W f\,dV$ integral hirukoitza

5. GAIA: Γ $\mathbb{R}^2$ / $\mathbb{R}^3$ kurba bat

$\int_\Gamma f\,ds$, $\int_\Gamma \vec{F}\,ds$ $\rightarrow$ $\vec{F}$-ren zirkulazioa Γ kurban zehar

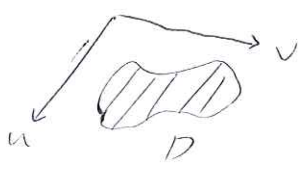
lerro integralak

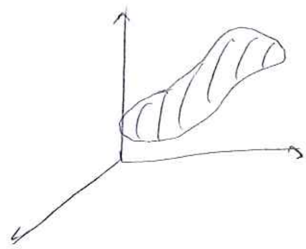
6. GAIA: GAINAZAL INTEGRALAK

6.1. GAINAZAL PARAMETRIZATUAK, AZALERA

– DEFINIZIOA:

$$\phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto \phi(u,v) = (x(u,v), y(u,v), z(u,v))$$





$\phi(D) = S$

$\phi \equiv$ GAINAZAL PARAMETRIZATUA /

$S = \phi(D)$ GAINAZALA

• ϕ C^1 bada $\Rightarrow$ S gainazal diferentziagarria / C^1 klasekoa

– ADIBIDEA:

1) Elipsoidea $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$\phi_1 : [0,2\pi] \times [0,\pi] \longrightarrow \mathbb{R}^3$$
$$(u,v) \longrightarrow \phi_1(u,v) = (a\cos u \sin v, b \sin u \sin v, c\cos v)$$

$$\phi_2 : [0,2\pi] \times [\tfrac{-\pi}{2}, \tfrac{\pi}{2}] \longrightarrow \mathbb{R}^3$$
$$(u,v) \longrightarrow \phi_2(u,v) = (a\cos u \cos v, b\sin u \sin v, c\cos v)$$

Elipsoidearen bi gainazal parametrizatu

2) Esfera (x_0, y_0, z_0) zentruarekin

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$$

$$\phi_1 : [0,2\pi] \times [0,\pi] \longrightarrow \mathbb{R}^3$$

$$(u,v) \longrightarrow \phi_1(u,v) = (x_0 + R\cos u \sin v, y_0 + R \sin u \sin v, z_0 R\cos v)$$

Esferaren gainazal parametrizatu bat

Begiratu ebela-n 119-120 orrialdeetan

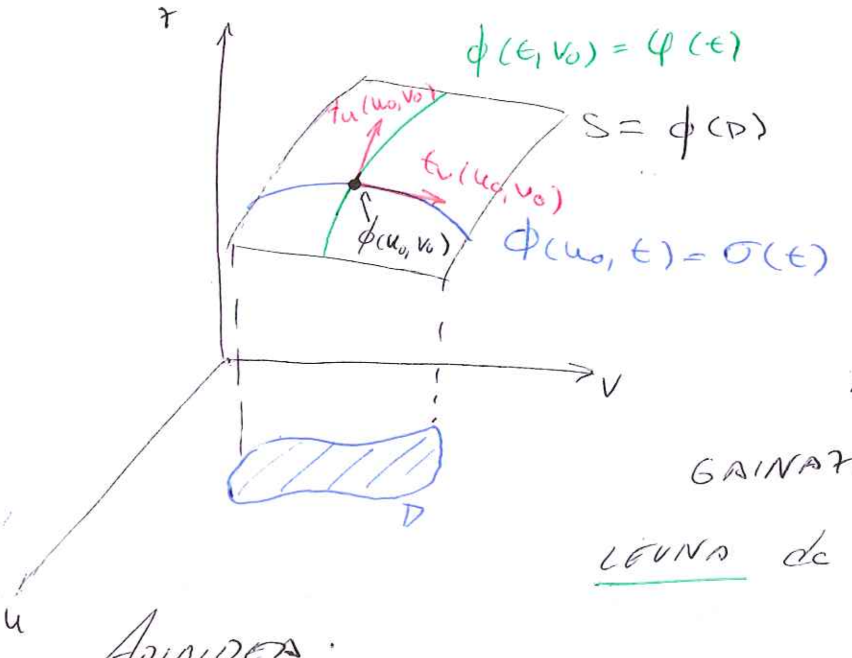
- $\phi : D \to \mathbb{R}^3$

$(u_0, v_0) \in D$ puntuan diferentziagarria

ABIADURA BEKTOREA

$$T_u(u_0,v_0) = \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right)$$

$$T_v(u_0,v_0) = \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right)$$



DEFINIZIOA:

$\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ C^1 gainazal parametrizatua $S = \phi(D)$

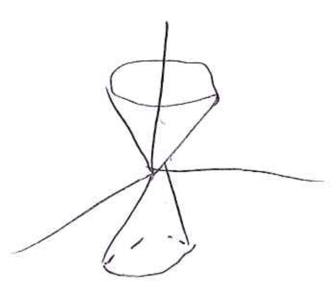
GAINAZALA $\phi(u_0 v_0)$ puntuan LEUNA da $t_u \times T_v(u_0,v_0) \neq \bar{0}$

Adibidea:

$x^2 + y^2 = z^2$ KONOA ⇒ Definitu

$\phi(u,v) = (u\cos v, u \sin v, u)$

$(u_0, v_0) = (0,0)$



$$T_u(0,0) = \left(\underbrace{\frac{\partial x}{\partial u}}_{\cos v}(0,0), \underbrace{\frac{\partial y}{\partial u}}_{\sin v}(0,0), \frac{\partial z}{\partial u}(0,0)\right) = (1,0,1)$$

$$T_v(0,0) = (-0\sin 0, 0\cos 0, 0) = (0,0,0)$$

$$T_u \times T_v(0,0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \Rightarrow \text{EZ DA LEUNA}$$

DEFINIZIOA

$\phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ C^1 gainazal parametrizatua $(u_0, v_0) \in D$ eta $\phi(u_0, v_0)$ puntuan gainazala leuna da $S = \phi(D)$-ren PLANO UKITZAILEA

$$\phi(u_0, v_0) = (x_0, y_0, z_0)$$

$$(x - x_0, y - y_0, z - z_0) \cdot \vec{N} = 0 \quad \text{non}$$

$$\vec{N} = T_u \times T_v(u_0, v_0)$$

OHARRA:

S gainazala $G(x,y,z)$ funtzio baten maila gainazal bat bezala adierazi badaiteke ($\exists K \in \mathbb{R}$ non S: $G = K$ ekuazioak definitzen du)

$$\Rightarrow \vec{n} = \frac{\nabla G}{\|\nabla G\|} \to \text{Unitarioa}$$

DEFINIZIOA

$\phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ gainazal parametrizatu leuna D osoan. $S = \phi(D)$

$$A(S) = \iint_D \|T_v \times T_u\| \, du\,dv \to S\text{-ren AZALERA}$$

INTEGRAL BIKOITZA

OHARRA: S zatika leuna

$$S = \bigcup_{i=1}^{k} S_i \qquad \mathring{S}_i \cap \mathring{S}_j = \phi \qquad \forall i \neq j$$

$$A(S) = \sum_{i=1}^{k} A(S_i)$$

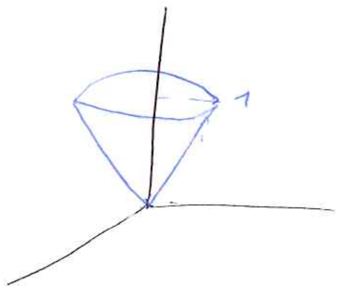
ADIBIDEA

$$\phi : \overbrace{[0,1]\times[0,2\pi]}^{D} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

Konoaren gainazal parametrizatua

$$\phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \rho)$$

$$x^2+y^2 = z^2 \qquad 0 \le z \le 1$$



$$T_\rho = (\cos\theta, \sin\theta, 1)$$

$$T_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$$

$$T_\rho \times T_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} = (-\rho\cos\theta, \rho\sin\theta, \rho)$$

$$\Rightarrow \|T_\rho \times T_\theta\| = \dots = \sqrt{2}\rho$$

$$A(S) = \iint_D \sqrt{2}\rho \, d\rho \, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{2}\rho \, d\rho \, d\theta = \dots = \sqrt{2}\pi$$

OHARRA:

$$\phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$(u,v) \longmapsto \phi(u,v) = (x,y,z)$$

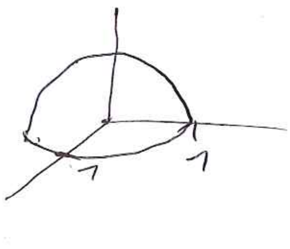
Aurkitu badaiteke $g : \tilde{D} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

C^1 klasekoa non $g(\tilde{D}) = S$

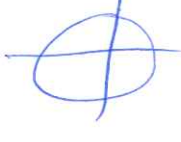
$$\Rightarrow A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2} \, du \, dv$$

ADIBIDEA:

$$S = \{x^2+y^2+z^2 = 1, \quad z \ge 0\}$$



Proiektatu OXY $\Rightarrow$



4

$$z = g(x,y) = \sqrt{1-x^2-y^2}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$g: D \subset \mathbb{R}^2 \to \mathbb{R}^3$

$g(D) = S$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\,dy =$$

$$= \int_0^{2\pi}\int_0^1 \frac{\rho\, d\rho\, d\theta}{\sqrt{1-\rho^2}} = [\dots] = 2\pi$$

6.2. LEHEN ETA BIGARREN MAILAKO GAINAZAL-INTEGRALAK

DEFINIZIOA:

$\phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ gainazal parametrizatu leuna,

$\phi(u,v) = (x(u,v), y(u,v), z(u,v))$ non $S = \phi(D)$.

f funtzio eskalar jarraitua.

$$\iint_S f\, dS = \iint_D f(\phi(u,v)) \cdot \|T_u \times T_v\|\, du\, dv \Leftarrow$$ f-ren S gaineko LEHEN MAILAKO GAINAZAL INTEGRALA

OHARRA:

1) $f(x,y,z) = 1 \to \iint_S 1\, dS = A(S)$

2) Bilatu ahal badugu $g: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ C^1 klasekoa non $g(D) = S$ $\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \cdot \sqrt{1+(g_x)^2+(g_y)^2}\, dx\,dy$ OXY proiektatu

OXZ-n proiektatu $\to$ $\iint_S f(x,y,z)\, dS = \iint_D f(x, g(x,z), z) \cdot \sqrt{1+(g_x)^2+(g_z)^2}\, dx\,dz$

[:]

ADIBIDEA

Kalkulatu $\iint_S z^2 dS$ non $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ $r=1$

Bi modu.

1) Definizioarekin

$\phi(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$

$\theta \in [0, 2\pi]$, $\varphi \in [0,\pi]$

$$T_\theta \times T_\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix} =$$

$$= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi, -\sin\varphi\cos\varphi)$$

$$\|T_\theta \times T_\varphi\| = [\cdots] = \sqrt{\sin^2\varphi} \underset{\sqrt{x^2}=|x|}{=} |\sin\varphi| \underset{\varphi\in[0,\pi]}{=} \sin\varphi$$

$$\iint_S z^2 dS = \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \cdot \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} \left[\frac{-\cos^3\varphi}{3}\right]_0^{\pi} d\theta = [\cdots] = \frac{4\pi}{3}$$

2) 2. Oharrarekin

$$S_1 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1, \ z\geq 0\}$$

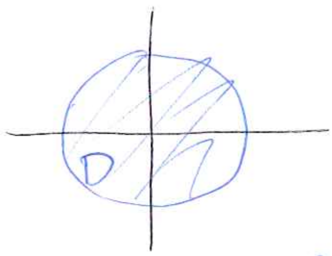
$$S_2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1, \ z\leq 0\}$$

$$S = S_1 \cup S_2$$

$$S_1 \to z = g_1(x,y) = \sqrt{1-x^2-y^2}$$

$$S_2 \to z = g_2(x,y) = -\sqrt{1-x^2-y^2}$$

$$g_1, g_2 : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$



$$g_1(D) = S_1$$
$$g_2(D) = S_2$$

$$\iint_S z^2 dS = \iint_{S_1} z^2 dS + \iint_{S_2} z^2 dS =$$

$$= \iint_D \left(\sqrt{1-x^2-y^2}\right)^2 \cdot \sqrt{1+(g_{1x})^2+(g_{1y})^2}\, dx\, dy +$$

$$+ \iint_D \left(-\sqrt{1-x^2-y^2}\right)^2 \cdot \sqrt{1+(g_{2x})^2+(g_{2y})^2}\, dx\, dy =$$

$$= 2 \cdot \iint_D (1-x^2-y^2) \cdot \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\, dx\, dy$$

$$= 2\iint_D (1-x^2-y^2)\cdot \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy = 2\iint_D (1-x^2-y^2)^{1/2}\,dx\,dy =$$

$x = \rho\cos\theta \quad \rho\in[0,1]$
$y = \rho\sin\theta \quad \theta[0,2\pi]$

$$= 2\int_0^{2\pi}\int_0^1 (1-\rho^2)^{1/2}\rho\,d\rho\,d\theta = [\dots] = \frac{4}{3}\pi$$

DEFINIZIOA

$\vec{F}:\mathbb{R}^3 \to \mathbb{R}^3$ funtzio bektorial jarraitua

$\phi: D\subset\mathbb{R}^2 \to \mathbb{R}^3$ gainazal parametrizatu leuna

$$\iint_\phi \vec{F}\,ds = \iint_D \vec{F}(\phi(u,v))\cdot T_u\times T_v\,du\,dv$$

↳ $\vec{F}$-ren ϕ-ren gaineko 2. MAILAKO GAINAZAL INTEGRALA

ADIBIDEA:

$\phi: D\in[0,2\pi]\times[0,\pi] \to \mathbb{R}^3$

$\phi(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$

$\vec{F}(x,y,z) = (x,y,z)$ (jarraitua).

Kalkulatu $\iint_\phi \vec{F}\,ds$

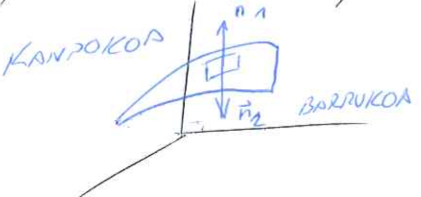
$T_\theta\times T_\varphi = (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)$

$$\iint_\phi \vec{F}\,ds = \iint_D \vec{F}(\phi(\theta,\varphi))\cdot T_\theta\times T_\varphi\,d\theta\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi} (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)\cdot(-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)\,d\varphi\,d\theta$$

$$= -\int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta = [\dots] = -4\pi$$

DEFINIZIOA: S gainazal norabidetua da, bi badittu, positiboa (kanpokoa) eta negatiboa (barrukoa)



MÖBIUS BANDA

KLEIN BOTILA

Bektore normal bakoitza gainazalaren alde batekin erlazionatzen da, horretarako S gainazal noranzkodunaren aldea finkatzeko, puntu bakoitzean $\vec{n}$ aukeratu behar dugu: $\vec{n}$-k S-ren alde positibotik kanpora begiratzen duena

DEFINIZIOA:

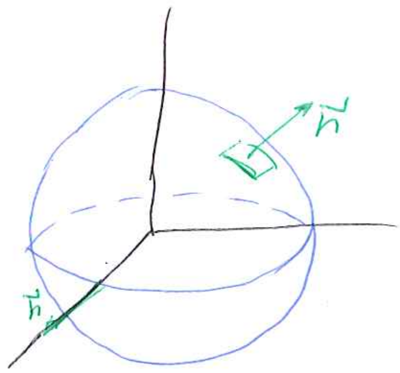
$\phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$

S gainazal parametrizatu leuna, ϕ bere parametrizazio bat eta $\vec{n}$ bektore normal unitarioa kanporantz begira

$$\vec{n} = \pm \frac{T_u \times T_v}{\|T_u \times T_v\|} \begin{cases} \oplus & \phi\text{-k orientazioa kontserbatu} \\ \ominus & \phi\text{-k orientazioa aldatu} \end{cases}$$

ADIBIDEA

$x^2+y^2+z^2=1$. esfera



Gogoratu

$G(x,y,z) = x^2+y^2+z^2$

esfera $G=1$ maila gainazala da

$$\Rightarrow \vec{n} = \frac{\nabla G}{\|\nabla G\|} = \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} =$$

$$= \frac{2(x,y,z)}{2\underbrace{\sqrt{x^2+y^2+z^2}}_{=1}} = (x,y,z)$$

$\vec{n}(1,0,0) = (1,0,0)$

$\phi(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$

$(\theta,\varphi) \in [0,2\pi] \times [0,\pi]$

$T_\theta \times T_\varphi = (-\sin^2\varphi\cos\theta, -\sin^2\varphi\cos\theta, \cos\varphi)$

$\|T_\theta \times T_\varphi\| = \sqrt{\sin^4\varphi + \cos^2\varphi}$

* 10.orr.

8

TEOREMA 6.1

S gainazal norabidetua $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ funtzio bektorial jarraitua ϕ_1 eta ϕ_2 bi gainazal parametrizatu izanik

$$\Rightarrow \begin{cases} (1)\ \phi_1 \text{ eta } \phi_2 \text{ orientazioa mantentzen badute} \\ \qquad \iint_{\phi_1} \vec{F} ds = \iint_{\phi_2} \vec{F} ds \\ (2)\ \phi_1 \text{ eta } \phi_2 \text{ orientazio ezberdina badute} \\ \qquad \iint_{\phi_1} \vec{F} ds = -\iint_{\phi_2} \vec{F} ds \end{cases}$$

DEFINIZIOA

- S gainazal norabidetua
- ϕ orientazioa mantentzen duen S-ren parametrizazioa
- $\vec{F}$ funtzio bektorial jarraitua

$$\iint_S \vec{F} ds = \iint_\phi \vec{F} ds \Rightarrow$$ $\vec{F}$-ren S gaineko 2. mailako GAINAZAL-INTEGRALA $\parallel$ $\vec{F}$-ren FLUXUA S-ren gainean

TEOREMA 6.2

- S gainazal norabidetua
- $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^2$ funtzio bektorial jarraitua

$$\Rightarrow \underbrace{\iint_S \vec{F} ds}_{\text{2. mailako gainazal-integrala}} = \underbrace{\iint \vec{F} \cdot \vec{n}\, ds}_{\text{1. mailako gainazal-integrala}}$$ non $\vec{n}$ S-ren bek. norm. unit.

OHARRAK:

1) S gainazala era honetan definitu ahal bada

$$\exists g: \Omega \subset \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto g(x,y) \qquad \vec{n} = \frac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$$

non $g(\Omega) = S$ $\quad \phi(x,y) = (x, y, g(x,y))$

$$\vec{F} = (F_1, F_2, F_3) \qquad \iint_S \vec{F} ds = \iint_S (-F_1 g_x - F_2 \cdot g_y + F_3)\, dx\, dy$$

$$\left[\frac{T_\theta \times T_\varphi}{\|T_\theta \times T_\varphi\|} \Bigg|_{\substack{(0,\pi/2)\\ \| \\ (1,0,0)}} = (-1,0,0) = -(1,0,0) \Rightarrow \text{ORIENTAZIOA ALDATU} \right]$$

$$\left(\begin{aligned} \iint \vec{F}\,ds &= \iint -F_1 g_x + F_2 - F_3 g_z \, dx\,dz \\ \iint \vec{F}\,ds &= \iint F_1 - F_2 g_y - F_3 g_z \, dy\,dz \end{aligned} \right)$$

2) $\vec{F} = (F_1, F_2, F_3)$ *jarraitua*

$$\iint \vec{F}\,ds = \iint F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

ADIBIDEA:

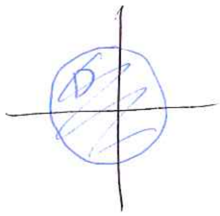
$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = 12,\ x^2+y^2 \leq 25\}$$

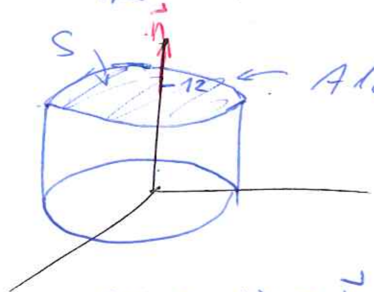
Kalkulatu $\iint_S \vec{F}\,ds$ non $\vec{F}(x,y,z) = (x,y,z)$

Hiru modu

1) Definizioarekin $\iint_S \vec{F}\,ds = + \iint \vec{F}\,ds$

$\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ non $\phi(x,y) = (x,y,12)$





← Alde positiboa

$T_x \times T_y = \dots = (0,0,1) = \vec{n}$

$\to \phi$ mantendu orientazioa

$$\iint \vec{F}\,ds \overset{def}{=} + \iint \vec{F}(\phi(x,y)) \cdot T_x \times T_y \, dx\,dy =$$

$$= \iint (x,y,12)\cdot(0,0,1)\,dx\,dy =$$

$$= \iint 12\,dx\,dy = 12 \underbrace{\iint 1\,dx\,dy}_{A(D)} = \boxed{12\pi \cdot 5^2}$$

2) Teorema 6.2

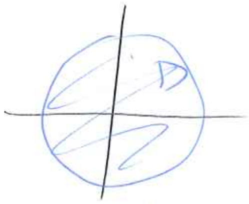
$$\iint_S \vec{F}\, dS = \iint \vec{F}\vec{n}\, dS = \iint (x,y,z)\cdot(0,0,1)\, dS =$$

$$= \iint_S z\, dS = 12\iint_S 1\, dS = \boxed{12\pi 5^2}$$

3) $g: D\subset \mathbb{R}^2 \to \mathbb{R}$

$z = g(x,y) = 12$

Projektato s Oxy planoch



$$\iint_S \vec{F}\, dS = \iint \Big(-\overset{F_1}{x}\underset{=0}{g_x} - \overset{F_2}{y}\underset{=0}{g_y} + \overset{F_3}{z}\Big)\,dx\,dy =$$

$$= \iint_S z\, dx\,dy = 12\iint_D dx\,dy = 12\cdot A(D) = \boxed{12\pi\cdot 5^2}$$

ANALISI BEKTORIALA ETA KONPLEXUA

6. Gaia: GAINAZAL-INTEGRALAK

Ariketak

1. Ondoren agertzen diren gainazalen bektore normal unitarioa kalkula ezazu eta emandako P puntuko plano ukitzailearen ekuazioa eman.

   (i) $x = \sin v,\ \ y = u,\ \ z = \cos v,\ \ u \in [-1, 3],\ \ v \in [0, 2\pi];\ P = (1, 0, 0)$.

   (ii) $x^2 + z^2 = r^2,\ \ -h \leq y \leq h\ \ h$ eta r positiboak izanik; $P = (0, 0, r)$.

   (iii) $x^2 + y^2 - z^2 = 1$ hiperboloidea; $P = (1, 1, 1)$.

   (iv) $\phi(r, \theta) = (r\cos\theta, r\sin\theta, \theta),\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 4\pi;\ P = (-1/2, 0, \pi)$.

   Em.: (i) $\vec{n} = (-\sin v, 0, -\cos v),\ \ x = 1$; (iii) $\vec{n} = (x/r, 0, z/r),\ \ z = r$;

   (iv) $\vec{n} = \dfrac{(x, y, -z)}{\sqrt{x^2 + y^2 + z^2}},\ \ z = x + y - 1$;

   (v) $\vec{n} = \dfrac{(\sin\theta, -\cos\theta, r)}{\sqrt{1 + r^2}},\ \ z = \pi - 2y$.

2. Ondorengo gainazalen azalerak kalkulatu:

   (i) $r(u, v) = ((a + b\cos u)\sin v, (a + b\cos u)\cos v, b\sin u),\ \ 0 < b < a,\ 0 \leq u \leq 2\pi,\ 0 \leq v \leq 2\pi$.

   (ii) $z \geq \sqrt{x^2 + y^2}$ eta $x^2 + y^2 + z^2 = 1$ arteko ebakidura.

   (iii) $x^2 + y^2 = ay$ zilindroaren barruan dagoen $x^2 + y^2 + z^2 = a^2$ esfera zatiaren azalera., $a > 0$ izanik.

   (iv) $z = x,\ \ z = 2x$ planoen artean dagoen $z^2 + y^2 = R^2$ zilindroaren zatia, $z \geq 0$ delarik.

   (v) $x^2 + y^2 + z^2 \leq 9$ esferaren barruan geratzen den $z = x^2 + y^2 - 7$ ekuazioko paraboloidearen zatia.

   Em.: (i) $4ab\pi^2$; (ii) $\pi(2 - \sqrt{2})$; (iii) $(2\pi - 4)a^2$; (iv) R^2; (v) $\frac{\pi}{6}(33\sqrt{33} - 21\sqrt{21})$.

3. Izan bitez $0 < a < b$ eta $f\colon [a, b] \to \mathbf{R}$ C^1 motako funtzio positiboa.

   (i) f-ren grafikoa y ardatzarekiko biraraztean sortzen den gainazalaren parametrizazioa eman eta frogatu bere azalera honela kalkula daitekela:

   $$A = 2\pi \int_a^b x\sqrt{1 + (f'(x))^2}\,dx.$$

   (ii) f-ren grafikoa x ardatzarekiko biraraztean sortzen den gainazalaren parametrizazioa eman eta frogatu bere azalera kalkulatzeko hurrengo formula erabili daitekela:

   $$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

4. Ondorengo (eremu eskalarren) gainazal-integralak kalkula itzazu:

(i) $\displaystyle\iint_S xyz\,dS$, S gainazala $(1,0,0)$, $(0,2,0)$ eta $(0,1,1)$ erpinetako triangelua izanik.

\+ (ii) $\displaystyle\iint_S x^2 z\,dS$, non S gainazala $x+z=R$ planoaren gainean geratzen den $x^2+y^2+z^2=R^2$ ekuazioko esferaren zatia den.

\+ (iii) $\displaystyle\iint_S z\,dS$, S gainazala $D=\{(x,y)\in\mathbf{R}^2 : x^2+y^2\leq 1\}$ eremuan definituriko $z=x^2+y^2$ funtzioaren grafikoa izanik.

\+ (iv) $\displaystyle\iint_S (x+y+z)\,dS$, S $x^2+y^2+z^2=a^2$, $z\geq 0$ esferaerdia izanik.

\+ (v) $\displaystyle\iint_S \frac{dS}{(x+y+1)^2}$, S gainazala $x+y+z\leq 1$, $x\geq 0$, $y\geq 0$, $z\geq 0$ tetraedroaren muga bada.

Em.: (i) $\frac{\sqrt{6}}{30}$; (ii) $\frac{5\sqrt{2}R^5\pi}{64}$; (iii) $\frac{\pi}{60}(25\sqrt{5}+1)$; (iv) πa^3; (v) $\frac{3-\sqrt{3}}{2}+(\sqrt{3}+1)\ln 2$.

5. Hurrengo (eremu bektorialen) gainazal-integralak kalkulatu:

\+ (i) $\displaystyle\iint_S (y-z)dydz+(z-x)dzdx+(x-y)dxdy$, S $x^2+y^2=z^2$, $0\leq z\leq h$ konoaren kanpoaldea da.

\+ (ii) $\displaystyle\iint_S \frac{dydz}{x}+\frac{dzdx}{y}+\frac{dxdy}{z}$, S gainazala $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ekuazioko elipsoidearen kanpoaldea da.

(iii) $\displaystyle\iint_S x^2dydz+y^2dzdx+z^2dxdy$, S $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ ekuazioa duen esferaren kanpoaldea da.

\+ (iv) $\displaystyle\iint_S (x,y,-z)\cdot dS$, S $[0,1]\times[0,1]\times[0,1]$ kubo unitarioaren mugaren kanpoaldea izanik.

\- (v) $\displaystyle\iint_S (xy,yz,zx)\cdot dS$, S 1. oktanteko $x^2+y^2+z^2=1$ esfera-zatiaren kanpoko aurpegia da.

\+ (vi) $\displaystyle\iint_S \vec{F}\cdot dS$, $S=\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+z^2=2,\ z\geq 1\}$ gainazala izanik, goranzko orientazioarekin, eta $\vec{F}(x,y,z)=(xz,xy,yz)$.

\+ (vii) $\displaystyle\iint_S (x,y,z)\cdot dS$ baldin eta S $z=1-x^2-y^2$ eta $x+z=1$ gainazalek mugatzen duten solidoaren muga bada, kanporako orientazioarekin.

Em.: (i) 0; (ii) $4\pi\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)$; (iii) $\frac{8\pi r^3}{3}(a+b+c)$; (iv) 1; (v) $\frac{3\pi}{16}$; (vi) $\frac{\pi}{4}$; (vii) $\frac{3\pi}{32}$.

6. GAINAZAL - INTEGRALAK

ARIKETAK

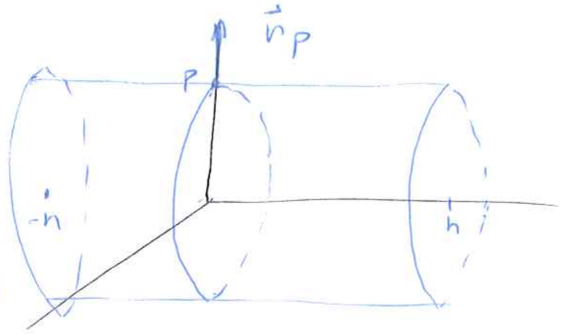
1. ARIKETA

ii) $x^2+z^2=r^2$

$-h \leq y \leq h$

$h, r > 0$

$P=(0,0,r)=\phi(0,\frac{\pi}{2})$



Bi era:

1) $\phi(t,\theta)=(r\cos\theta,\ t,\ r\sin\theta)$

$t \in [-h,h]$ $\rightarrow$ $T_t \times T_\theta = \dots = (r\cos\theta,\ 0,\ r\sin\theta)$

$\theta \in [0,2\pi]$ $\quad \|T_t \times T_\theta\| = r$

$$\vec{n}=\frac{T_t \times T_\theta}{\|T_t \times T_\theta\|}=(\cos\theta,\ 0,\ \sin\theta)_P \underset{\theta=\frac{\pi}{2}}{=} \boxed{(0,0,1)}$$

Plano ukitzailea $\rightarrow P=(0,0,r)$

$$(x-0,\ y-0,\ z-r)\cdot \underset{(0,0,1)}{\vec{n}} = 0 \Rightarrow z-r=0 \Rightarrow \boxed{z=r}$$

2) Gainazala $\rightarrow$ maila gainazal bat bezala adierazi

$$G(x,y,z)=x^2+z^2=\underset{\text{ETEA}}{r^2}$$

$$\vec{n}=\frac{\nabla G}{\|\nabla G\|}=\frac{(2x,0,2z)}{\sqrt{4x^2+4z^2}}=\frac{(x,0,z)}{\sqrt{x^2+z^2}}=\frac{(x,0,z)}{r}\Big|_{P=(0,0,r)}$$

$$\Rightarrow \vec{n}=(0,0,1)$$

iii) $x^2+y^2-z^2=1$ hiperboloidea $\quad P(1,1,1) \xrightarrow{?} P(u,v)$

Bi modu

1) $\phi(u,v)=(\cos u\cosh v,\ \sin u\cosh v,\ \sinh v)$

$T_u \times T_v = \dots$

$$2)\ G(x,y,z) = \underbrace{x^2+y^2-z^2=1}_{\text{HIPERBOLOIDEA}}$$

$$\Rightarrow \vec{n} = \frac{\nabla G}{\|\nabla G\|} = \frac{(2x, 2y, -2z)}{\sqrt{4x^2+4y^2+4z^2}} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\Bigg|_{P(1,1,1)} =$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

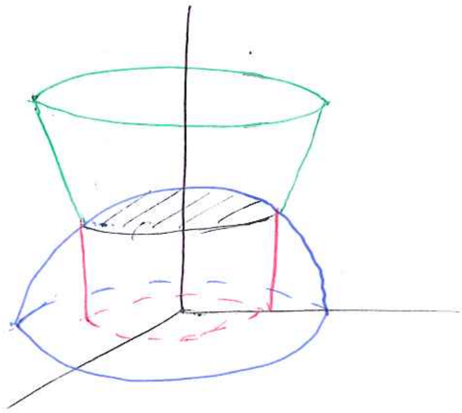
Plano ukitzailea

$$(x-1, y-1, z-1)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = 0$$

$$\boxed{x+y-z=1}$$

2. ARIKETA

$$ii)\ \begin{cases} z \geq \sqrt{x^2+y^2} & \text{KONOA} \\ x^2+y^2+z^2=1 & \text{ESFERA} \end{cases} \qquad \text{EBAKIDURA} \Rightarrow x^2+y^2=\frac{1}{2}$$

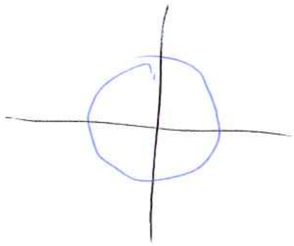


$$A(S) = \iint_D \sqrt{1+g_x^2+g_y^2}\,dx\,dy$$

$$x^2+y^2+z^2=1 \Rightarrow z = g(x,y) = \sqrt{1-x^2-y^2}$$

$$g: D \subset \mathbb{R}^2 \to \mathbb{R}$$

S-ren OXY planoan proiekzioa



$$A(S) = \iint_D \sqrt{1+g_x^2+g_y^2}\,dx\,dy =$$

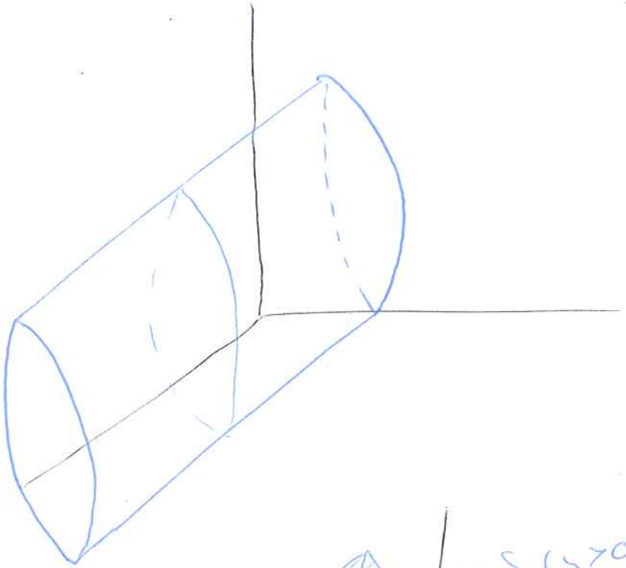
$$= \iint \sqrt{1+\left(\frac{-2x}{2\sqrt{1-x^2-y^2}}\right)^2+\left(\frac{-2y}{2\sqrt{1-x^2-y^2}}\right)^2}\,dx\,dy =$$

$$= [\dots] = \iint_D \sqrt{\frac{1}{1-x^2+y^2}}\,dx\,dy = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \sqrt{\frac{1}{1-\rho^2}}\,\rho\,d\rho\,d\theta =$$

$$= [\dots] = \pi\left[2-\sqrt{2}\right]$$

$$\begin{cases} x = \rho\cos\theta & \theta\in[0,2\pi) \\ y = \rho\sin\theta & \rho\in[0,1/\sqrt{2}] \end{cases} \quad |J| = \rho$$

iv) $\begin{cases} z = x \\ z = 2x \\ z^2 + y^2 = R^2 \\ z \geq 0 \end{cases}$

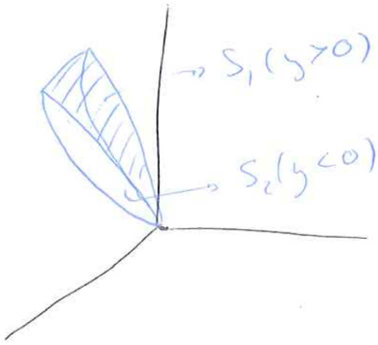


S-ren zilindroaren zati bat da $\to z^2+y^2=R^2$

$A(S) = 2\,A(S_1)$

$\downarrow$

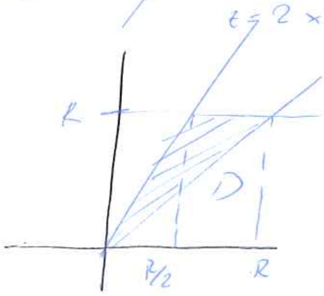
$y = +\sqrt{R^2-z^2} = g(x,z)$



$y = g(x,z) = \sqrt{R^2-z^2}$

$g : D \subset \mathbb{R}^2 \to \mathbb{R}$

OXZ proiektatu



$$A(S) = 2A(S_1) = 2\iint_D \sqrt{1+g_x^2+g_z^2}\,dx\,dz$$

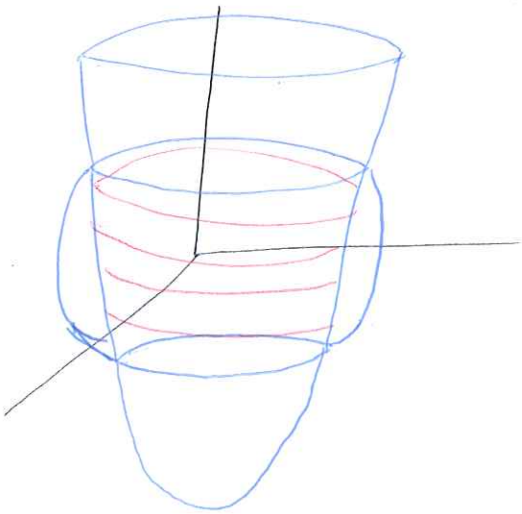
$$= 2\iint_D \sqrt{1+\frac{z^2}{R^2-z^2}}\,dx\,dz =$$

$$= 2\iint_D \sqrt{\frac{R^2}{R^2-z^2}}\,dx\,dz = 2R\iint_D \frac{1}{\sqrt{R^2-z^2}}\,dx\,dz =$$

$$= 2R\int_0^R\int_{z/2}^{z} \frac{1}{\sqrt{R^2-z^2}}\,dx\,dz = 2R\int_0^R \frac{1}{\sqrt{R^2-z^2}}\left[x\right]_{z/2}^{z} dz =$$

$$= [\ldots] = R^2$$

v) $\begin{cases} x^2+y^2+z^2 \leq 9 \\ z = x^2+y^2-7 \end{cases}$

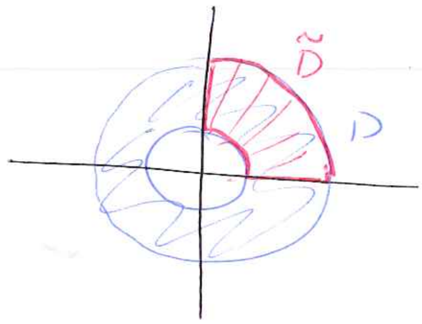


$z = g(x,y) = x^2+y^2-7$

OXY planoan proiektatu

EBAKI PUNTUAK

$$\begin{cases} x^2+y^2+z^2=9 \\ z=x^2+y^2-7 \end{cases} \rightarrow \begin{matrix} 8=x^2+y^2 \\ 5=x^2+y^2 \end{matrix}$$



$g: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$

$g(D)=S$

$$A(S)=\iint_D \sqrt{1+g_x^2+g_y^2}\,dx\,dy=$$

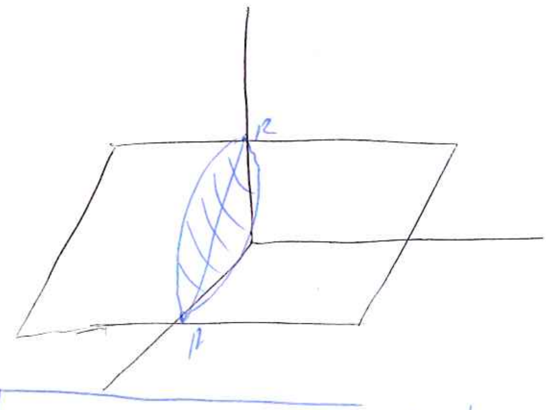
$$=4\iint_{\tilde{D}} \sqrt{1+(2x)^2+(2y)^2}\,dx\,dy= \quad \begin{matrix} x=\rho\cos\theta \\ y=\rho\sin\theta \\ \theta\in[0,\pi/2] \\ \rho\in[\sqrt{5},\sqrt{8}] \end{matrix}$$

$$=4\cdot\int_0^{\pi/2}\int_{\sqrt{5}}^{\sqrt{8}} \sqrt{1+4\rho^2}\cdot\rho\,d\rho\,d\theta=[\ldots]=\frac{\pi}{6}\left[\sqrt{33}^3-\sqrt{21}^3\right]$$

4. ARIKETA

ii) $\iint_S x^2 z\,dS$

$\begin{cases} x+z=R\text{-ren gainean} \\ x^2+y^2+z^2=R^2\text{-ren zatia} \end{cases}$

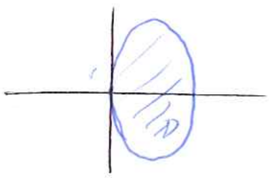


$$\iint_S x^2z\,dS=\iint_D f(x,y,g(x,y))\cdot\sqrt{1+g_x^2+g_y^2}\,dx\,dy=$$

$$=\iint_D x^2\sqrt{R^2-x^2-y^2}\cdot\sqrt{1+\left(\frac{-2x}{2\sqrt{R^2-x^2-y^2}}\right)^2+\left(\frac{-2y}{2\sqrt{R^2-x^2-y^2}}\right)^2}\,dx\,dy=$$

$$=\ldots=\iint_D Rx^2\,dx\,dy=$$

$D\rightarrow$ Proiektatu S OXY-n



$$\begin{cases} z=R-x \\ x^2+y^2+z^2=R^2 \end{cases} \rightarrow \frac{(x-R/2)^2}{R^2/4}+\frac{y^2}{R^2/2}=1 \Rightarrow$$

$$\begin{cases} x = \frac{R}{2} + \frac{R}{2}\rho\cos\theta \\ y = 0 + \frac{R}{\sqrt{2}}\rho\sin\theta \end{cases} \qquad |J| = \rho\,\frac{R}{2}\,\frac{R}{\sqrt{2}}$$

$$\theta \in [0, 2\pi] \qquad \rho \in [0, 1]$$

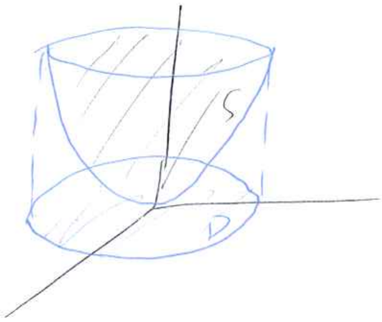
$$k\int_0^{2\pi}\int_0^1 R\left(\frac{R}{2} + \frac{R}{2}\rho\cos\theta\right)^2 \rho\,\frac{R^2}{2\sqrt{2}}\,d\rho\,d\theta = \dots = \frac{5\sqrt{2}R^5\pi}{64}$$

iii) $\iint_\Sigma z\,dS =$

$z = g(x,y) = x^2 + y^2$

$g : D \subset \mathbb{R}^2 \to \mathbb{R}$

$\Rightarrow$



$$= \iint_D (x^2+y^2)\sqrt{1+g_x^2+g_y^2}\,dx\,dy =$$

$$= \iint_D (x^2+y^2)\sqrt{1+4x^2+4y^2}\,dx\,dy =$$

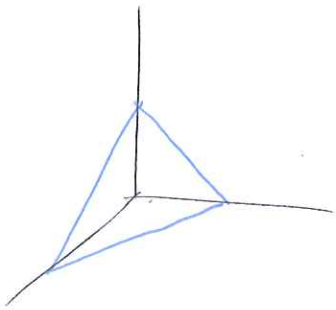
$$= \int_0^{2\pi}\int_0^1 \rho^2\sqrt{1+4\rho^2}\,\rho\,d\rho\,d\theta =$$

$$= \int_0^{2\pi}\left[\int_0^1 \rho^2\sqrt{1+4\rho^2}\,\rho\,d\rho\right]d\theta =$$

$$\left[\begin{matrix} 1+4\rho^2 = t^2 \\ 8\rho\,d\rho = 2t\,dt \end{matrix}\right]$$

$$= [\dots] = \frac{\pi}{60}\left[25\sqrt{5}+1\right]$$

v) $\iint_\Sigma \frac{1\,dS}{(x+y+1)^2} = f(x,y,z)$



$S = S_1 \cup S_2 \cup S_3 \cup S_4 \to y = 0$

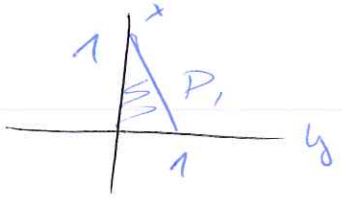
S_1: $z = 1-x-y$, S_2: $x = 0$, S_3: $z = 0$

$\underline{S_1}$

$g_1 : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad \overset{Oxy}{\Longrightarrow}$

$z = g(x,y) = 1 - x$

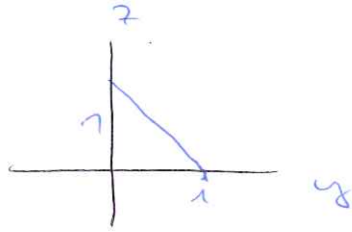
$g_1(D_1) = S_1$



$\underline{S_2}$

$x = g(y,z) = 0$

$g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad g_2(D_2) = S_2$

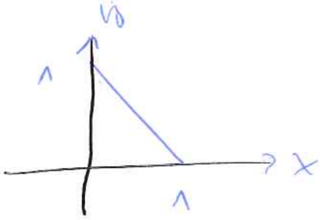


$\underline{S_3}$

$z = g_3(x,y) = 0$

$g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$g_3(D_3) = S_3$

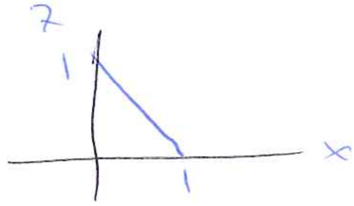


$\underline{S_4}$

$y = g(x,z) = 0$

$g_4 : D_4 \longrightarrow \mathbb{R}$

$g_4(D_4) = S_4$



$$\iint_S \frac{1}{(x+y+1)^2}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4}$$

$$= \iint_{D_1} \frac{1}{(x+y+1)^2} \sqrt{1+(-1)^2+(-1)^2}\, dx\, dy \; +$$

$$+ \iint_{D_2} \frac{1}{(0+y+1)^2} \sqrt{1+0}\, dy\, dz + \iint_{D_3} \frac{1}{(x+y+1)^2} \sqrt{1}\, dx\, dy \; +$$

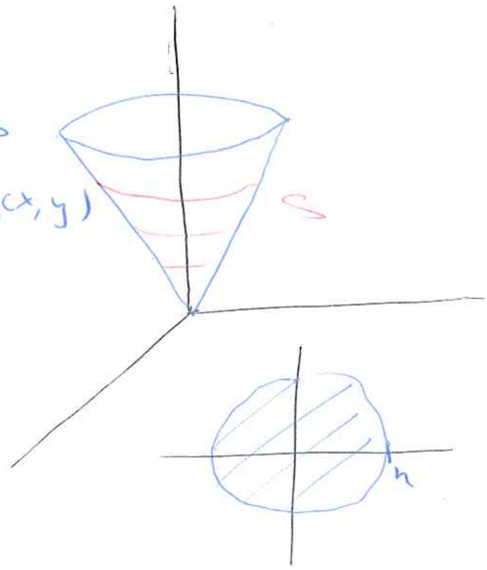
$$+ \iint_{D_4} \frac{1}{(x+0+1)^2} \sqrt{1}\, dx\, dz = [\dots] = \frac{3-\sqrt{3}}{2} + \ln 2 (-1+\sqrt{2})$$

5. ARIKETA

2. MAILAKO GAINAZAL INTEGRALA

i) $\iint_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy$

$\vec{F}(x,y,z) = (\underbrace{y-z}_{F_1}, \underbrace{z-x}_{F_2}, \underbrace{x-y}_{F_3})$



$x^2+y^2=z^2 \longrightarrow$

$\hookrightarrow z = \sqrt{x^2+y^2} = g(x,y)$

OXY planoan

$g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$g(D) = S$

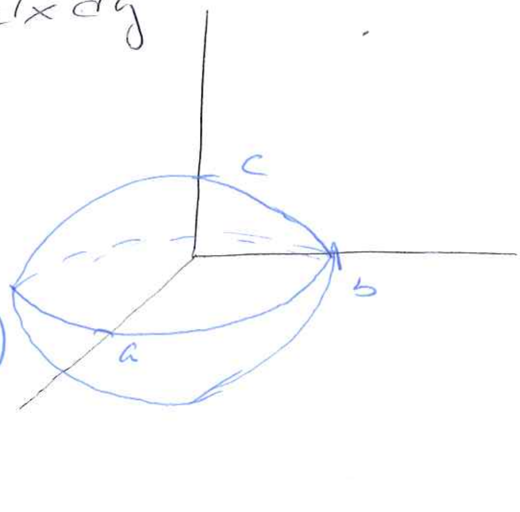
$D = \begin{cases} z = h \\ x^2+y^2 = z^2 \end{cases} \Rightarrow x^2+y^2 = h^2$

$$I = \iint_D -F_1 g_x - F_2 g_y + F_3 \, dx\,dy =$$

$$= \iint_D -(y-z)\frac{2x}{2\sqrt{x^2+y^2}} - (z-x)\frac{2y}{2\sqrt{x^2+y^2}} + (x-y)\,dx\,dy =$$

$$[\dots] = \iint_D 2x - 2y \, dx\,dy = \begin{cases} = [\text{POLARRAK}] = \dots = 0 \\ = \left[\begin{matrix} f(x,y) \text{ BAKOITIA} \\ \text{y ALDAGAIETAN} \end{matrix}\right] \overset{\text{PROP } 3.6}{=} 0 \end{cases}$$

ii) $\iint_S \frac{1}{x}\,dy\,dz + \frac{1}{y}\,dz\,dx + \frac{1}{z}\,dx\,dy$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

S-ren parametrizazioa

$\phi(\theta,\varphi) = (a\cos\theta\sin\varphi, b\sin\theta\sin\varphi, c\cos\varphi)$

$\theta \in [0, 2\pi]$

$\varphi \in [0, \pi]$

$T_\theta = (-a\sin\theta\sin\varphi,\ b\cos\theta\sin\varphi,\ 0)$

$T_\varphi = (a\cos\theta\cos\varphi,\ b\sin\theta\cos\varphi,\ -c\sin\varphi)$

$T_\theta \times T_\varphi = (-bc\cos\theta\sin^2\varphi,\ -ac\sin\theta\sin^2\varphi,\ -ab\sin\varphi\cos\varphi)$

$P_{(a,0,0)} \longrightarrow \begin{cases} \theta = 0 \\ \varphi = \pi/2 \end{cases} \Rightarrow (T_\theta \times T_\varphi)_{(0,\pi/2)} = (-bc,0,0) = bc(-1,0,0)$

$\Rightarrow \phi$ Ez du orientazioa aldatzen

$$I = -\iint_D \vec{F}(\phi(\theta,\varphi)) \cdot T_\theta \times T_\varphi =$$

$$= -\int_0^{2\pi}\int_0^{\pi} \frac{-bc\cos\theta\sin^2\varphi}{a\cos\theta\sin\varphi} + \frac{-ac\sin\theta\sin^2\varphi}{b\sin\theta\sin\varphi} + \frac{-cb\sin\varphi\cos\varphi}{c\cos\varphi}\,d\varphi\,d\theta =$$

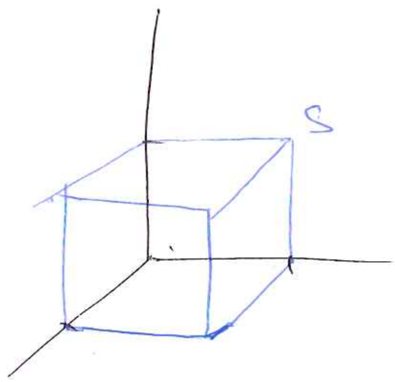
$$= -\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right) \cdot \int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta = [\dots] =$$

$$= \left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right) 4\pi$$

iv) $\iint_S (x, y, -z)\,dS \qquad S \equiv [0,1]\times[0,1]\times[0,1]$

G GAINAZALA

Ez da C^1ko leuna



$S_1: z = 0$

$\vec{n}_1 = (0,0,-1)$

$S_2: z = 1$

$\vec{n}_2 = (0,0,1)$

$S_3: y = 0$

$\vec{n}_3 = (0,-1,0)$

$S_4: y = 1$

$\vec{n}_4 = (0,1,0)$

$S_5: x = 0$

$\vec{n}_5 = (-1,0,0)$

$S_6: x = 1$

$\vec{n}_6 = (1,0,0)$

$$I \overset{\text{TEOR 6.2}}{=} \sum_{i=1}^{6} \iint_{S_i} \underbrace{\vec{F}\cdot\overset{\text{KTEAK DIRELAKO}}{\vec{n}_i}\, dS}_{\text{LEHEN MAILAKOA}} =$$

$$= \iint_{S_1} (x,y,-z)(0,0,-1)\,dS + \iint_{S_2} (x,y,-z)(0,0,1)\,dS +$$

$$+ \iint_{S_3} (x,y,-z)(0,-1,0)\,dS + \iint_{S_4} (x,y,-z)(0,1,0)\,dS +$$

$$+ \iint_{S_5} (x,y,-z)(-1,0,0)\,dS + \iint_{S_6} (x,y,-z)(1,0,0)\,dS =$$

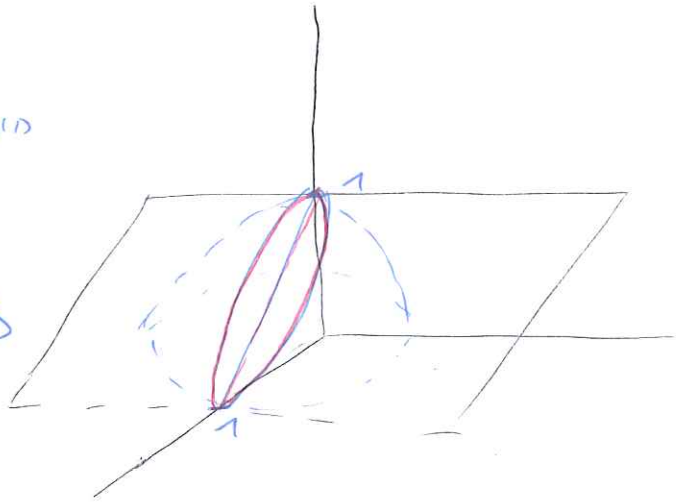
$$= \iint_{S_1} \underset{0}{z}\,dS + \iint_{S_2} -\underset{1}{z}\,dS + \iint_{S_3} -\overset{0}{y}\,dS + \iint_{S_4} \overset{1}{y}\,dS + \iint_{S_5} -\overset{0}{x}\,dS + \iint_{S_6} \overset{1}{x}\,dS =$$

$$= -\iint_{S_2} 1\,dS + \iint_{S_4} 1\cdot dS + \iint_{S_6} 1\,dS =$$

$$= -\overset{1}{A(S_2)} + \overset{1}{A(S_4)} + \overset{1}{A(S_6)} = \boxed{1}$$

viii) $\iint_S (x,y,z)\,dS$ *Konporantz*

$$S \equiv \begin{cases} z = 1 - x^2 - y^2 \Rightarrow \text{PARABOLOID} \\ x + z = 1 \Rightarrow \text{PLANOA} \end{cases}$$



$$I = \iint_{S_1} (x,y,z)\,dS + \iint_{S_2} (x,y,z)\,dS$$

$\underline{\underline{S_1}}$ $z = g(x,y) = 1 - x^2 - y^2$

$g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\xrightarrow{OXY}$ $\begin{cases} z = 1 - x^2 - y^2 \\ x + z = 1 \end{cases}$

$\longrightarrow [\ldots] \Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$

$$I_{S_1} = \iint_D -x(-2x) - y(-2y) + z\, dx \overset{\text{POLARRAK}}{=} \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} (1+\rho^2)\rho\, d\rho\, d\theta =$$

$$= [\ldots] = \frac{11}{32}\pi$$

$$I_2 \overset{\text{TEOR } 6.2}{=} \iint_{S_2} (x,y,z)\cdot\vec{n}\, dS =$$

$$G(x,y,z) = x+z = 1$$

$$\vec{n} = \frac{\nabla G}{\|\nabla G\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \qquad z = g(x,y) = 1-x$$

$$g: D \subseteq \mathbb{R}^2 \to \mathbb{R} \quad g(D) = S_2$$

BEHERANTZ

$$= \iint_{S_2} -\frac{1}{\sqrt{2}} \underbrace{(x+z)}_{1} dS = -\frac{1}{\sqrt{2}} A(S_2) \overset{\downarrow}{=}$$

$$= -\frac{1}{\sqrt{2}} \iint_D \overbrace{\sqrt{1+g_x^2+g_y^2}}^{\sqrt{2}}\, dx\, dy = \frac{-\sqrt{2}}{\sqrt{2}} \underbrace{\iint_D dx\,dy}_{\pi\left(\frac{1}{2}\right)^2} = \frac{-\pi}{4}$$

$$I = I_1 + I_2 = \frac{11\pi}{32} - \frac{\pi}{4} = \frac{3\pi}{32}$$

1. Ariketa

i) $x = \sin v, \; y = u, \; z = \cos v \qquad u \in [-1,3], \; v \in [0, 2\pi]$

$$P = (1,0,0)$$

$$\phi(u,v) = (\sin v, u, \cos v) \leftarrow P = \left(0, \frac{\pi}{2}\right)$$

$$T_u = (0,1,0) \qquad T_v = (\cos v, 0, -\sin v)$$

$$T_u \times T_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ \cos v & 0 & -\sin v \end{vmatrix} = (-\sin v, 0, -\cos v)$$

$$\vec{N} = T_u \times T_v (u_0, v_0) = (-\sin v, 0, -\cos v)\Big|_{(0,\frac{\pi}{2})} = (-1,0,0)$$

$$\vec{n} = \frac{T_u \times T_v}{\|T_u \times T_v\|} \Rightarrow \boxed{\vec{n} = (-\sin v, 0, -\cos v)}$$

$$(x-1, y, z)\cdot(-1,0,0) = 0 \Rightarrow -x+1 = 0$$

$$\boxed{x = 1}$$

iv) $\phi(r,\theta)=(r\cos\theta, r\sin\theta, \theta)$ $\quad r\in[0,1]\quad \theta\in[0,4\pi]$

$P(-1/2, 0, \pi) \longrightarrow P=(\frac{1}{2},\pi)$

$T_r=(\cos\theta, \sin\theta, 0) \qquad T_\theta=(-r\sin\theta, r\cos\theta, 1)$

$$T_r\times T_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, r)$$

$\vec{N} = T_r\times T_\theta(\frac{1}{2},\pi) = (0, +1, \frac{1}{2})$

$(x+\frac{1}{2}, y, z-\pi)\cdot(0,1,\frac{1}{2})=0 \Rightarrow y+\frac{1}{2}(z-\pi)=0$

$$\boxed{z=y-2\pi}$$

$\|T_r\times T_\theta\| = \sqrt{\sin^2\theta+\cos^2\theta+r^2} = \sqrt{1+r^2}$

$$\vec{n} = \frac{T_r\times T_\theta}{\|T_r\times T_\theta\|} \Rightarrow \boxed{\vec{n} = \frac{(\sin\theta, -\cos\theta, r)}{\sqrt{1+r^2}}}$$

2. ARIKETA

i) $r(u,v) = ((a+b\cos u)\sin v, (a+b\cos u)\cos v, b\sin u)$

$0<b<a \qquad u\in[0,2\pi] \qquad v\in[0,2\pi]$

$$A(S) = \iint_D \|T_v\times T_u\|\,du\,dv$$

$T_v = ((a+b\cos u)\cos v, -(a+b\cos u)\sin v, 0)$

$T_u = (-b\sin u\sin v, -b\sin u\cos v, b\cos u)$

$$T_v\times T_u = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (a+b\cos u)\cos v & -(a+b\cos u)\sin v & 0 \\ -b\sin u\sin v & -b\sin u\cos v & b\cos u \end{vmatrix} =$$

$=(-(a+b\cos u)\sin v\, b\cos u, -(a+b\cos u)\cancel{\cos v}\,b\cos u, (a+b\cos u)\,b\sin u) =$

$= -(a+b\cos u)b(\sin v\cos u, \cos v\cos u, \sin u)$

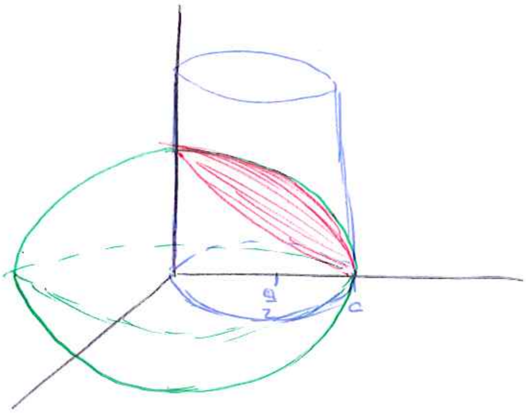
$\|T_v\times T_u\| = (a+b\cos u)b\sqrt{\underbrace{\sin^2 v\cos^2 u+\cos^2 v\cos^2 u}_{1}+\underbrace{\sin^2 u}_{1}} = (a+b\cos u)b$

11

$$A(S) = \int_0^{2\pi}\int_0^{2\pi} (a+b\cos u)\, b\, du\, dv = \int_0^{2\pi} abu + b^2\sin u \Big]_0^{2\pi} dv =$$

$$= \int_0^{2\pi} 2\pi ab\, dv = \boxed{4\pi^2 ab}$$

iii) $x^2+y^2=ay \qquad x^2+y^2+z^2=a^2 \qquad a>0$



$$x^2+\left(y-\frac{a}{2}\right)^2=\frac{a^2}{4}$$

$$z=g(x,y)=\sqrt{a^2-x^2-y^2}$$

PROIEKTATU OXX

$$x=\rho\cos\theta$$

$$y=+\rho\sin\theta$$

$$x^2+y^2=ay \Rightarrow \rho^2=a\rho\sin\theta$$

$$\Rightarrow \rho=a\sin\theta$$

$$A(S)=\iint_D \sqrt{1+g_x^2+g_y^2}\,dx\,dy=$$

$$=\iint_D \sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right)^2}\,dx\,dy=\iint_D\sqrt{\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}}\,dx\,dy=$$

$$=\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy=4\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{a}{\sqrt{a^2-\rho^2}}\,\rho\, d\rho\, d\theta=$$

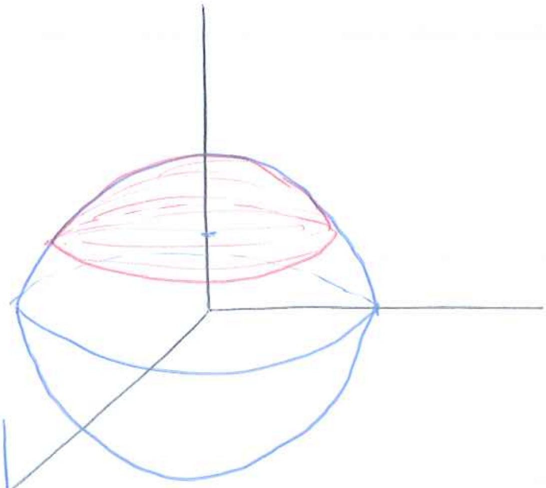
$$=4\int_0^{\pi/2}\Big[-a\sqrt{a^2-\rho^2}\Big]_0^{a\sin\theta} d\theta=4\int_0^{\pi/2} -a^2\sqrt{1-\sin^2\theta}+a^2\, d\theta$$

$$=4\int_0^{\pi/2} -a^2\cos+a^2 d\theta=2\left[a^2\sin\theta-a^2\theta\right]\Big|_0^{\pi/2}=-4\left[a^2-\frac{a^2\pi}{2}\right]=a^2(-4+2\pi)$$

5. ARIKETA

vi) $\iint_S \vec{F}\cdot dS \qquad S=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=2,\ z\geq 1\}$

$\vec{F}(x,y,z)=(xz, xy, yz)$



$\phi(\theta,\varphi)=(\sqrt{2}\cos\theta\sin\varphi, \sqrt{2}\sin\theta\sin\varphi, \sqrt{2}\cos\varphi)$

$T_\theta=(-\sqrt{2}\sin\theta\sin\varphi, \sqrt{2}\cos\theta\sin\varphi, 0)$

$T_\varphi=(\sqrt{2}\cos\theta\cos\varphi, \sqrt{2}\sin\theta\cos\varphi, -\sqrt{2}\sin\varphi)$

$$T_\theta\times T_\varphi=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sqrt{2}\sin\theta\sin\varphi & \sqrt{2}\cos\theta\sin\varphi & 0 \\ \sqrt{2}\cos\theta\cos\varphi & \sqrt{2}\sin\theta\cos\varphi & -\sqrt{2}\sin\varphi \end{vmatrix}=$$

$=(-2\cos\theta\sin^2\varphi, -2\sin\theta\sin^2\varphi, -2\sin\varphi\cos\varphi)=$

$=-2(\cos\theta\sin^2\varphi, \sin\theta\sin^2\varphi, \sin\varphi\cos\varphi)$.

$F(\phi(\theta,\varphi))=(2\cos\theta\sin\varphi\cos\varphi, 2\cos\theta\sin\theta\sin^2\varphi, 2\sin\theta\sin\varphi\cos\varphi)$

$\sqrt{2}\cos\varphi=1 \Rightarrow \cos\varphi=\frac{1}{\sqrt{2}} \Rightarrow \varphi=\frac{\pi}{4}$

$\theta\in[0,2\pi] \qquad \varphi\in[0,\frac{\pi}{4}]$

$F(\phi(\theta,\varphi))\cdot T_\theta\times T_\varphi=$

$=-4(\cos^2\theta\sin^3\varphi\cos\varphi+\cos\theta\sin\theta\sin^4\varphi+\frac{1}{4}\sin\theta\sin^2 2\varphi)$

$$I=4\int_0^{2\pi}\int_0^{\pi/4}\cos^2\theta\sin^3\varphi\cos\varphi+\cos\theta\sin\theta\left[\frac{1}{4}-\frac{\cos 2\varphi}{2}+\frac{1}{8}+\frac{\cos 4\varphi}{8}\right]+$$

$$+\sin\theta\frac{1}{4}\left[\frac{1-\cos 4\varphi}{2}\right]d\varphi\, d\theta=$$

$$=4\int_0^{2\pi}\left[\cos 2\theta\frac{1}{4}\sin^4\varphi+\cos\theta\sin\theta\left(\frac{\varphi}{4}-\frac{\sin 2\varphi}{4}+\frac{\varphi}{8}+\cancel{\frac{\sin 4\varphi}{32}}^{0}\right)+\right.$$

$$\left.+\sin\theta\frac{1}{4}\left(\frac{\varphi}{2}-\cancel{\frac{\sin 4\varphi}{8}}^{0}\right)\right]_0^{\pi/4}d\theta=$$

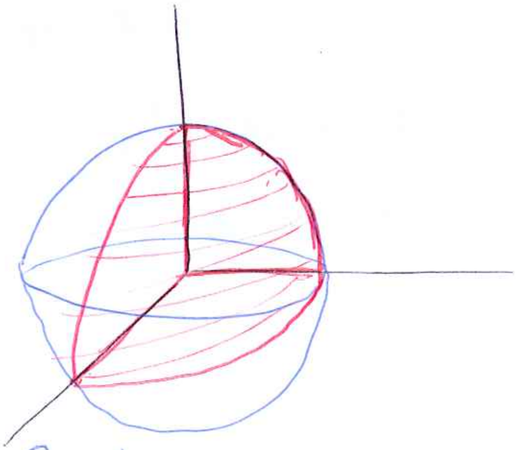
$$= 4\int_0^{2\pi} \cos^2\theta \frac{1}{16} + \cos\theta\sin\theta\left(\frac{\pi}{16} - \frac{1}{4} + \frac{\pi}{32}\right) + \sin\theta \frac{\pi}{32}\, d\theta =$$

$$= 4\left[\left(\frac{\theta}{2} + \cancel{\frac{\sin 2\theta}{4}}^{\,0}\right)\frac{1}{16} - \frac{1}{2}\cos^2\theta\left(\frac{3\pi}{32} - \frac{1}{4}\right) - \cos\theta \frac{\pi}{32}\right]_0^{2\pi} =$$

$$= 4 \cdot \left[\frac{\pi}{16} - \cancel{\frac{1}{2}\left(\frac{3\pi}{32} - \frac{1}{4}\right)} - \cancel{\frac{\pi}{32}} - 0 + \cancel{\frac{1}{2}\left(\frac{3\pi}{32} - \frac{1}{4}\right)} + \cancel{\frac{\pi}{32}}\right] =$$

$$= 4 \cdot \frac{\pi}{16} = \boxed{\frac{\pi}{4}}$$

v) $\iint_S (xy, yz, zx) \cdot dS \qquad S \equiv x^2 + y^2 + z^2 = 1$



$\vec{F}(x,y,z) = (xy, yz, zx)$

$\phi(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$

$T_\theta = (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0)$

$T_\varphi = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)$

$$T_\theta \times T_\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix} =$$

$$= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi) =$$

$$= -(\cos\theta\sin^2\varphi, \sin\theta\sin^2\varphi, \sin\varphi\cos\varphi)$$

$F(\phi(\theta,\varphi)) = (\cos\theta\sin\theta\sin^2\varphi, \sin\theta\cos\varphi\sin\varphi, \cos\theta\cos\varphi\sin\varphi)$

$F(\phi(\theta,\varphi)) \cdot T_\theta \times T_\varphi =$

$$= -\left(\cos^2\theta\sin\theta\sin^4\varphi + \sin^2\theta\sin^3\varphi\cos\varphi + \cos\theta\frac{1}{4}\sin 2\varphi\right)$$

$\theta \in [0, \pi/2] \qquad \varphi \in [0, \pi/2]$

$$I = \int_0^{\pi/2}\int_0^{\pi/2} \cos^2\theta\sin\theta\left(\frac{1-\cos 2\varphi}{2}\right)^2 + \sin^2\theta\sin^3\varphi\cos\varphi + \frac{1}{4}\cos\theta\sin^2 2\varphi \, d\varphi d\theta =$$

$$\hookrightarrow \frac{1+\cos 4\varphi}{2} \qquad \frac{1-\cos 4\varphi}{2}$$

$$= \int_0^{\pi/2}\left[\cos^2\theta\sin\theta\,\frac{1}{4}\left(\varphi - \cancel{\sin 2\varphi}^{\,0} + \frac{\varphi}{2} + \cancel{\frac{\sin 4\varphi}{8}}^{\,0}\right) + \frac{1}{4}\sin^2\theta\,\underbrace{\sin^4\varphi}_{\frac{1}{\sqrt{2}}} + \right.$$

$$\left. + \frac{1}{8}\cos\theta\left(\varphi - \cancel{\frac{\sin 4\varphi}{2}}^{\,0}\right)\right]_0^{\pi/2} d\theta =$$

$$\frac{3\pi}{4} \qquad \frac{1-\cos 2\theta}{2} \qquad \frac{1}{4}\cos\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\sin\theta\,\frac{1}{4}\left(\frac{\pi}{2}+\frac{\pi}{4}\right) + \frac{1}{16}\sin^2\theta + \frac{\pi}{16}\cos\theta \quad d\theta$$

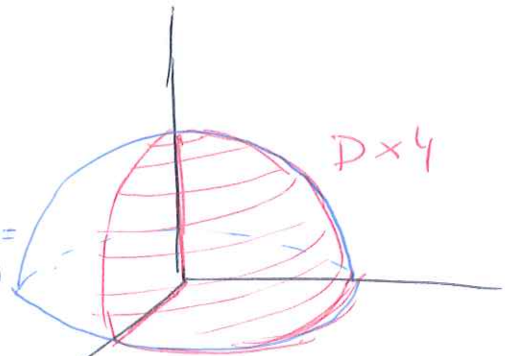
$$= \left[-\frac{\pi}{16}\cos^3\theta + \frac{\theta}{32} - \frac{\cancel{\sin 2\theta}^{\,0}}{\cancel{4\cdot 16}} + \frac{\pi}{16}\sin\theta\right]_0^{\pi/2} =$$

$$= 0 + \frac{\pi}{64} + \frac{\pi}{16}^{4'} + \frac{^4\pi}{16} = \frac{3\pi}{16}\cdot\frac{3}{4} \;???$$

4. Ariketa

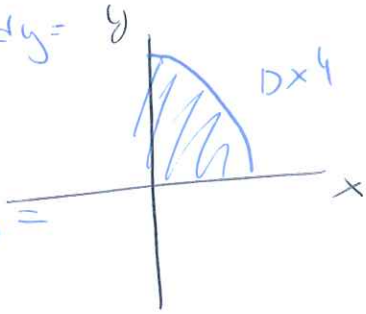
iv) $\iint_S (x+y+z)\,dS, \quad S \equiv x^2+y^2+z^2=a^2 \quad z\geq 0$

$g(x,y)=\sqrt{a^2-x^2-y^2}=z$



$$\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dx\,dy =$$

$$= \iint_D (x+y+\sqrt{a^2-x^2-y^2})\cdot\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,dx\,dy =$$



$$= \iint_D (x+y+\sqrt{a^2-x^2-y^2})\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy =$$

$$= \iint_D \frac{ax}{\sqrt{a^2-x^2-y^2}} + \frac{ay}{\sqrt{a^2-x^2-y^2}} + a \quad dx\,dy =$$

$$= 4a \int_0^a \int_a^{\sqrt{a^2-y^2}} \frac{x}{\sqrt{a^2-x^2-y^2}} + \frac{y}{\sqrt{a^2-x^2-y^2}} + 1 \, dx \, dy =$$

$$= 4a \int_0^a \left[\sqrt{a^2-x^2-y^2} + y \arcsin\left(\frac{x}{\sqrt{a^2-y^2}}\right) + x \right]_0^{\sqrt{a^2-y^2}} dy =$$

$$= 4a \int_0^a \cancel{\sqrt{a^2-a^2+y^2-y^2}} + y \arcsin 1 + \cancel{\sqrt{a^2-y^2}} - \cancel{\sqrt{a^2-y^2}} + y \cdot 0 + 0 \, dy =$$

$$= 4a \int_0^a \frac{\pi}{2} y \, dy = \cancel{4} a \frac{\pi}{\cancel{4}} y^2 \Big|_0^a = \boxed{\pi a^3}$$

7. ANALISI BEKTORIALEKO TEOREMAK

7.1. ERAGILE BEKTORIALAK

- DEFINIZIOA

$$\vec{u} = (u_1, u_2, u_3) \qquad \in \mathbb{R}^3$$
$$\vec{v} = (v_1, v_2, v_3)$$

ESKALARRA $\Rightarrow$ $\vec{u} \cdot \vec{v} = \ldots$

BEKTORIALA $\Rightarrow$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- DEFINIZIOA

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \Rightarrow \text{NABLA ERAGILEA}$$

- DEFINIZIOA

$f : \mathbb{R}^3 \to \mathbb{R} \quad C^1$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \Rightarrow f\text{-ren GRADIENTEA}$$

- DEFINIZIOA

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \quad C^1$

$$\vec{F} = (F_1, F_2, F_3)$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \Rightarrow f\text{-ren DIBERGENTZIA}$$

$$\operatorname{rot} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \Rightarrow f\text{-ren ERROTAZIONALA}$$

PROPIETATEAK (136. orr liburuk)

TEOREMA 7.1:

$f : \mathbb{R}^3 \to \mathbb{R} \quad C^2$ klasekoa $\Rightarrow \operatorname{rot}(\nabla f) = \vec{0}$

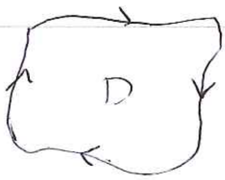
TEOREMA 7.2:

$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ C^2 Klasekoa

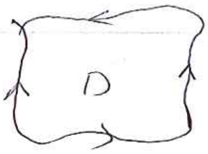
$div(rot\,\vec{F}) = 0$

7.2. GREENEN TEOREMA

$D \subset \mathbb{R}^2$ eremua bada



C^- orientazio negatiboa (clockwise)



C^+ orientazio positiboa (anticlockwise)

TEOREMA 7.5: GREENEN TEOREMA

$D \subset \mathbb{R}^2$ 3. motako eremu elementala (3. gaia)

C kurba bere muga eta $P, Q: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

C^1 klaseko funtzioak. Orduan,

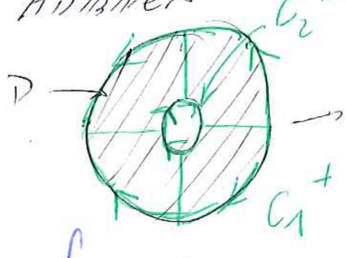
$$\underbrace{\int_{C^+} P\,dx + Q\,dy}_{\downarrow} = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\downarrow} dx\,dy$$

INTEGRAL BIKOITZA (7.GAIA)

2. MAILAKO LERRO-INTEGRALA (5.GAIA)

OHARRA: Ez da beharrezkoa D eremua 3. motakoa izatea. Green aplikatzeko, nahikoa da D 3.motako eremu elementalen bildura izatea.
Kasu honetan kontuz aukeratu behar da mugaren orientazioa Green bete dadin.

ADIBIDEA



$\rightarrow$ Ez da 3.motakoa $\rightarrow$ $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$

$$= \int_{C_2^-} P\,dx + Q\,dy + \int_{C_1^+} P\,dx + Q\,dy$$

TEOREMA 7.6: PLANOKO EREMU BATEN AZALERA

$D \subset \mathbb{R}^2$ 3. mailako eremu elementalen bildura finitua bada eta ∂D bere muga orientazio positiboarekin.

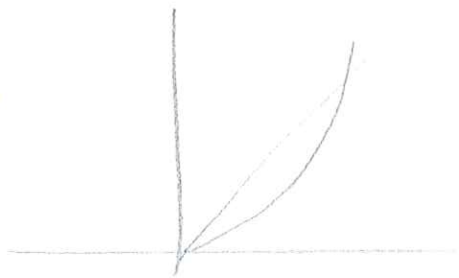
$$A(D) = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

TEOREMA 7.7: GREENEN TEOREMA ERA BEKTORIALEAN

- $D \subset \mathbb{R}^2$ Greenen teoremako hipotesi berdinek
- ∂D bere muga
- $\vec{F}: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ Klasekoa

$$\Longrightarrow \underbrace{\int_{\partial D} \vec{F}\,ds}_{\text{2. MAILAKO LERRO INTEGRALA.}} = \underbrace{\iint_D \text{rot}\,\vec{F}\cdot \overset{(0,0,1)}{\vec{k}}\,dx\,dy}_{\text{INTEGRAL BIKOITZA}}\; dz$$

ADIBIDEA



$$\vec{F}(x,y) = (xy^2, x+y)$$

$$D \Rightarrow \begin{cases} y = x^2 \\ y = x \\ x \geq 0 \end{cases} \text{ mugatuta}$$

Kalkulatu

$$\iint_D \text{rot}\,\vec{F}\cdot\vec{k}\,dx\,dy$$

Bi modu

1) Zuzenean

$$\text{rot}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & x+y & 0 \end{vmatrix} = \left(0, 0, \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(xy^2)\right) = (0,0,1-2xy)$$

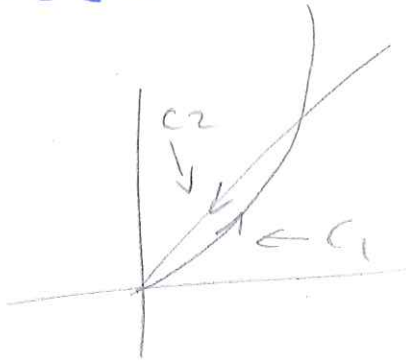
$$\iint_D \text{rot}\,\vec{F}\cdot\vec{k}\,dx\,dy = \iint_D (0,0,1-2xy)\cdot(0,0,1)\,dx\,dy =$$

$$= \iint_D 1-2xy\,dx\,dy = \int_0^1\int_{x^2}^{x} 1-2xy\,dy\,dx = \dots = \frac{1}{12}$$

2. TEOREMA 7.7 erabiliz:

$$\iint_D \operatorname{rot} \vec{F}\cdot\vec{k} = \int_{\partial D} \vec{F} ds = \int_{C_1} \vec{F} ds + \int_{C_2} \vec{F} dS =$$

$C_1^* = \sigma_1(t) = (t, t^2) \quad t \in [0,1]$
$\overset{\shortparallel}{x}\ \ \overset{\shortparallel}{y}$



$\sigma_1(0) = (0,0)$ Orientazioa
$\sigma_1(1) = (1,1)$ mantentzen du

$C_2 = \sigma_2(t) = (t, t)$
$\overset{\shortparallel}{x}\ \overset{\shortparallel}{y}$

$\sigma_2(0) = (0,0)$ Orientazioa
$\sigma_1(1) = (1,1)$ aldatzen du

$$= \int_{\sigma_1} \vec{F}(\sigma_1(t))\cdot\sigma_1(t)\,dt - \int_{\sigma_2} \vec{F}(\sigma_2(t))\cdot\sigma_2'(t)\,dt =$$

$$= \int_0^1 (t\cdot t^4\cdot 1 + (t+t^2)2t)\,dt - \int_0^1 (t\cdot t^2\cdot 1 + (t+t)\cdot 1)\,dt =$$

$$= [\dots] = \frac{1}{12}$$

TEOREMA 7.8: DIBERGENTZIAREN TEOREMA PLANOAN

- $D \subset \mathbb{R}^2$ Greenen teoremako hipotesiekin
- ∂D bere muga
- $\vec{n}$ ∂D-ren bektore normal unitarioa
  Kanporantz norabidetua
- $\sigma:[a,b]\to\mathbb{R}^2$ $\sigma(t) = (x(t), y(t))$ orientazioa mantentzen duen ∂D kurbaren parametrizazioa

$$\Rightarrow \vec{n} = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} \qquad \vec{F}: D \to \mathbb{R}^2 \ C^1 \text{ klasekoa}$$

$$\int_{\partial D} \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}\cdot dA$$

7.3. STOKESEN TEOREMA

TEOREMA 7.9: STOKESEN TEOREMA GRAFIKOETARAKO

- $D \subseteq \mathbb{R}^2$ Greenen teoremako hipotesiekin
- $g: D \subset \mathbb{R}^2 \to \mathbb{R}$ C^1 klasekoa $\wedge$ $g(D) = S$
- $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3 \to C^1$ klasekoa
- ∂S orientazio positiboa duen S-ren muga

$$\Rightarrow \iint_S \operatorname{rot} \vec{F} dS = \int_{\partial S} \vec{F} dS \longrightarrow \text{2. MAILAKO LERRO-INTEGRALA}$$

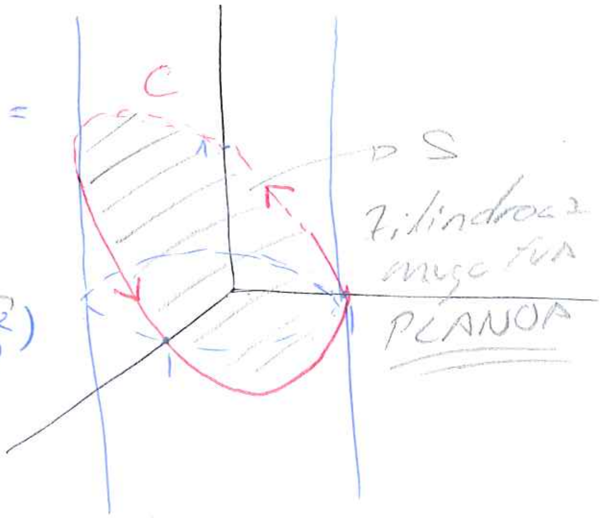
2.MAILAKO GAINAZAL INTEGRALA

ADIBIDEA

$$\int_C -y^3 dx + x^3 dy - z^3 dz \overset{\text{STOKES}}{\underset{\text{TEOR 7.9.}}{=}} \quad \text{non } C = \begin{cases} x^2+y^2=1 \\ x+y+z=1 \end{cases} \text{ ebakidura}$$

orientazio positiboa

$$= \iint_S \operatorname{rot} \vec{F} dS = \iint_S (0,0,3x^2+3y^2) dS =$$

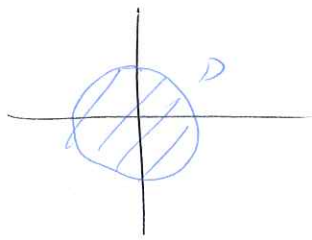


$$\operatorname{rot} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (\overset{G_1}{0}, \overset{G_2}{0}, \overset{G_3}{3x^2+3y^2})$$

$$z = \overset{\text{PLANOA}}{g(x,y)} = 1 - x - y$$

$g: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\quad g(D) = S$

S-ren proiekzioa OXY



$$= \iint_D -G_1 g_x - G_2 g_y + G_3 \, dx\, dy =$$

$$= \iint_D -0 g_x - 0 g_y + (3x^2+3y^2) dx\, dy \overset{\text{POLARRAK}}{=}$$

$x = \rho\cos\theta$ $\quad |J| = \rho$

$y = \rho \sin\theta$

5

$\rho \in [0,1]$

$\theta \in [0,2\pi]$

$$= \int_0^{2\pi}\int_0^1 3\rho^2 \rho \, d\rho \, d\theta = [\dots] = \frac{3\pi}{2}$$

TEOREMA 7.10: STOKES-EN TEOREMA

$\phi: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ injektiboa

$S = \phi(D)$ gainazal parametrizatu norabidetua

∂S bere muga norabidetua

$$\Rightarrow \iint_S \text{rot}\, \vec{F}\, dS = \int_{\partial S} \vec{F} ds$$

2. MAILAKO GAINAZAL INT. ↳ 2. MAILAKO LERRO INT.

7.4. EREMU KONTSERBAKORRAK

TEOREMA 7.11:

$\vec{F}$ C^1 klaseko eremu bektoriala $\mathbb{R}^3 - \{$puntu kopuru finitua$\}$ eremuan definitua.

BALIOKIDEAK DIRA:

i) $\int_C \vec{F} ds = 0$ $\forall C$ kurba itxi sinpleetarako

ii) $\int_{C_1} \vec{F} ds = \int_{C_2} \vec{F} ds$ $C_1 \wedge C_2$ kurba sinple norabidetuak eta mutur berdinekin

iii) $\exists f$ funtzio eskalarra non $\nabla f = \vec{F}$ den (f, $\vec{F}$-ren potentziala) $\vec{F}$-ren eremuko puntuetan

iv) $\text{rot}\, \vec{F} = \bar{0}$

ii) 

iii) $\int_C \nabla f ds = f(\sigma(b)) - f(\sigma(a))$

$\sigma: [a,b]$

DEFINIZIOA

$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ C^1 klasekoa

$\mathbb{R}^3 - \{$puntu kopuru finitu bat$\}$

$\vec{F}$ KONTSERBAKORRA da $\int_c \vec{F} ds = 0$ $\forall c$ kurba sinple itxirako

DEFINIZIOA:

$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ IRROTAZIONALA da, $\mathrm{rot}\,\vec{F} = \bar{0}$ bada.

OHARRAK:

1) $\vec{F}$ irrotazionala $\overset{\text{TEOR 7.11}}{\Longleftrightarrow}$ $\vec{F}$ kontserbakorra

2) $\vec{F}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\vec{F} = (P, Q)$ C^1 klasekoa

$\vec{F} = (P, Q, 0) \Rightarrow \mathrm{rot}\,\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot \underset{\vec{k}}{\overset{(0,0,1)}{}}$

KOROLARIOA 7.12:

$\vec{F}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ C^1 klasekoa

$\vec{F} = (P, Q)$

$\frac{\partial P}{\partial y} \underset{\uparrow}{=} \frac{\partial Q}{\partial x} \Rightarrow \exists f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ C^1: $\nabla f = \vec{F}$

puntu guztietan

ADIBIDEA

$\sigma: [1, 2] \longrightarrow \mathbb{R}^2$

$\sigma(t) = \left(e^{t-1}, \sin\frac{\pi}{t}\right)$

$\vec{F}(x,y) = (2x\cos y\,\hat{\imath} - x^2 \sin y\,\hat{\jmath}) = (2x\cos y, -x^2\sin y, 0)$

Kalkulatu $\int_\sigma \vec{F} ds$

$\int_\sigma \vec{F} ds = \int_1^2 \vec{F}(\sigma(t)) \cdot \sigma'(t) dt =$

$= \int_1^2 \left[2e^{t-1} \cdot \cos\left(\sin\frac{\pi}{t}\right), -\left(e^{t-1}\right)^2 \sin\left(\sin\frac{\pi}{t}\right)\right] \cdot$

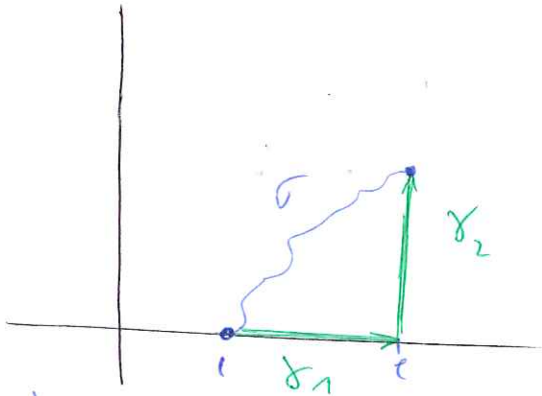
$\cdot \left[e^{t-1}, \frac{-\pi}{t^2}\cos\frac{\pi}{t}\right] dt = $ [KONPLIKATUA

2. AUKERA

$$\text{rot}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\cos y & -x^2\sin y & 0 \end{vmatrix} = [\dots] = 0 \Rightarrow \vec{F} \text{ irrotazionale}$$

- $\sigma(t)$ itxia?

$$\sigma(1) = (1,0)$$
$$\sigma(2) = (e,1) \Rightarrow \underline{EZ}$$



TEOR 7.11

$$\text{rot}\,\vec{F} = 0 \iff \int_\sigma \vec{F}\,ds = \int_\gamma \vec{F}\,ds$$

$\gamma_1(t) = (t, 0)$, $t \in [1, e]$ $\qquad$ $\gamma_2(t) = (e, t)$, $t \in [0, 1]$

$$\gamma = \gamma_1 \cup \gamma_2$$

$$\int_\sigma \vec{F}ds = \int_\gamma \vec{F}ds = \int_1^e \vec{F}(\gamma_1(t))\gamma_1'(t)\,dt + \int_0^1 \vec{F}(\gamma_2(t))\gamma_2'(t)\,dt =$$

$$= \int_1^e 2t\cos 0\,dt + \int_0^1 (-e^2\sin t)\,dt =$$

$$= t^2\Big]_1^e + e^2[\cos t]_0^1 = \boxed{e^2\cos 1 - 1}$$

3. AUKERA

$$\text{rot}\,\vec{F} = \bar{0} \overset{\text{TEOR 7.11}}{\iff} \exists f \text{ non } \nabla f = \vec{F}$$

Bilatu

$$\nabla f = \vec{F} \Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (P, Q)$$

$$\frac{\partial f}{\partial x} = 2x\cos y \overset{\text{integratu}}{\Rightarrow} f(x,y) = x^2\cos y + h(y)$$

$$\frac{\partial f}{\partial y} = -x^2\sin y + h'(y) = -x^2\sin y$$

aukeratu

$$h'(y) = 0 \Rightarrow h(y) = k = 0$$

$$f(x,y) = x^2\cos y$$

$$\int_\sigma \nabla f\, ds = f(\sigma(b)) - f(\sigma(a))$$

$$\int_\sigma \vec{F} ds = \int_\sigma \nabla f\, ds = f(\underbrace{\sigma(2)}_{(e,1)}) - f(\underbrace{\sigma(1)}_{(1,0)}) =$$

$f(x,y) = x^2\cos y$

$$= e^2\cos 1 - 1\cos 0 = \boxed{e^2\cos 1 - 1}$$

TEOREMA 7.13

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ C^1 klasekoa

$\operatorname{div}\vec{F} = 0 \implies \exists\, \vec{G}: \mathbb{R}^3 \to \mathbb{R}^3$ (C^1 klasekoa)

non $\operatorname{rot}\vec{G} = \vec{F}$

Nola? $\vec{G} = (G_1, G_2, G_3)$

$$\begin{cases} G_1(x,y,z) = \int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt \\ G_2(x,y,z) = -\int_0^z F_1(x,y,t)\,dt \\ G_3(x,y,z) = 0 \end{cases}$$

5. GAUSSEN TEOREMA

Lehenago Greenen teoremaren ataleen, dibergentziaren teorema ikusi genuen,

$$\int_{\partial D} \vec{F}\cdot\vec{n}\, ds = \iint_D \operatorname{div}\vec{F}\, dA$$

TEOREMA 7.14: GAUSSEN DIBERGENTZIAREN TEOREMA

- $\Omega \subset \mathbb{R}^3$ IV motako eremu elementala (4. gaia)
- $\partial\Omega$, Ω bornatzen duen gainazal itxi norabidetua orientazio positiboarekin.
- $\vec{F}: \Omega \to \mathbb{R}^3$ C^1 klasekoa

$$\implies \underbrace{\iint_{\partial\Omega} \vec{F} ds}_{\text{2. MAI. GAINZ. INT}} \overset{\text{6. GAIA}}{=} \underbrace{\iint_{\partial\Omega} \vec{F}\cdot\vec{n}\, ds}_{\text{1. GAINZ. INT.}} = \underbrace{\iiint \operatorname{div}\vec{F}\, dV}_{\text{INT HIRUKOITZA}}$$

OHARRA: Ω, IV motakoa ez denean baina IV motako eremu elementalen bildura bezala adierazi daitekeenean Gauss aplika daiteke.

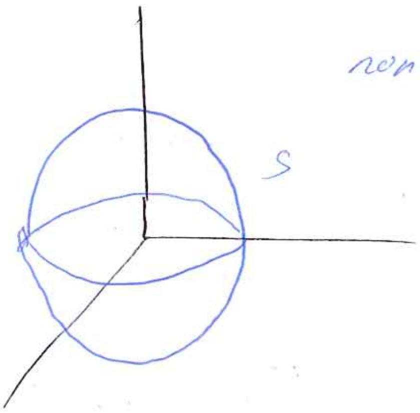
ADIBIDEA

$\vec{F}(x,y,z) = (2x, y^2, z^2)$ Kalkulatu $\iint_S \vec{F}\cdot\vec{n}\,ds$

$S \to x^2+y^2+z^2=1$ kanpoko orientazioarekin

$\iint_S \vec{F}\cdot\vec{n}\,ds = \Big\{$ − 6. gaikoa erabiliz

$= \iiint_\Omega \operatorname{div}\vec{F}\,dV =$



non $S = \partial\Omega$

$\operatorname{div}\vec{F} = \frac{\partial F_1}{x} + \frac{\partial F_2}{\partial y} + \frac{F_3}{\partial z}$

$= \iiint_\Omega 2+2y+2z\,dx\,dy\,dz =$

$= \underbrace{2\iiint_\Omega 1\,dx\,dy\,dz}_{B(\Omega)} + 2\iiint_\Omega \cancel{y\,dx\,dy\,dz}^{\,0} + 2\iiint_\Omega \cancel{z\,dx\,dy\,dz}^{\,0} =$

Simetrikoa OXZ-rekiko — bakoitia y-rekiko

OXY-rekiko simetrikoa — z-rekiko bakoitia

$= 2\cdot\frac{4\pi r^3}{3} = \boxed{\frac{8\pi}{3}}$

OHARRA: $\iint_S \vec{F}ds \to \vec{F}$-ren barrurako fluxua S gainazalean zehar.

DEFINIZIOA: $\vec{F}$ C^1 klasekoa: $\vec{F}$ DIFERENTZIA GABEKOA da $\operatorname{div}\vec{F}=0$ bada puntu guztian

OHARRA:

1) $\operatorname{div}\vec{F}=0 \iff \iint_S \vec{F}ds = 0$

2) $\vec{F}$ fluxu baten abiadura adierazten badu eta $\operatorname{div}\vec{F}=0$ bada, kanporantz eta barrurantz doan fluxuaren kantitatea berdina da. (FLUXU KONPRIMAEZINA)

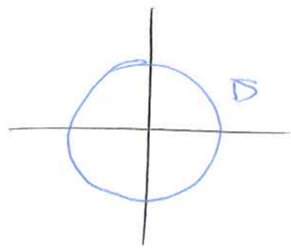
ADIBIDEA

Kalkulatu $\iint_S \vec{F} ds = \begin{cases} \overset{G.GAIN}{=} \iint_{S_1} \vec{F} ds + \iint_{S_2} \vec{F} ds + \iint_{S_3} \vec{F} ds \\ \overset{GAUSS}{=} \iiint_W \operatorname{div} \vec{F} dV \end{cases}$

$\vec{F}(x,y,z)=(xy^2, xz, y)$

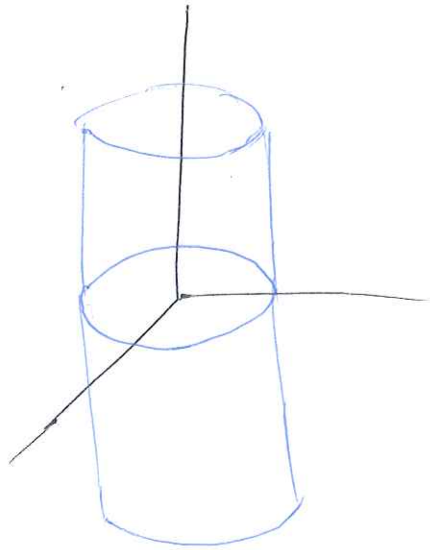
$W=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 1, \; -1\le z\le 1$

$\partial W = S$ itxia



$\iiint_W \operatorname{div} \vec{F} dV = \iiint_W (x^2+y^2)dx\,dy\,dz =$

$= \int_0^{2\pi}\int_0^1\int_{-1}^1 \rho^2 \rho \, dz \, d\rho \, d\theta = \int_0^{2\pi}\int_0^1 \rho^3 [z]_{-1}^1 d\rho \, d\theta = \dots = \pi$



OHARRA:

$\operatorname{div}\vec{F} = x^2+y^2 \ge 0 \quad \forall (x,y,z)\in W$

$\downarrow$

Fluxua ahkra gitea da.

Kalkulatu 2 integralak eta bardindu

ANALISI BEKTORIALA ETA KONPLEXUA

7. Gaia: ANALISI BEKTORIALEKO TEOREMAK

Ariketak

1. Egiaztatu Greenen teorema ondoko adibideetan, bi integralak kalkulatuz:

   (i) $I = \oint_\gamma (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$,

   hemen γ kurba itxia $(0,0)$ eta $(1,1)$ puntuen arteko $y = \sqrt{x}$ eta $y = x^2$ kurba-arkuez osatua eta norantza positibokoa da.

   (ii) $I = \oint_\gamma (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$

   γ kurba itxia $x = 0,\ y = 0, x+y = 1$ kurbez osatutako triangelua, norantza positibokoa, denean.

   (iii) $I = \oint_\gamma (2x - y^3)\, dx - xy\, dy$,

   γ kurba itxia $x^2 + y^2 = 1,\ \ x^2 + y^2 = 9$ zirkunferentziek osatutako zirkuitua, norantza positibokoa da.

   (iv) $I = \oint_\gamma xy^2\, dy - x^2 y\, dx$,

   γ $x^2 + y^2 = a^2$ zirkunferentzia, norantza positibokoa da.

   (v) $I = \oint_\gamma (x + y)\, dx - (x - y)\, dy$,

   non γ $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1$ ekuazioko elipsea norantza positibokoa den.

2. Greenen teorema erabiliz, kalkula ezazu

$$I = \int_{AmO} (e^x \sin y - my)\, dx + (e^x \cos y - m)\, dy$$

integrala, AmO kurba $A = (a, 0)$ puntutik $O = (0, 0)$ puntura doan $x^2 + y^2 = ax$ ekuazioko goiko zirkunferentziaerdia denean.

Em.: $\pi m a^2/8$.

3. Kalkula ezazu $I = \oint_\gamma (x^2 - 2xy)\, dx + (x^2 y + 3)\, dy$, γ $y^2 = 8x$ eta $x = 2$ kurben arkuez osatuta dagoelarik, erlojuaren orratzen kontrako norantzan hartuta.

Em.: 128/5.

4. Izan bitez R planoko eskualde elementala, Γ bere muga, erlojuaren orratzen kontrako orientazioarekin eta ρ konstantea R-ren dentsitatea. Kalkula ezazu R-ren azalera suposatuz R-ren masa-zentrua $(2, 5)$ puntua dela eta

$$\int_{\Gamma^+} (\arctan x - y^2)dx + (\ln y + x^2)dy = 21.$$

Em.: 3/2.

38.4

5. Stokesen teorema erabiliz, kalkula itzazu ondoko integralak:

(i) $I = \oint_\gamma (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$, non $\gamma(t) = (a\sin^2 t, 2a\sin t\cos t, a\cos^2 t)$, $0 \le t \le \pi$ den.

(ii) $I = \oint_\gamma (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$, non γ kurba $\dfrac{x}{a} + \dfrac{z}{h} = 1 \quad a, h > 0$ planoaren eta $x^2+y^2 = a^2$ zilindroaren arteko ebakidura den, gainetik ikusita norantza positibokoa.

(iii) $I = \oint_\gamma z\,dx + x\,dy + y\,dz$, non γ $x^2+y^2 = 4$, $z = 0$ kurba den, norantza positibokoa.

(iv) $I = \oint_\gamma (y^2+z^2)\,dx + (x^2+z^2)\,dy + (x^2+y^2)\,dz$. Hemen γ kurba $x^2+y^2+z^2 = 2Rx$, ($z \ge 0$) esferaerdiaren eta $x^2+y^2 = 2rx$, $(0 < r < R)$ zilindroaren arteko ebakidura da eta, gainetik ikusita, norantza positibokoa.

(v) $I = \oint_\gamma 2z\,dx - x\,dy + 3y\,dz$. Hemen γ kurba itxia $1 - z = x^2 + y^2$ gainazalaren eta $x \ge 0$, $y \ge 0$, $z \ge 0$ planoerdien arteko ebakibura-kurbez osaturik dago eta gainetik ikusita norantza positibokoa da.

Em.: (i) 0; (ii) $-2a(h+a)\pi$; (iii) 4π; (iv) $2\pi R r^2$; (v) $\dfrac{10}{3} - \dfrac{\pi}{4}$.

6. Izan bedi C kurba $\dfrac{x^2}{2} + \dfrac{y^2}{2} + z^2 = 1$ elipsoidearen goiko erdiaren eta $x^2+y^2-y = 0$ zilindroaren arteko ebakidura, goitik ikusita erlojuaren orratzen kontrako orientazioa duela. Kalkula ezazu $\vec{F}(x,y,z) = (y^3, xy + 3xy^2, z^4)$ eremuaren zirkulazioa C-n zehar.

Em.: $\pi/8$.

7. Izan bitez $\vec{F}(x,y,z) = (x^2, 2xy + x, z)$ eremu bektoriala, $x^2+y^2 = 1$ kurba $= XY$ planoan, erlojuaren orratzen kontrako orientazioarekin hartuta, eta $x^2+y^2 \le 1$ diska $z = 0$ planoan, gorako orientazioa duela.

(i) Kalkulatu $\vec{F}$-ren fluxua diskan zehar.

(ii) Kalkulatu $\vec{F}$-ren zirkulazioa kurban zehar.

(iii) Kalkulatu $\vec{F}$-ren errotazionalaren fluxua diskan zehar. Betetzen al da Stokesen teorema?

Em.: (i) 0; (ii) π; (iii) π.

8. Froga ezazu ondorengo eremuak kontserbakorrak direla, eta $\displaystyle\int_C \vec{F}\cdot ds$ kalkulatu:

(i) $\vec{F}(x,y) = (xy^2 + 3x^2y, (x+y)x^2)$ da, C kurba $(1,1)$, $(0,2)$, $(3,0)$ erpinetako triangelua izanik, norantza positiboarekin

(ii) $\vec{F}(x,y) = (\cos xy^2 - xy^2\sin xy^2, -2x^2y\sin xy^2)$ da, eta C-ren parametrizazio bat $\sigma(t) = (e^t, e^{t+1})$, $-1 \le t \le 0$ ibilbidea.

(iii) $\vec{F}(x,y,z) = (x^3 - 3xy^2, y^3 - 3x^2y, z)$ da, eta C kurba $(0,0,0)$, $(0,0,1)$, $(0,1,1)$ eta $(1,1,1)$ puntuak lotzen dituen edozein ibilbide da.

Em.: (i) 0; (ii) $\cos e^2 - \dfrac{1}{e}\cos\dfrac{1}{e}$; (iii) $-1/2$.

$-(\cos e^t e^{2t+2} - e^t e^{2t+2} \sin e^t e^{2t+2}, -2 e^{2t} e^{t+2} \sin e^t e^{t+2})$

9. Egiaztatu $\vec{F}(x,y,z) = \big(yz(2x+y+z), xz(x+2y+z), xy(x+y+2z)\big)$ eremua kontserbakorra dela, eta kalkulatu $\vec{F}$-ren potentziala.

Em.: $f(x,y,z) = xyz(x+y+z)$.

10. Izan bedi $\vec{F}(x,y,z) = (6xy\cos z, 3x^2\cos z, -3x^2 y\sin z)$.

(i) Froga ezazu $\vec{F}$ kontserbakorra dela.

(ii) Aurki ezazu $\vec{F}$ eremuaren potentziala.

(iii) Kalkula ezazu $\displaystyle\int_\sigma \vec{F}\cdot ds$, non $\vec{\sigma}(t) = (\cos^3 t, \sin^3 t, 0)$, $t\in[0,\pi/2]$ den.

Em.: (ii) $f(x,y,z) = 3x^2 y\cos z + k$; (iii) 0.

11. Izan bitez $P(x,y) = \dfrac{-y}{x^2+y^2}$ eta $Q(x,y) = \dfrac{x}{x^2+y^2}$.

(i) Frogatu $\displaystyle\int_C \frac{xdy - ydx}{x^2+y^2} = 2\pi$ dela, jatorrian zentratutako eta 1 erradioko zirkunferentzian zehar, erlojuaren orratzen kontrako orientazioarekin.

(ii) Kontserbakorra al da $\vec{F}(x,y) = (P,Q)$ eremua?

(iii) Frogatu $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ dela. Berdintza hau eremu kontserbakorren teoremaren aurkakoa al da?

12. Gaussen teorema erabiliz, kalkula itzazu ondoko integralak:

(i) $I = \displaystyle\iint_S (x,y,z)\cdot dS$. Hemen S gainazala $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 + \left(\dfrac{z}{c}\right)^2 = 1$ elipsoidearen kanpoko aurpegia da.

(ii) $I = \displaystyle\iint_S (ax, by, cz)\cdot dS$, non S gainazal itxiak V bolumeneko gorputza mugatzen duen (kanpoko aurpegia).

(iii) $I = \displaystyle\iint_{\partial\Omega} (xy, yz, zx)\cdot dS$, non $\partial\Omega$ $\Omega = \{(x,y,z)\ :\ x^2+y^2+z^2\le 1\ x,y,z\ge 0\}$ solidoa mugatzen duen gainazalaren kanpoaldea den.

(iv) $I = \displaystyle\iint_S (x^2,y^2,z^2)\cdot dS$, non S gainazala $\{(x,y,z)\ :\ 0\le x,y,z\le a\}$ kuboaren kanpoaldeko gainazal mugatzailea den.

(v) $I = \displaystyle\iint_{\partial\Omega} (yx, -2y^2, z^2)\cdot dS$, $\partial\Omega$ gainazala $\Omega = \{(x,y,z)\ :\ x^2+y^2\le 4, 0\le z\le 3\}$ solidoaren kanpoaldeko aurpegia izanik.

(vi) $\displaystyle\iint_S (xy, yz, zx)\cdot dS$, S $\Omega = \{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+z^2\ge 4,\ z\le 4-(x^2+y^2),\ z\ge 0\}$ eremuaren muga den gainazalaren kanpoko aurpegia da.

Em.: (i) $4\pi abc$; (ii) $(a+b+c)V$; (iii) $3\pi/16$; (iv) $3a^4$; (v) 36π; (vi) $27\pi/2$.

+ 13. Izan bedi S hurrengo bi baldintzek definitutako gainazala: $x^2+y^2=1$, $0\le z\le 1$ denean eta $x^2+y^2+(z-1)^2=1$, $z\ge 1$ denean. Kalkula ezazu $\displaystyle\iint_S \operatorname{rot}\vec{F}\cdot dS$ integrala baldin eta $\vec{F}(x,y,z)=(zx+z^2y+x,\, z^3yx+y,\, z^4x^2)$ bada eta S-n kanporako bektore normala hartzen bada.

Em.: 0.

+ + 14. Izan bedi W lehen oktantean hiru plano koordenatuek, $2x+y=6$ planoak eta $z=4-x^2$ zilindro parabolikoak mugatzen duten solidoa eta izan bedi $\vec{F}(x,y,z)=(y,2x,z)$ eremu bektoriala.

(i) Kalkula ezazu $\vec{F}$ eremuaren lerro-integrala Γ kurbaren gainean, Γ plano batean ez dagoen W-ren mugaren zatiaren muga izanik. Orientazioa aukera dezakezu.

(ii) Kalkula ezazu $\vec{F}$ eremuaren gainazal-integrala W gorputzaren mugan zehar, bektore normalak kanporantz begiratzen duela.

Em.: (i) 8; (ii) 24.

++ 15. Izan bitez $\vec{F}(x,y,z)=(1,0,1)$ eremua eta $S=\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+z^2=2,\ z\ge 1\}$ gainazala. Kalkula ezazu $\vec{F}$ eremuaren gainazal-integrala S-ren gainean gorako orientazioarekin, definizioa erabiliz eta dibergentziaren teorema erabiliz.

Em.: π.

+ 16. Kalkula ezazu $\displaystyle\iint_S \operatorname{rot}\vec{F}\cdot dS$ gainazal-integrala $\vec{F}(x,y,z)=(y,z,x)$ bada eta S $x^2+y^2=1$ ekuazioko zilindroaren zatia, $z=0$ eta $z=x+2$ planoen artean geratzen dena, kanporako bektore normala kontsideratuz.

Em.: $-\pi$.

$(\cos\theta, \sin\theta, 0)$

$$\int_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta + \sin^2\theta}\, d\theta =$$

$\vec{F}(x,y) = (P,Q)$ $\qquad = \int_0^{2\pi} 1\, d\theta = 2\pi$

ii) $\vec{F}$ KONTSERBAKORRA

$\vec{F}$ Kontserbakorra balitz, $\int_C \vec{F}\,ds = 0$ izan beharko luke $\forall C$ kurba sinple itxirako

Gure kasuan, C kurba itxia eta sinplea da eta $\int_C \vec{F}\,ds = 2\pi \neq 0$ ematen digu

$\Rightarrow \vec{F}$ ez da kontserbakorra

iii) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+y\,2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \dots = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Baldintza hori ematu kontserbakorren teoremaren aurkakoa da?

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2 \quad C^1$

$\vec{F}(P,Q)$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \underset{*}{\Rightarrow} \operatorname{rot}\vec{F} = \vec{0}$

Ez da $*$ kontraesaten,

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ berdintza ez delako $\forall (x,y) \in \mathbb{R}^2$ betetzen $((0,0)$-n ez da betetzen)

ii) f non $\nabla f = \vec{F}$

$$\frac{\partial f}{\partial x} = 6xy\cos z \Rightarrow f(x,y,z) = 3x^2y\cos z + h(y,z)$$

$$\frac{\partial f}{\partial y} = 3x^2\cos z + \frac{\partial H}{\partial y} = 3x^2\cos z \Rightarrow \frac{\partial h}{\partial y} = 0$$

$$\Rightarrow h(y,z) = k(z)$$

$$\Rightarrow f(x,y,z) = 3x^2y\cos z + k(z) = \boxed{3x^2y\cos z}$$

$$\frac{\partial f}{\partial z} = -3x^2y\sin z + k'(z) = -3x^2y\sin z$$

$$\Rightarrow k'(z) = 0 \Rightarrow k(z) = C = 0 \ [\text{aukeratu}]$$

iii). Kalkulatu $\int_\sigma \vec{F}ds$ non

$$\sigma(t) = (\cos^3 t, \sin^3 t, 0) \qquad t \in [0, \pi/2]$$

$$\int_\sigma \vec{F}ds = \int_0^{\pi/2} \vec{F}(\sigma(t)) \cdot \sigma'(t)dt =$$

$$= \int_0^{\pi/2} (6\cos^3 t\sin^3 t \cdot 1,\ 3\cos^6 t \cdot 1,\ -3\cos^6 t \cdot \sin^3 t \cdot 0) \cdot$$
$$\cdot (3\cos^2 t \cdot (-\sin t),\ 3\sin^2 t\cos t, 0)dt = \dots \text{astuna}$$

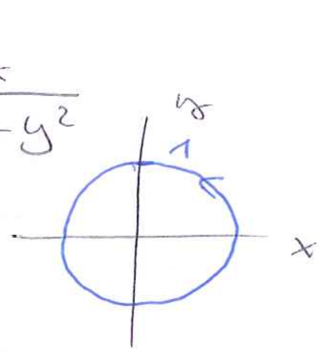
$$\int_\sigma \vec{F}ds = \int_\sigma \nabla f ds = f(\sigma(\pi/2)) - f(\sigma(0)) =$$

$$= f(0,1,0) - f(1,0,0) = \boxed{0 - 0 = 0}$$

11. ARIKETA

$$P(x,y) = \frac{-y}{x^2+y^2} \wedge Q(x,y) = \frac{x}{x^2+y^2}$$

i) Frogatu $\displaystyle\int_c \frac{xdy - ydx}{x^2+y^2} = 2\pi$



$$c \to \sigma(\theta) = (\cos\theta, \sin\theta) \qquad \theta \in [0, 2\pi]$$

$$\sigma(0) = (1,0) \ ; \ \sigma(\pi/2) = (0,1) \Rightarrow$$

ORIENTAZIOA MANTENDU

B1 nodu

1) $\int_C \vec{F}ds = \int_{C_1} \vec{F}ds \stackrel{def}{=} \int_0^1 \vec{F}(\sigma(t))\cdot \sigma'(t)dt =$

$= \int_0^1 (t^3 - 3t^3, t^3 - 3t^3, t)\cdot(1,1,1)dt =$

$= \int_0^1 -4t^3 + t\, dt = [\dots] = \boxed{-\frac{1}{2}}$

2) $\int_C \vec{F}ds = \int_C \nabla f\, ds = f(\sigma(b)) - f(\sigma(a))$

f ? non $\nabla f = \vec{F}$

$\frac{\partial f}{\partial x} = x^3 - 3xy^2 \rightarrow f(x,y,z) = \frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + h(y,z)$

$\frac{\partial f}{\partial y} = y^3 - 3x^2y \rightarrow \cancel{-3x^2y} + h'(y,z) = y^3 \cancel{-3x^2y}$

$h'(y,z) = y^3$

$\frac{\partial h}{\partial y} = y^3 \Rightarrow h(y,z) = \frac{y^4}{4} + k(z)$

$\Rightarrow f(x,y,z) = \frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{y^4}{4} + k(z)$

$\frac{\partial f}{\partial z} = k'(z) = z \Rightarrow k(z) = \frac{1}{2}z^2 + C$

$\hookrightarrow C=0$ aukeratu

$f(x,y,z) = \frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{y^4}{4} + \frac{1}{2}z^2$

$\int_C \vec{F}ds = \int_C \nabla f ds = f(1,1,1) - f(0,0) =$

$= \frac{1}{4} - \frac{3}{2} + \frac{1}{4} + \frac{1}{2} = \boxed{-\frac{1}{2}}$

10 ARIKETA

$\vec{F}(x,y,z) = (6xy\cos z, 3x^2\cos z, -3x^2y\sin z)$

i) $rot\vec{F} = \dots = 0 \Rightarrow \vec{F}$ kontserbakorra

iii) $\iint_S rot\vec{F}ds \overset{6.\text{GAIA}}{=} \iint \underbrace{rot\,\vec{F}}_{(0,0,2y+1)}\cdot\vec{n}\,ds = \iint_S (0,0,2y+1)(0,0,1)ds =$

$= \iint_S 2y+1\,ds = \iint_D 2y+1\,dx\,dy = 2\underbrace{\iint_D \overset{\text{BAKOITIA}}{y}\,dx\,dy}_{\text{SIMETRIKOA}} + \underbrace{\iint_D 1\,dx\,dy}_{A(D)} = \pi$

STOKES

$$\underbrace{\int_{\partial S}\vec{F}ds}_{\pi \quad \text{ii)}} \underset{BAI}{\overset{?}{=}} \underbrace{\iint_S rot\vec{F}ds}_{\pi \quad \text{iii)}}$$

8. ARIKETA

$$\int_C \vec{F}\cdot ds$$

i) $\vec{F}(x,y) = (xy^2+3x^2y,\ (x+y)x^2)$

$$rot\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2+3x^2y & (x+y)x^2 & 0 \end{vmatrix} = \bar{0} \underset{\text{def}}{\Rightarrow} \vec{F} \text{ KONTSERBAKORRA}$$

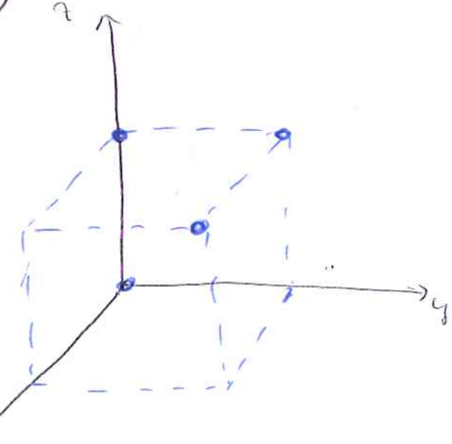
* Kurba itxi eta sinpleetako $\int_C \vec{F}\,ds = 0$

$$\Rightarrow \int_C \vec{F}\,ds = 0$$

iii) $\vec{F}(x,y,z) = (x^3-3xy^2,\ y^3-3x^2y,\ z)$

$C \Rightarrow (0,0,0),\ (0,0,1),\ (0,1,1),\ (1,1,1)$

$$rot\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3-3xy^2 & y^3-3x^2y & z \end{vmatrix} = \bar{0}$$



$\Rightarrow \vec{F}$ KONTSERBAKORRA $\Rightarrow \forall C, C_1$

Kurbak bi mutur berdinekin

$\Rightarrow \int_C \vec{F}ds = \int_{C_1}\vec{F}ds$ $\qquad C_1: \sigma(t) = (t,t,t)\quad t\in[0,1]$

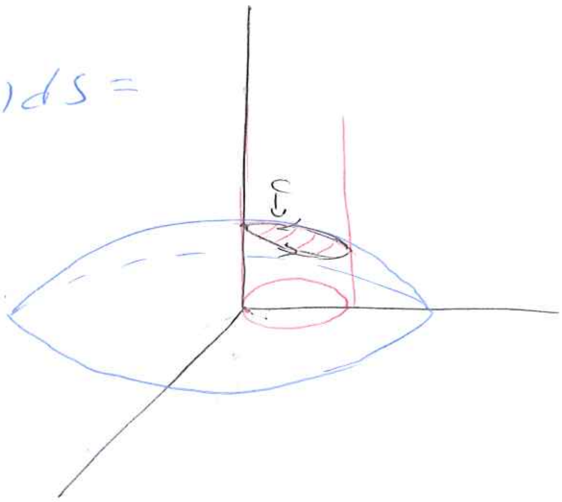
$\sigma(0) = (0,0,0)$
$\sigma(1) = (1,1,1)$ $\Rightarrow$ ORIENTAZIOA MANTENDU

6. ARIKETA

$$C = \begin{cases} \frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1 & z \geq 0 \\ x^2 + y^2 - y = 0 \longrightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} \end{cases}$$

$$\vec{F} = (y^3, xy + 3xy^2, z^4)$$

$$\vec{F}\text{-ren zirkulazioa } C\text{-n zehar} = \int_C \vec{F}\, ds$$



$$\int_C \vec{F}\, ds \overset{\text{STOKES}}{\underset{C = \partial S}{=}} \iint_S \text{rot}\, \vec{F}\, ds = \iint (0,0,y)\, dS =$$

$$= \left[\begin{array}{l} z = g(x,y) = \sqrt{1 - \frac{x^2+y^2}{2}} \\ g: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R} \end{array}\right]$$

$$= \iint_D -0 \cdot g_x - 0 g_y + y\, dx\, dy =$$

$$= \iint_D y\, dx\, dy =$$

$x = \rho \cos\theta$

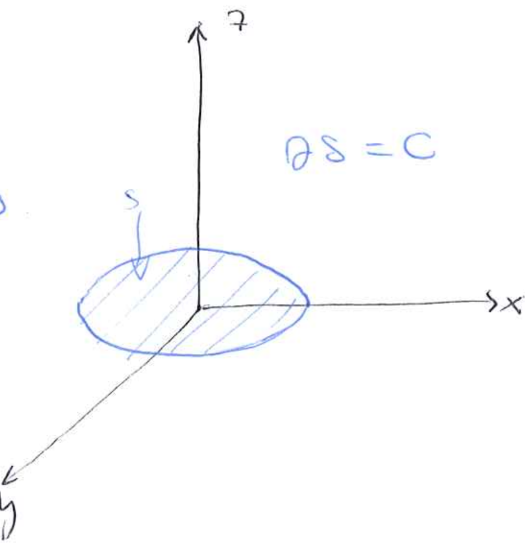
$y = \rho \sin\theta$ $\quad \theta \in [0, \pi]$

$|J| = \rho$ $\quad \rho \in [0, \text{zirkunf}] = [0, \sin\theta]$

$$= \int_0^{\pi} \int_0^{\sin\theta} \rho \sin\theta\, \rho\, d\rho\, d\theta = [\ldots] = \frac{\pi}{8}$$

7. ARIKETA

$$\vec{F}(x,y,z) = (x^2, 2xy + x, -z)$$



$$i) \iint_S \vec{F}\, ds \overset{\text{6. GAIA}}{=} \iint_S \vec{F} \cdot \vec{n}\, ds = \iint \overset{0}{\overset{\shortparallel}{z}}\, ds = 0$$

[FLUXUA]

$$ii) \int_C \vec{F}\, ds = \int_0^{2\pi} F(\sigma(\theta)) \cdot \sigma'(\theta)\, d\theta =$$

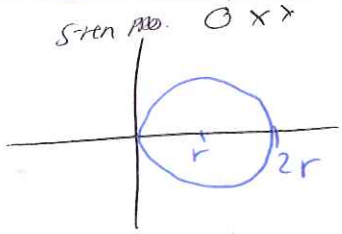
$$= \int_0^{2\pi} (\cos^2\theta, 2\cos\theta \sin\theta + \cos\theta, 0) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta =$$

$$= [\ldots] = \pi$$

6

$$= 2\iint_D -(y-z)g_x - (z-x)g_z + (x-y)\,dx\,dy =$$

S-ten pro. Oxy



$$z = g(x,y) = \sqrt{2Rx - x^2 - y^2}$$

$$g: D \subset \mathbb{R}^2 \to \mathbb{R} \qquad g(D) = S$$

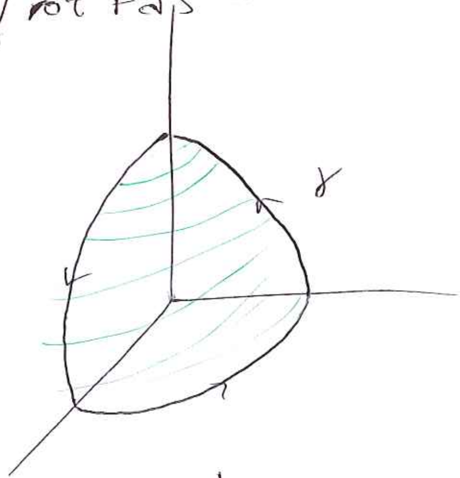
$$= 2\iint_D \left(-y + \sqrt{2Rx - x^2 - y^2}\right) \cdot \frac{2R - 2x}{2\sqrt{2Rx - x^2 - y^2}} +$$

$$+ \left(-2\sqrt{2Rx - x^2 - y^2} + x\right) \cdot \frac{-2y}{2\sqrt{2Rx - x^2 - y^2}} + (x-y)\,dx\,dy$$

$$= \ldots = 2\pi R r^2$$ — Kortejicnck —

v) $I = \oint_\gamma 2z\,dx - x\,dy + 3y\,dz \overset{\text{STOKES}}{=} \iint_S \operatorname{rot} \vec{F}\, d\vec{S} =$

$\vec{F} = (2z, -2x, 3y)$ S non $\partial S = \gamma$

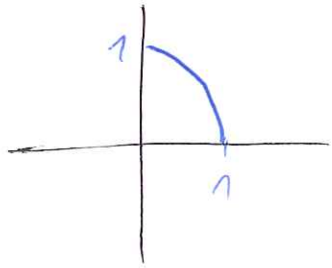


$\operatorname{rot} \vec{F} = (3, 2, -1)$

$z = g(x,y) = 1 - x^2 - y^2$

$g: D \subset \mathbb{R}^2 \to \mathbb{R}$

S-ten projekcija Oxy ploncn



$$= \iint_S (3, 2, -1)\,d\vec{S} = \iint_D -G_1 g_x - G_2 g_y + G_3\,dx\,dy =$$

$$= \iint_D -3(-2x) - 2\cdot(-2y) - 1\,dx\,dy =$$

$$= \int_0^{\pi/2}\int_0^1 (6\rho\cos\theta + 4\rho\sin\theta - 1)\rho\,d\rho\,d\theta = \ldots = \frac{10}{3} - \frac{\pi}{4}$$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\quad |J| = \rho \quad \theta \in [0, \pi/2] \quad \rho \in [0, 1]$

$$(2,5) \Rightarrow \begin{cases} 2 = \bar{x} = \dfrac{\iint_R \tilde{\rho}\, x\, dx\,dy}{\iint_R \tilde{\rho}\, dx\,dy} \Rightarrow 2 = \dfrac{\iint_R x\, dx\,dy}{A(R)} \Rightarrow 2A(R) = \iint x\,dx\,dy \quad (1) \\ 5 = \bar{y} = \dfrac{\iint_R \tilde{\rho}\, y\, dx\,dy}{\iint_R \tilde{\rho}\, dx\,dy} \Rightarrow 5\cdot A(R) \overset{(2)}{=} \iint_R y\, dx\,dy \end{cases}$$

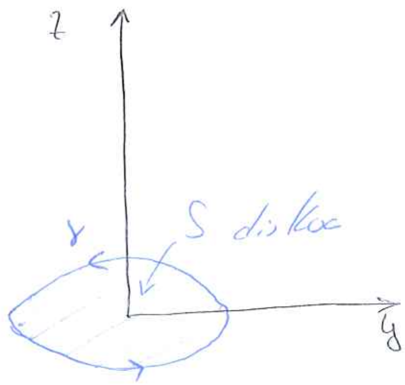
$$\text{Lehen, } 21 = \iint_R 2x + 2y\, dx\,dy = 2\iint_R x\, dx\,dy + 2\iint_R y\, dy\,dx \overset{(1)}{\underset{(2)}{=}}$$

$$= 2\cdot 2A(R) + 2\cdot 5\cdot A(R) \Rightarrow 21 = 14A \Rightarrow \boxed{A(R) = \frac{3}{2}}$$

5. Ariketa

iii) $I = \oint_\gamma z\,dx + x\,dy + y\,dz \overset{\text{STOKES}}{=} \iint_S \operatorname{rot} \vec{F}\, ds = \iint_S (1,1,1)\, ds = *$

$\gamma = \begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$ non $\partial S = \gamma$.



$$\operatorname{rot} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1,1,1)$$

$$\pi \cdot 2^2 = A(S) = \iint_S 1\, ds = \Delta$$

TEOR 6.2

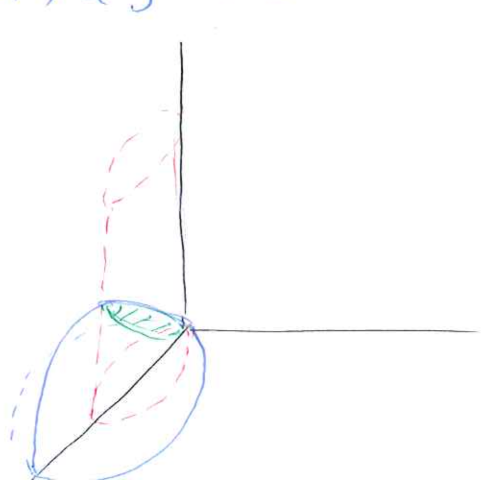
$$* = \iint_S (1,1,1)\cdot \underset{(0,0,1)}{\vec{n}}\, ds = \Delta$$

iv) $I = \oint_\gamma (y^2 + z^2)\,dx + (x^2 + z^2)\,dy + (x^2 + y^2)\,dz =$

$$\vec{F} = (y^2 + z^2,\ x^2 + z^2,\ x^2 + y^2)$$

$$\gamma \begin{cases} x^2 + y^2 + z^2 = 2Rx & \to (x-R)^2 + y^2 + z^2 = R^2 \\ x^2 + y^2 = 2rx \quad (z \geq 0) & \to (x-r)^2 + y^2 = r^2 \end{cases}$$

$0 < r < R$



$$\operatorname{rot} \vec{F} = (2y - 2z,\ 2z - 2x,\ 2x - 2y)$$

$$= \iint_S \operatorname{rot} \vec{F}\, ds = 2\iint_S (y - x,\ z - x,\ x - y)\, ds$$

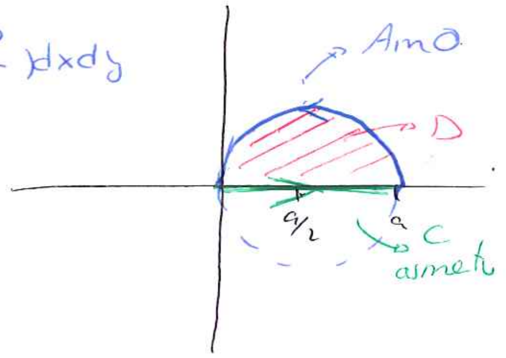
2. ARIKETA

$$I=\int (e^x\sin y-my)\,dx+(e^x\cos y-m)\,dy$$

$AmO \equiv A=(a,0)$ puntutik $O=(0,0)$-ra

doan $x^2+y^2=ax$ goiko zirkunferentzia erdia

$$I+\int_C \underbrace{P\,dx+Q\,dy}_{0\ [\text{egin}]}=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$



$$\bullet \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=$$

$$=\iint_D m\,dx\,dy=\int_0^{\pi}\int_0^{a/2} m\,\rho\,d\rho\,d\theta=[\ldots]=\frac{ma^2\pi}{8}$$

$$\begin{cases} x=\frac{a}{2}+\rho\cos\theta \\ y=0+\rho\sin\theta\end{cases}\quad |J|=\rho\qquad \begin{matrix}\rho\in[0,\frac{a}{2}]\\ \theta\in[0,\pi]\end{matrix}$$

$$\boxed{I=\frac{ma^2\pi}{8}}$$

$$\int_{AmO}\underbrace{(e^x\sin y-my)}_{P}\,dx+\underbrace{(e^x\cos y-m)}_{Q}\,dy+$$

$$+\int_{C_1}P\,dx+Q\,dy=\int_0^a[(e^x\sin 0-m\cdot 0)\cdot 1+(e^t\cos 0-m)\cdot 0]\,dt=0$$

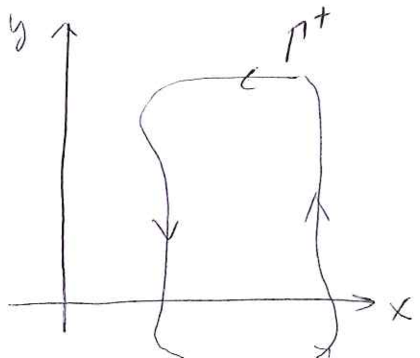
$\gamma_1(t)=(t,0)\quad t\in[0,a]\qquad y=0$

Green $\parallel$

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\int_0^{a/2} m\rho\,d\rho\,d\theta=\frac{ma^2\pi}{8}$$

$$I+\phi=\frac{\pi a^2 m}{8}$$

4. ARIKETA



$A(R)=?$

$$2A=\int_{\Gamma^+}\underbrace{(\arctan x-y^2)}_{P}\,dx+\underbrace{(\ln y+x^2)}_{Q}\,dy=$$

$$\overset{\text{GREEN}}{=}\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\iint 2x+2y\,dx\,dy$$

3

iii) $I = \oint_\gamma (2x - y^3)\,dx - xy\,dy$, =

$\gamma \equiv \{ x^2+y^2=1,\ x^2+y^2=9$ zirkuitua, positiboranzt $\}$

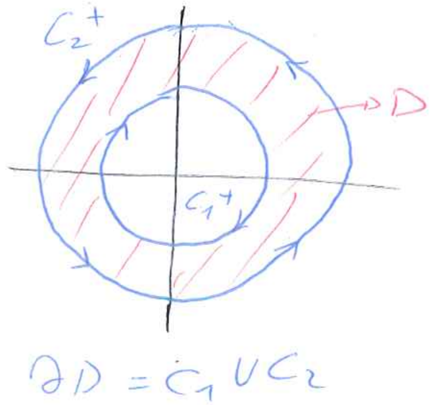
C_1 : $\sigma_1 : [0, 2\pi] \longrightarrow \mathbb{R}$

$\sigma_1 = (\cos\theta, \sin\theta)$

$\sigma_1(0) = (1, 0)$

$\Rightarrow$ Orientazioa aldatu

$\sigma(\pi/2) = (0, 1)$



C_2 : $\sigma_2 : [0, 2\pi] \longrightarrow \mathbb{R}$

$\sigma_2(\theta) = (3\cos\theta, 3\sin\theta)$

$\partial D = C_1 \cup C_2$

$\sigma_2(0) = (3,0)$

$\sigma_2(\pi/2) = (0, 3)$

$$= -\int_0^{2\pi} [(2\cos\theta - \sin^3\theta)(-\sin\theta) - \cos\theta\sin\theta\cdot\cos\theta]\,d\theta +$$

$$+ \int_0^{2\pi} [(6\cos\theta - 27\sin^3\theta)(-3\sin\theta) - 3\cos\theta \cdot 3\sin\theta \cdot 3\cos\theta]\,d\theta =$$

$$= \int_0^{2\pi} -16\sin\theta\cos\theta + 80\sin^4\theta - 26\sin\theta\cos^2\theta\,d\theta =$$

$$= -16 \frac{\sin^2\theta}{2}\Big|_0^{2\pi} - 26\left[\frac{\cos^3\theta}{3}\right]_0^{2\pi} + 80\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta =$$

$$= [\dots] = 60\pi$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D (-y + 3y^2)\,dx\,dy =$$

$$= \iint_D 3y^2\,dx\,dy \overset{\text{POLARRAK}}{=} \int_0^{2\pi}\int_1^3 3\rho^2 \sin^2\theta\,\rho\,d\rho\,d\theta = [\dots] = 60\pi$$

$\theta \in [0, 2\pi]$
$\rho \in [1, 3]$

$\Rightarrow$ Egiaztatu dugu Greenen Teorema.

7. Analisi Bektorialeko Teoremak

Ariketak

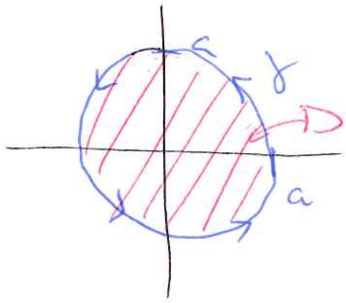
1. Ariketa

iv) $I = \oint_\gamma xy^2dy - x^2ydx$

$\gamma \equiv x^2+y^2=a^2$ zirkunferentzia, noranzko positiboan

GREEN $\Rightarrow$ $\oint_\gamma Qdy + Pdx = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy : \partial D = \gamma$

$I = \oint_\gamma xy^2dy - x^2ydx \overset{def}{=}$



$\sigma : [0, 2\pi] \to \mathbb{R}$

$\sigma(\theta) = (a\cos\theta, a\sin\theta)$

$\sigma(0) = (a,0)$
$\sigma(\pi/2) = (0,a)$ $\Rightarrow$ Noranzkoa mantendu

$= + \int_0^{2\pi} [a\cos\theta(a\sin\theta)^2 \underbrace{a\cos\theta}_{dy} - (a\cos\theta)^2 a\sin\theta \underbrace{(-a\sin\theta)}_{dx}] d\theta =$

$= 2a^4 \int_0^{2\pi} \sin^2\theta \cdot \cos^2\theta \, d\theta = 2a^4 \cdot \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2} \cdot \frac{1+\cos 2\theta}{2}\right)^2 d\theta =$

$= [\dots] = \frac{a^4\pi}{2}$

$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = \iint_D (y^2+x^2) dxdy \overset{\text{POLARRAK}}{=}$

$= \int_0^{2\pi}\int_0^a \rho^2 \rho \, d\rho \, d\theta = \boxed{\frac{\pi a^4}{2}}$

$\Rightarrow$ Egiaztatu dugu Greenen teorema

12. ARIKETA

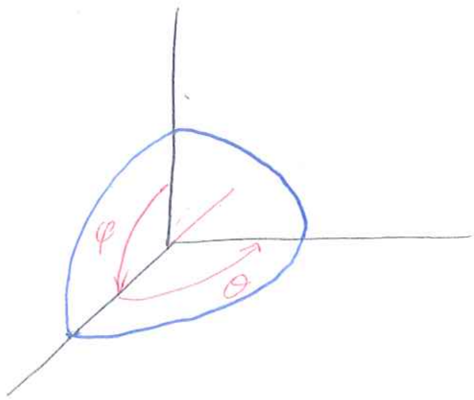
ii) $I = \iint_S (ax, by, cz) dS \overset{GAUSS}{=} \iiint_V div \vec{F} dV =$

$= \iiint_V (a+b+c) dV = (a+b+c) \iiint_V 1 dV = (a+b+c) \cdot B(V)$

iii) $I = \iint_{\partial\Omega} (xy, yz, zx) dS \overset{GAUSS}{=} \iiint_\Omega div \vec{F} dV =$

$= \iiint_\Omega (y+z+x) dx dy dz =$

ALD - ALD : ESFERIKOAK



$x = \rho \cos\theta \sin\varphi$

$y = \rho \sin\theta \sin\varphi \qquad \theta, \varphi \in [0, \pi/2]$

$z = \rho \cos\varphi \qquad \rho \in [0,1]$

$|J| = \rho^2 \sin\varphi$

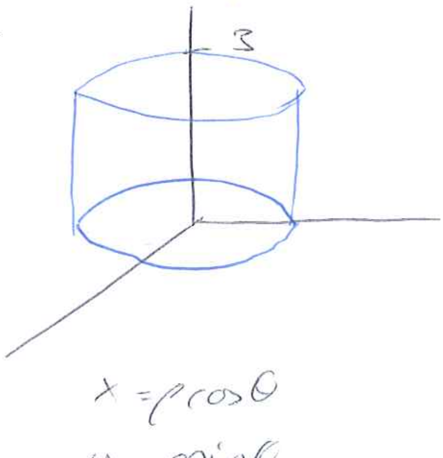
$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho \sin\theta \sin\varphi + \rho\cos\varphi + \rho \cos\theta \sin\varphi) \rho^2 \sin\varphi \, d\rho d\varphi d\theta =$

$= [\dots] = \boxed{\frac{3\pi}{16}}$

v) $I = \iint_{\partial\Omega} (yx, -2y^2, z^2) \cdot dS =$

$\Omega = \{(x,y,z) : x^2+y^2 \leq 4, \ 0 \leq z \leq 3\}$ $\quad \partial\Omega = S_1 \cup S_2 \cup S_3$

$z=0 \quad z=3 \quad$ zilindroa



$= \iiint_\Omega div \vec{F} = \iiint_\Omega (y - 4y + 2z) dV =$

$= \iiint (-3y + 2z) dx dy dz =$

$= \int_0^{2\pi} \int_0^2 \int_0^3 (-3\rho\sin\theta + 2\rho\cos\theta) \rho \, dz d\rho d\theta =$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

11

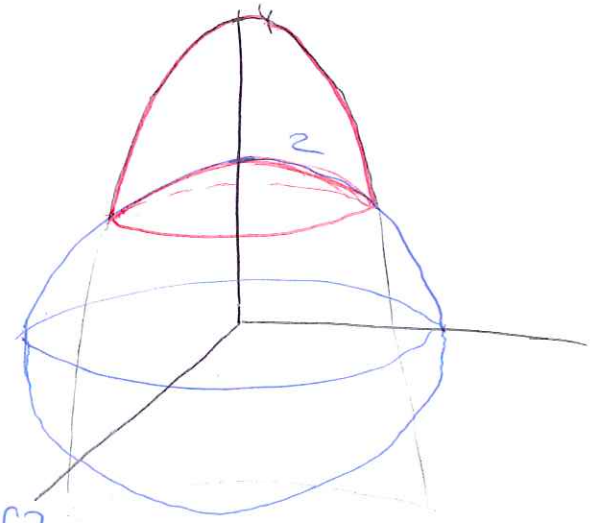
$$= \int_0^{2\pi}\int_0^{2}\int_0^{3} -3\rho^2\sin\theta + 2\rho^2\cos\theta \, dz\, d\rho\, d\theta = [\cdots] = \boxed{36\pi}$$

vi) $\iint_S (xy, yz, zx) \cdot dS$

$S = \partial\Omega \Rightarrow \Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \geq 4,$

$z \leq 4-(x^2+y^2),\ z \geq 0\}$

$S = \underset{\text{parabo}}{S_1} \cup \underset{\text{esfera}}{S_2}$



$$\begin{cases} x^2+y^2+z^2 = 4 \\ z = 4-(x^2+y^2) \end{cases}$$

$z^2 + 4 = 4 + z$

$z^2 - z = 0 \Rightarrow z = 0, 1$

$z = 1 \Rightarrow x^2+y^2 = 3$ [zirkunf]

$$\iint_S (xy, yz, zx)\, dS = \iiint_\Omega \operatorname{div} \vec{F}\, dV$$

$$= \iiint_\Omega (y+z+x)\, dV = \begin{bmatrix} x = \rho\cos\theta & \theta \in [0, 2\pi] \\ y = \rho\sin\theta & \rho \in [0, \sqrt{3}]\ |J| = \rho \\ z = z & z \in [\text{zirk}, \text{parabo}] \end{bmatrix}$$

$x^2+y^2+z^2 = 4 \Rightarrow z = \sqrt{4-\rho^2}$

$z = 4-(x^2+y^2) \Rightarrow z = 4-\rho^2$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{4-\rho^2}}^{4-\rho^2} (\rho\sin\theta + z + \rho\cos\theta)\rho\, dz\, d\rho\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left[\rho^2\sin\theta z + \rho\frac{z^2}{2} + \rho^2\cos\theta z \right]_{\sqrt{4-\rho^2}}^{4-\rho^2} d\rho\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \cancel{\rho^2(4-\rho^2)\sin\theta - \rho^2\sqrt{4-\rho^2}\sin\theta} + \frac{\rho}{2}(4-\rho^2)^2 - \frac{\rho}{2}(4-\rho^2) +$$

$$+ \cancel{\rho^2\cos\theta(4-\rho^2) - \rho^2\cos\theta\sqrt{4-\rho^2}}\, d\rho\, d\theta =$$

$$= \int_0^{\sqrt{3}} \int_0^{2\pi} \frac{\rho}{2}(4-\rho^2)^2 - \frac{\rho}{2}(4-\rho^2)\, d\theta\, d\rho =$$

$$= [\dots] = \boxed{\frac{\pi \cdot 27}{4}}$$

13. ARIKETA

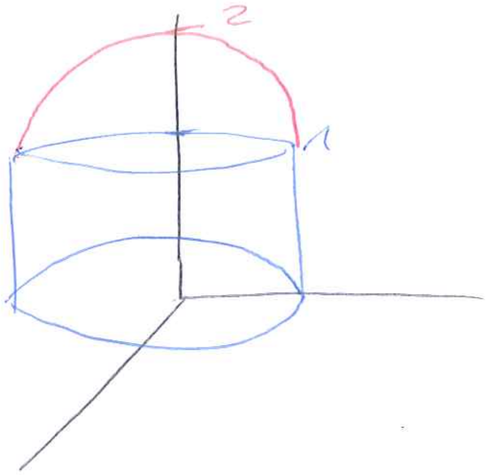
zilindroaren muga, ez horren

$$S = \begin{cases} x^2+y^2=1, & 0 \le z \le 1 \\ x^2+y^2+(z-1)^2=1, & z \ge 1 \end{cases}$$

Kalkulatu $\iint_S rot \vec{F} dS$

$$\vec{F}(x,y,z) = (zx + z^2 y + x,\ z^3 y x + y,\ z^4 x^2)$$

$$S = S_1 \cup S_2$$



HIRU AUKERA:

1) Definizioarekin

$$\iint_S rot \vec{F} ds = \iint_{S_1} rot \vec{F} ds + \iint_{S_2} rot \vec{F} ds$$

2) Gauss aplikatu S itxiz

$$\tilde{S} = S \cup S_3 \Rightarrow S_3 = \{z=0,\ x^2+y^2 \le 1\}$$

behetik tapa jarri

$\tilde{S}$ itxia eta $\iint_{\tilde{S}} rot \vec{F} dS \overset{\text{GAUSS}}{=} \iiint_{\Omega} \underbrace{div(rot \vec{F})}_{=0} ds = 0$

non $\partial \Omega = \tilde{S}$

$$\iint_{\tilde{S}} rot \vec{F} ds = \iint_S rot \vec{F} ds + \overbrace{\iint_{S_3} rot \vec{F} ds}^{\iint_{S_3} rot \vec{F} \cdot \vec{n}\, ds,\ \vec{n}=(0,0,-1)} = 0$$

$$\iint_S rot \vec{F} ds = -\iint_{S_3} rot \vec{F} \cdot \vec{n}\, ds$$

3) STOKES erabili

$$\iint_S \text{rot}\,\vec{F}\,ds = \int_{\partial S} \vec{F}\,ds = \int_0^{2\pi} \vec{F}(\sigma(\theta))\,\sigma'(\theta)\,d\theta =$$

$$\sigma(\theta) = (\cos\theta, \sin\theta, 0)$$

$$\begin{cases} x^2+y^2=1 \\ z=0 \end{cases} \qquad \theta \in [0, 2\pi] \qquad \sigma(0) = (1,0,0) \qquad \sigma(\pi/2) = (0,1,0)$$

$$= \int_0^{2\pi} (\cos\theta, \sin\theta, 0)\cdot(-\sin\theta, \cos\theta, 0)\,d\theta =$$

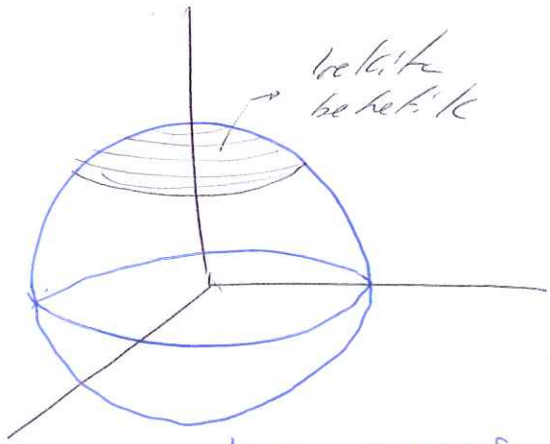
$$= \int_0^{2\pi} 0\,d\theta = 0$$

15. ARIKETA

$$\vec{F}(x,y,z) = (1,0,1)$$

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 2, \; z \geq 1\}$$

$$\iint_S \vec{F}\,ds =$$



1) DEFINIZIOA ERABILIZ

$$g: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

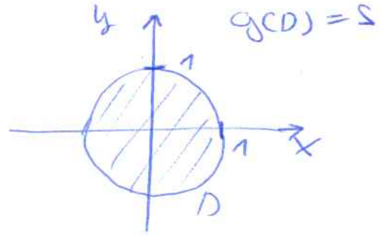
$$z = g(x,y) = \sqrt{2-x^2-y^2}$$

PROIEKTATU OXY PLANOAN

$$z=1 \rightarrow x^2+y^2+1 = 2$$

$$x^2+y^2 = 1$$

$$g(D) = S$$



$$\iint_S \vec{F}\,ds = \iint_D -F_1 g_x - F_2 g_y + F_3\,dx\,dy =$$

$$= \iint_D +1\cdot\frac{+\cancel{2}x}{\cancel{2}\sqrt{2-x^2-y^2}} - 0\cdot g_y + 1\,dx\,dy =$$

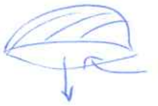
$$= \iint_D x(2-x^2-y^2)^{-1/2} + 1\,dx\,dy = \begin{bmatrix} x = \rho\cos\theta & \theta\in[0,2\pi] \\ y = \rho\sin\theta & \rho\in[0,1] \end{bmatrix}$$

$$= \int_0^{2\pi}\int_0^1 (\rho\cos\theta(2-\rho^2)^{-1/2} + 1)\rho \, d\rho \, d\theta =$$

$$= \int_0^1\int_0^{2\pi} (\rho^2(2-\rho^2)^{-1/2}\cos\theta + \rho) \, d\theta \, d\rho =$$

$$= \int_0^1 \left[\theta\rho\right]_0^{2\pi} d\rho = \int_0^1 2\pi\rho \, d\rho = \pi\rho^2\Big|_0^1 = \boxed{\pi}$$

2. GAUSS ERABILIZ



$\Rightarrow$ ITXI BEHAR

$\tilde{S} = S \cup S_1 \quad : \quad S_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 1, \; z=1\}$

$$\iint_{\tilde{S}} \vec{F} \, ds \overset{GAUSS}{=} \iiint_{\Omega \to \partial\Omega = \tilde{S}} \operatorname{div}\vec{F} \, dV = \iiint_{\Omega} 0 \, dV = 0$$

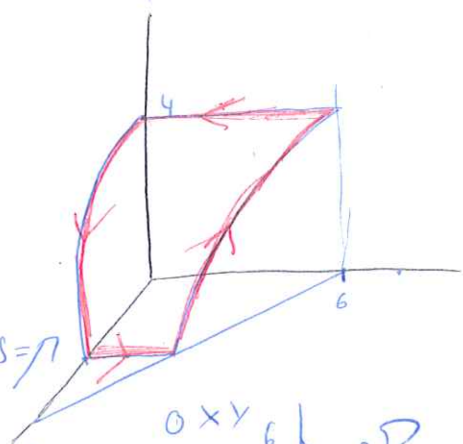
$$\iint_S \vec{F} \, ds + \iint_{S_1} \vec{F} \, ds = 0 \Rightarrow \iint_S \vec{F} \, ds = -\iint_{S_1} \vec{F} \, ds =$$

$$= -\iint_{S_1} \underbrace{\vec{F}}_{(1,0,1)} \cdot \underbrace{\vec{n}}_{(0,0,-1)} \, ds = -\iint_{S_1} -1 \, ds = \iint_{S_1} \underbrace{1 \, dS}_{A(S_1)} = \boxed{\pi}$$

14. ARIKETA

$$\begin{cases} 2x+y=6 \\ z = 4-x^2 \end{cases}$$

$\vec{F}(x,y,z) = (y, 2x, z)$

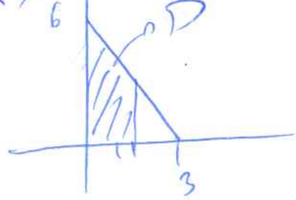


i) $\displaystyle\int_\Gamma \vec{F} \, ds \overset{STOKES}{=} \iint rot\,\vec{F} \, dS =$ non $\partial S = \Gamma$

$z = g(x,y) = 4-x^2$

$rot\,\vec{F} = (0,0,1) \qquad g : D \subset \mathbb{R}^2 \to \mathbb{R}$

OXY



$$= \iint_D 1 \, dx \, dy = \int_0^2 \int_0^{6-2x} 1 \, dy \, dx = \int_0^2 6-2x \, dx = 8$$

15

ii) $\iint_{\partial W} \vec{F}\, ds \overset{\text{Gauss}}{=} \iiint_W \underbrace{\operatorname{div} \vec{F}}_{1}\, dV = \iiint_W 1\, dx\, dy\, dz =$

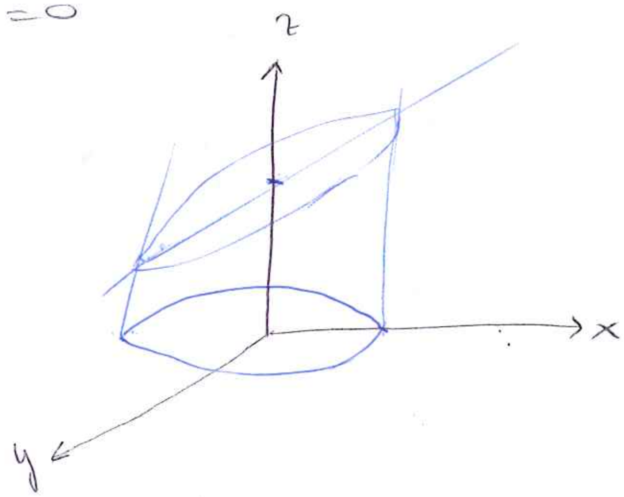
$= \int_0^2 \int_0^{6-2x} \int_0^{4-x^2} dz\, dy\, dx = \int_0^2 \int_0^{6-2x} 4 - x^2\, dy\, dx =$

$= \int_0^2 (4-x^2)\cdot(6-2x)\, dx \doteq [\dots] = \boxed{24}$

16. Ariketa

$\iint_S \operatorname{rot} \vec{F} \cdot dS$ $\qquad S \equiv \begin{cases} x^2+y^2=1 \\ z = x+2 \\ z=0 \end{cases}$

$\vec{F}(x,y,z) = (y, z, x)$



$\iint_S (-1,-1,-1)\, dS$

S ez da itxia $\xrightarrow{\quad}$ tapak

itxia $\leftarrow \tilde{S} = S \cup \underset{z=0}{S_1} \cup \underset{z=x+2}{S_2}$

$\iint_{\tilde{S}} \operatorname{rot}\vec{F}\, ds = \iiint_\Omega \underbrace{\operatorname{div}(\operatorname{rot}\vec{F})}_{0}\, dV = 0$

$\shortparallel$

$\iint_S \operatorname{rot}\vec{F}\, ds + \iint_{S_1} \operatorname{rot}\vec{F}\, ds + \iint_{S_2} \operatorname{rot}\vec{F}\, ds = 0$

$\iint_{\underset{z=0}{S_1}} \operatorname{rot}\vec{F}\, ds = \iint_{S_1} (-1,-1,-1) \cdot \underset{(0,0,-1)}{\vec{n}}\, dS = \iint_{S_1} dS = A(S) = \pi$

$\iint_{S_2} \operatorname{rot}\vec{F}\, ds = \iint_{S_2} (-1,-1,-1) \cdot \frac{(-1,0,+1)}{\sqrt{2}}\, dS = \iint_{S_2} 0 = 0$

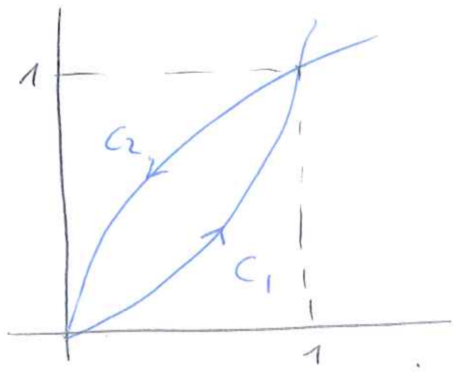
$z = x+2 \to x - z + 2 = 0 \Rightarrow (1,0,-1)$

$\iint_S \operatorname{rot}\vec{F}\, ds = -\iint_{S_1} \operatorname{rot}\vec{F}\, ds = \boxed{-\pi}$

1. ARIKETA

i) $I = \oint_{\gamma} (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$P = 3x^2 - 8y^2 \qquad Q = 4y - 6xy$



$C_1 : \sigma_1(t) = (t, t^2)$

$\sigma(0) = (0,0)$
$\sigma(1) = (1,1)$ $\Rightarrow$ ORIENTAZIOA MANTENDU

$C_2 : \sigma_2(t) = (t, \sqrt{t})$

$\sigma(0) = (0,0)$
$\sigma(1) = (1,1)$ $\Rightarrow$ ORIENTAZIOA ALDATU

$$\int_{C_1} Pdx + Qdy - \int_{C_2} Pdx + Qdy =$$

$$= \int_0^1 3t^2 - 8y^4 + (4t^2 - 6t^3)2t\,dt - \int_0^1 3t^2 - 8t + (4\sqrt{t} - 6t\sqrt{t})\frac{1}{2\sqrt{t}}\,dt =$$

$$= \int_0^1 3t^2 - 8t^4 + 8t^3 - 12t^4\,dt - \int_0^1 3t^2 - 8t + 2 - 3t\,dt =$$

$$= \left[t^3 - \frac{8}{5}t^5 + 4t^4 - \frac{12}{5}t^5\right]_0^1 - \left[t^3 - \frac{11}{2}t^2 + 2t\right]_0^1 = 1 - \frac{5}{2} = \frac{3}{2}$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy =$$

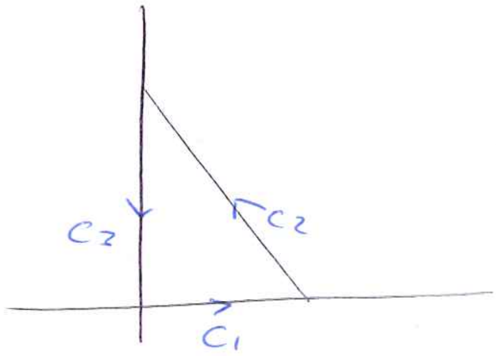
$$= \iint_D -6y + 16y\,dxdy = \iint_D 10y\,dxdy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dydx =$$

$$= \int_0^1 \left[5y^2\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 5x - 4x^4\,dx = \left[\frac{5}{2}x^2 - x^5\right] = \frac{5}{2} - 1 = \frac{3}{2}$$

ii) $I = \oint_\gamma (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

$x=0,\ y=0,\ x+y=1$

$P = 3x^2-8y^2 \qquad Q = 4y-6xy$



$C_1: \sigma_1(t) = (t,0) \qquad t\in[0,1]$

$\sigma_1(0) = (0,0)$

$\sigma_2(1) = (1,0)$ $\Rightarrow$ ORIENTAZIOA MANTENDU

$C_2: \sigma_2(t) = (t, 1-t) \qquad t\in[0,1] \qquad \sigma_2'(t) = (1,-1)$

$\sigma_2(0) = (0,1)$

$\sigma_2(1) = (1,0)$ $\Rightarrow$ ORIENTAZIOA ALDATU

$C_3: \sigma_3(t) = (0,t) \qquad t\in[0,1]$

$\sigma_3(0) = (0,0)$

$\sigma_3(1) = (0,1)$ $\Rightarrow$ ORIENTAZIOA ALDATU

$$\int_0^1 3t^2\,dt - \int_0^1 3t^2 - 8(1-2t+t^2) - 4(1-t) + 6t(1-t)\,dt +$$

$$-\int_0^1 4t\,dt =$$

$$= \left[\cancel{t^3} - \cancel{t^3} + 8\left(t - t^2 + \frac{1}{3}t^3\right) + 4t - 2t^2 - 3t^2 + \cancel{2t^3} - \cancel{2t^2}\right]_0^1 dt =$$

$$= \frac{8}{3} - 1 = \boxed{\frac{5}{3}}$$

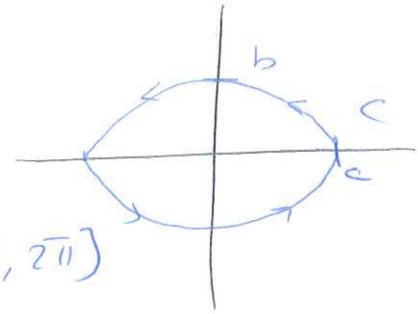
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D 10y\,dy\,dx = \int_0^1\int_0^{1-x} 10y\,dy\,dx =$$

$$= \int_0^1 \left[5y^2\right]_0^{1-x} dx = \int_0^1 5(1-2x+x^2)\,dx = \left[5\left(\cancel{x-x^2} + \frac{1}{3}x^3\right)\right]_0^1 =$$

$$= \boxed{\frac{5}{3}}$$

v) $I = \oint_\gamma (x+y)dx - (x-y)dy$

$\gamma \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



$C: \sigma(\theta) = (a\cos\theta, b\sin\theta) \quad \theta \in [0, 2\pi)$

$\left.\begin{array}{l}\sigma(0) = (a,0) \\ \sigma(\frac{\pi}{2}) = (0,b)\end{array}\right\} \Rightarrow$ ORIENTAZIOA NAINTENDU

$P = x+y \qquad Q = y-x$

$$\int_0^{2\pi} (a\cos\theta + b\sin\theta)(-a\sin\theta) - (a\cos\theta - b\sin\theta) b\cos\theta \, d\theta =$$

$$= \int_0^{2\pi} -a^2\cos\theta\sin\theta \underbrace{- ab\sin^2\theta - ab\cos^2\theta}_{-ab} - b^2\sin\theta\cos\theta \, d\theta =$$

$$= \left[\cancel{\frac{1}{2}a^2\cos^2\theta} - \frac{ab\theta}{2} - \cancel{\frac{ab\cos 2\theta}{4}} - \frac{ab\theta}{2} - \cancel{\frac{ab\sin 2\theta}{2}} - \cancel{\frac{b^2\sin^2\theta}{2}} \right]_0^{2\pi} = \boxed{-2\pi ab}$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D -1-1\,dx\,dy = \iint_D -2\,dx\,dy =$$

$$= \int_0^{2\pi}\int_0^1 -2ab\rho\,d\rho\,d\theta = \int_0^{2\pi} -ab\,d\theta = \boxed{-2\pi ab}$$

5. ARIKETA

i) $I = \oint_\gamma (y+z)dx + (z+x)dy + (x+y)dz$ non

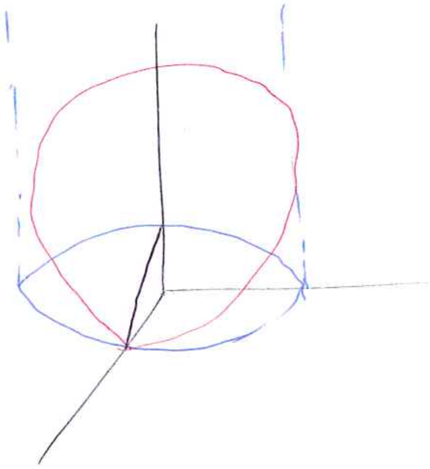
$\gamma(t) = (a\sin^2 t, 2a\sin t\cos t, a\cos^2 t), \quad 0 \le t \le \pi$

$$\mathrm{rot}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = (1-1, 1-1, 1-1) = \bar{0}$$

STOKES $\Rightarrow \int_{\partial S} \vec{F}\,ds = \iint_S \mathrm{rot}\vec{F}\,ds = 0$

ii) $I = \oint_\gamma (y-z)dx + (z-x)dy + (x-y)dz$

$\gamma = \frac{x}{a} + \frac{z}{h} = 1 \ , \ x^2+y^2=a^2$



$\vec{F}(y-z, z-x, x-y)$

$$\text{rot}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} =$$

$= (-1-1, -1-1, -1-1) = -2(1,1,1)$

$z = h - \frac{h}{a}x = g(x,y) \Rightarrow g_x = -\frac{h}{a} \quad g_y = 0$

STOKES $\Rightarrow \int_{\partial S} \vec{F} ds = \iint_S \text{rot}\vec{F} ds$

$I = \iint_D -2\frac{h}{a} - 2 dxdy = \int_0^{2\pi}\int_0^a -2\frac{h}{a} - 2\rho d\rho d\theta =$

$= -\int_0^{2\pi} ah + a^2 d\theta = \boxed{-2\pi a(h+a)}$

9. ARIKETA Kalkulatu potentziala $\nabla f = \vec{F}$

$\vec{F}(x,y,z) = (yz(2x+y+z), xz(x+2y+z), xy(x+y+2z))$

$$\text{rot}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz(2x+y+z) & xz(x+2y+z) & xy(x+y+2z) \end{vmatrix} =$$

$= (x^2 + 2xy + 2zx - x^2 - 2yx - 2xz,$

$2xz + 2zy + z^2 - 2zx - 2yz + \cancel{z}^2,$

$2zx + 2yz + z^2 \ 2xz - 2zy - z^2) = (0,0,0) \Rightarrow$ <u>KONTS.</u>

$$\vec{F}(x,y,z) = (2xyz + y^2z + z^2y,\ x^2z + 2yxz + z^2x,\ x^2y + xy^2 + 2zxy)$$

$$\frac{\partial f}{\partial x} = 2xyz + y^2z + z^2y$$

$$\Rightarrow f = x^2yz + y^2zx + z^2yx + h(y,z)$$

$$\frac{\partial f}{\partial y} = x^2z + 2yzx + z^2x + h'(y,z) = x^2z + 2yxz + z^2x$$

$$\Rightarrow h(y,z) = A + g(z)$$

$$\frac{\partial f}{\partial z} = x^2y + y^2x + 2zyx + g'(z) = x^2y + xy^2 + 2zxy$$

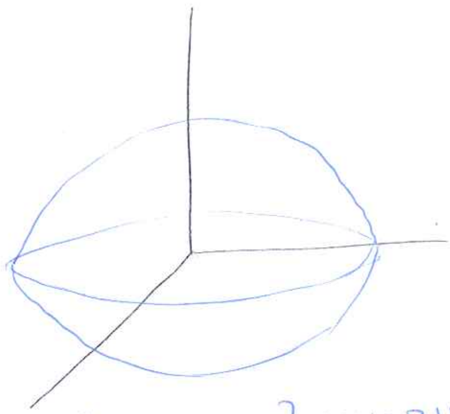
$$\Rightarrow g(z) = B$$

Aukerațu $A = B = 0 \Rightarrow \boxed{f(x,y,z) = x^2yz + y^2zx + z^2yx}$

12. ARIKETA

i) $I = \iint_S (x,y,z)\cdot dS$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Gauss $\Rightarrow \iint_S \vec{F}\,dS = \iiint_V \mathrm{div}\,\vec{F}\,dV'$

ALD. ALD: ZILINDRIKOAK

$x = a\rho\cos\theta$

$y = b\rho\sin\theta$

$z = cz$

$J = abc\rho$

$z^2 = 1-\rho^2$

$$I = \iiint_\Omega 3\,dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-\rho^2}}^{\sqrt{1-\rho^2}} 3abc\rho\,dz\,d\rho\,d\theta =$$

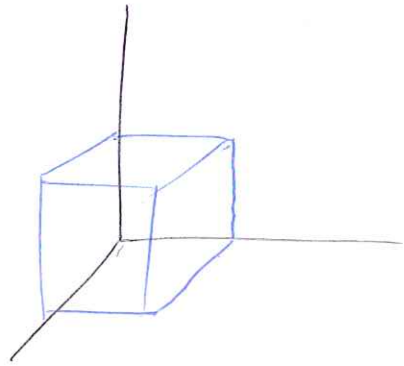
$$= \int_0^{2\pi}\int_0^1 6abc\,\rho\sqrt{1-\rho^2}\,d\rho\,d\theta =$$

$$= 6abc\int_0^{2\pi}\left[-\frac{1}{3}(1-\rho^2)^{3/2}\right]_0^1 d\theta =$$

$$= 2abc\int_0^{2\pi} 1-0\,d\theta = \boxed{4\pi abc}$$

iv) $I = \iint_S (x^2, y^2, z^2) \cdot dS \qquad S: 0 \le x, y, z \le a$

GAUSS: $\iint_S F ds = \iiint_V \operatorname{div} F dV'$



$I = \iiint_V (2x + 2y + 2z) dV =$

$= \int_0^a \int_0^a \int_0^a 2x + 2y + 2z \, dx \, dy \, dz =$

$= \int_0^a \int_0^a \left[x^2 + (2y + 2z)x\right]_0^a dy dz = \int_0^a \int_0^a a^2 + (2y + 2z) a \, dy dz =$

$= \int_0^a \left[a^2 y + a y^2 + 2azy\right]_0^a dz = \int_0^a a^3 + a^3 + 2za^2 \, dz =$

$= \left[a^3 z + a^3 z + z^2 a^2\right]_0^a = \boxed{3a^4}$

ANALISI BEKTORIALA ETA KONPLEXUA

Fisika eta Ingeniaritza Elektronikoko Graduetako 2. kurtsoa - 46. Taldea

Lehen azterketa partziala. 2019ko Urtarrilaren 15a.

+ 1. **(2.25 puntu)** Izan bedi $f(x,y,z) = x^2+y^2+z^2+x+y+z$ funtzioa. Aurkitu f-ren mutur absolutuak $D = \{(x,y,z) \in \mathbf{R}^3 : x^2+y^2+z^2 \leq 1, y+z=1\}$ eremuan.

+ 2. **(2 puntu)** Izan bedi ondorengo sistema,

$\frac{1}{2}+\frac{1}{4}+\frac{1}{2}$

$$\begin{cases} xe^{u+v} + uv - 1 = 0, \\ ye^{u-v} - 2uv - 1 = 0. \end{cases}$$

Frogatu sistema horrek $u = u(x,y)$ eta $v = v(x,y)$ funtzioak inplizituki definitzen dituela $(x,y,u,v) = (1,1,0,0)$ puntuaren ingurune batean. Kalkulatu u eta v funtzioen Taylorren lehen mailako polinomioak $(1,1)$ puntuan.

+ 3. **(2.25 puntu)** Kalkulatu bi era ezberdinetan $\int_C 3x^2ydx + dy$ lerro integrala, non C kurba $y=3x$, $y=x$ eta $x+y=4$ zuzenez osatuta dagoen, erlojuaren orratzen aurkako norantza harturik.

   a) Zuzenean.

   b) Analisi Bektorialeko teorema egoki bat erabiliz.

+ 4. **(1.5 puntu)** Izan bedi C kurba $y+2z$ eta $x^2+\frac{y^2}{4}+z^2=1$ gainazalen ebakidura. Kalkulatu ondorengo integrala,

$= 0$

$$\int_C (-y^3 + \cos(e^x))dx + ydy + zdz.$$

+ 5. **(2 puntu)** Kalkulatu $\vec{F}(x,y,z) = (x^3y+e^z, -x^2y^2, -x^2yz+2z)$ funtzioaren S gainazalarekiko fluxua non S gainazala honako solidoaren muga den, $\Omega = \{x^2+y^2-z^2 \leq 1, z \leq 2, z \geq -2\}$.

$3x^2y - 2x^2y - x^2y + 2$

$2y + 1 - \lambda = 0$

$2z + 1 - \lambda = 0$

$y + z = 1$

$\boxed{2017-01-17}$

1. ARIKETA

$f(x,y,z) = x+2y-z$

$W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1 \ , \ x+y+z \geq 0\}$

• W-ren BARRUAN

$f \ C^1$ denez $\to \nabla f = \bar{0}$ beher

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 1 \neq 0 \\ \dfrac{\partial f}{\partial y} = 2 \neq 0 \quad \Rightarrow \text{EZIN} \\ \dfrac{\partial f}{\partial z} = -1 \neq 0 \end{cases}$$

$\Rightarrow$ W-ren barruan ez dago puntu kritikorik

• W-ren MUGAN

• $x^2+y^2+z^2=1$ mugan

$h(\lambda,x,y,z) = f(x,y,z) - \lambda(g_1(x,y,z) - c_1)$

$g_1 = x^2+y^2+z^2 \qquad c_1 = 1$

$h(\lambda,x,y,z) = x+2y-z-\lambda(x^2+y^2+z^2-1)$

$$\nabla h = 0 \Rightarrow \begin{cases} \dfrac{\partial h}{\partial \lambda} = -x^2-y^2-z^2+1 = 0 \Rightarrow \dfrac{1}{4\lambda^2}+\dfrac{1}{\lambda^2}+\dfrac{1}{4\lambda^2} = 1 \\ \dfrac{\partial h}{\partial x} = 1-2x\lambda = 0 \Rightarrow x = \dfrac{1}{2\lambda} \\ \dfrac{\partial h}{\partial y} = 2-2y\lambda = 0 \Rightarrow y = \dfrac{1}{\lambda} \\ \dfrac{\partial h}{\partial z} = -1-2z\lambda = 0 \Rightarrow z = \dfrac{-1}{2\lambda} \end{cases}$$

$\dfrac{1}{2\lambda^2}+\dfrac{1}{\lambda^2} = 1$

$\dfrac{3}{2\lambda^2} = 1 \Rightarrow \lambda = \sqrt{\dfrac{3}{2}}$

$\lambda = +\sqrt{\frac{3}{2}} \Rightarrow P\left(\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right)$

$\lambda = -\sqrt{\frac{3}{2}} \Rightarrow P\left(\frac{-1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right)$

$\nabla g = (2x, 2y, 2z) = 0 \Rightarrow (0,0,0) \in \partial W$

- $x + y + z = 0$

$h(\mu, x, y, z) = f(x,y,z) - \mu(g_2(x,y,z) - C_2)$

$g_2 = x + y + z \qquad C_2 = 0$

$h(\mu, x, y, z) = x + 2y - z - \mu(x + y + z)$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial \mu} = -x - y - z = 0 \\ \frac{\partial h}{\partial x} = 1 - \mu = 0 \\ \frac{\partial h}{\partial y} = 2 - \mu = 0 \\ \frac{\partial h}{\partial z} = -1 - \mu = 0 \end{cases}$$

$\nexists \mu$ non $\nabla h = \bar{0}$

$\nabla g_2 = (1,1,1) \neq 0$

- EBAKIDURA

$g_1 = x^2 + y^2 + z^2$

$g_2 = x + y + z$

$h(\lambda, \mu, x, y, z) = f(x,y,z) - \lambda g_1 - \mu g_2$

$h(\lambda, \mu, x, y, z) = x + 2y - z - \lambda(x^2 + y^2 + z^2) - \mu(x + y + z)$

$$\nabla h = \bar{0} \Rightarrow \begin{array}{l} \frac{\partial h}{\partial \lambda} = -x^2 - y^2 - z^2 = 0 \\ \frac{\partial h}{\partial \mu} = -x - y - z = 0 \\ \frac{\partial h}{\partial x} = 1 - 2\lambda x - \mu = 0 \rightarrow x = \frac{1-\mu}{2\lambda} \\ \frac{\partial h}{\partial y} = 2 - 2\lambda y - \mu = 0 \rightarrow y = \frac{2-\mu}{2\lambda} \\ \frac{\partial h}{\partial z} = -1 - 2\lambda z - \mu = 0 \rightarrow y = \frac{-1-\mu}{2\mu} \end{array}$$

$$x+y+z=0 \Rightarrow \frac{1-\mu}{2\lambda}+\frac{2-\mu}{2\lambda}+\frac{-1-\mu}{2\lambda}=0$$

$$1-\mu+2-\mu-1-\mu=0 \Rightarrow 2=3\mu \Rightarrow \mu=\frac{2}{3}$$

$$x=\frac{1}{6\lambda} \qquad y=\frac{4}{6\lambda}=\frac{2}{3\lambda} \qquad z=\frac{-5}{6\lambda}$$

$$x^2+y^2+z^2=0 \Rightarrow \frac{1}{36\lambda^2}+\frac{16}{36\lambda^2}+\frac{25}{36\lambda^2}=0 \Rightarrow \text{EZ DAGO}$$

$f=x+2y-z$

PUNTUAK: $(0,0,0)$, $\left(\frac{1}{\sqrt6}, \sqrt{\frac{2}{3}}, \frac{-1}{\sqrt6}\right)$, $\left(\frac{-1}{\sqrt6}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt6}\right)$

$\frac{2}{\sqrt6}$

$$f(P_1)=0+2\cdot0-0=0$$

$$f(P_2)=\frac{1}{\sqrt6}+2\sqrt{\frac{2}{3}}+\frac{1}{\sqrt6}=\frac{6}{\sqrt6}$$

$$f(P_3)=\frac{-1}{\sqrt6}-2\sqrt{\frac{2}{3}}-\frac{1}{\sqrt6}=-\frac{6}{\sqrt6}$$

$\Rightarrow$ $\left(\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{-1}{\sqrt6}\right)$ MAXIMO ABSOLUTUA

$\left(\frac{-1}{\sqrt6}, \frac{-2}{\sqrt6}, \frac{1}{\sqrt6}\right)$ MINIMO ABSOLUTUA

2. ARIKETA

$$\begin{cases} e^u+e^v+u+v-2x+2y=2 \\ \sin u+\cos v+xy=1 \end{cases}$$

i) $u=u(x,y)$ $\quad v=v(x,y)$ $\quad (x_0,y_0,u,v)=(0,0,0,0)$

H1) $\vec{F}_1(0,0,0,0)=e^0+e^0+0+0-2=1+1-2=0$ ✓

$\vec{F}_2(0,0,0,0)=\sin 0+\cos 0+0-1=1-1=0$ ✓

H2) $\frac{\partial F_1}{\partial u}=e^u+1 \qquad \frac{\partial F_1}{\partial v}=e^v+1$

$$\frac{\partial F_1}{\partial u} = \cos u \qquad \frac{\partial F_2}{\partial v} = -\sin v$$

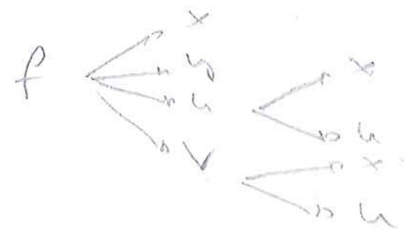
$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u + 1 & e^v + 1 \\ \cos u & -\sin v \end{vmatrix} =$$

$$= -\sin v\,(e^u+1) - \cos u\,(e^v+1)\Big|_{\bar{0}} = 0 - 2 = -2 \neq 0$$

TEOR 2.2

$\Rightarrow$ $(0,0,0,0)$ ingurune batean $\exists u, v$ C^1 klasekoak non $u = u(x,y)$ eta $v = v(x,y)$ sistemaren soluzioak diren.

b) $v(x,y)$ 2. TAYLOR $(0,0)$-n



$$e^u + e^v + u + v - 2x + 2y - 2 = 0$$

(1) $\frac{\partial}{\partial x} \Rightarrow \underset{1}{\underset{\|}{e^u}} \frac{\partial u}{\partial x} + \underset{1}{\underset{\|}{e^v}} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} - 2 = 0$

(2) $\frac{\partial}{\partial y} \Rightarrow \underset{1}{\underset{\|}{e^u}} \frac{\partial u}{\partial y} + \underset{1}{\underset{\|}{e^v}} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + 2 = 0$

$$\sin u + \cos v + xy - 1 = 0$$

(3) $\frac{\partial}{\partial x} = \underset{1}{\underset{\|}{\cos u}} \frac{\partial u}{\partial x} - \underset{0}{\underset{\|}{\sin v}} \frac{\partial v}{\partial \cancel{x}} + \underset{0}{\underset{\|}{y}} = 0$

(4) $\frac{\partial}{\partial y} = \underset{1}{\underset{\|}{\cos u}} \frac{\partial u}{\partial y} - \underset{0}{\underset{\|}{\sin v}} \frac{\partial v}{\partial y} + \underset{0}{\underset{\|}{x}} = 0$

(1) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} - 2 \Rightarrow -\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} - 2$

(2) $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + 2 \Rightarrow -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + 2$

(3) $\wedge$ (4) $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y} \Rightarrow \boxed{\frac{\partial v}{\partial x} = 2 \quad \frac{\partial v}{\partial y} = -2}$

(1)

$$\frac{\partial^2}{\partial x^2} \Rightarrow e^u\left(\frac{\partial u}{\partial x}\right)^2+e^u\frac{\partial^2 u}{\partial x^2}+e^v\left(\frac{\partial v}{\partial x}\right)^2+e^v\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}=0$$

$$\xrightarrow{(0,0)} \frac{\partial^2 u}{\partial x^2}+4+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}=-2$$

$$\frac{\partial^2}{\partial y^2} \Rightarrow e^u\left(\frac{\partial u}{\partial y}\right)^2+e^u\frac{\partial^2 u}{\partial y^2}+e^v\left(\overset{-2}{\frac{\partial v}{\partial y}}\right)^2+e^v\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}=0$$

$$\xrightarrow{(0,0)} \frac{\partial^2 u}{\partial y^2}+4+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}=0$$

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}=-2$$

(2)

$$\frac{\partial^2}{\partial x} \Rightarrow \cancel{-\sin u\left(\frac{\partial u}{\partial x}\right)^2}+\cos u\frac{\partial^2 u}{\partial x^2}-\cos v\left(\overset{2}{\frac{\partial v}{\partial x}}\right)^2\cancel{-\sin v\frac{\partial^2 v}{\partial x^2}}=0$$

$$\xrightarrow{(0,0)} \frac{\partial^2 u}{\partial x^2}-4=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}=4 \rightarrow \boxed{\frac{\partial^2 v}{\partial x^2}=-6}$$

$$\frac{\partial^2}{\partial y} \Rightarrow \cancel{-\sin u\left(\frac{\partial u}{\partial y}\right)^2}+\cos u\frac{\partial^2 u}{\partial y^2}-\cos v\left(\frac{\partial v}{\partial y}\right)^2\cancel{-\sin v\left(\frac{\partial^2 v}{\partial y^2}\right)^2}=0$$

$$\xrightarrow{(0,0)} \frac{\partial^2 u}{\partial y^2}-4=0 \Rightarrow \frac{\partial^2 u}{\partial y^2}=4 \rightarrow \boxed{\frac{\partial^2 v}{\partial y^2}=-6}$$

(2)

$$\frac{\partial^2}{\partial x\partial y} \Rightarrow \cancel{-\sin u\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}}+\cos u\frac{\partial^2 u}{\partial x\partial y}-\cos v\underset{-2}{\frac{\partial v}{\partial y}}\underset{2}{\frac{\partial v}{\partial x}}\cancel{-\sin v\frac{\partial^2 v}{\partial y^2}}+1=0$$

$$\xrightarrow{(0,0)} \frac{\partial^2 u}{\partial x\partial y}+4=0 \Rightarrow \frac{\partial^2 u}{\partial x\partial y}=-4$$

(1)

$$\frac{\partial^2}{\partial x\partial y} \Rightarrow e^u\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+e^u\frac{\partial^2 u}{\partial y\partial x}+e^v\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}+e^v\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y\partial x}=0$$

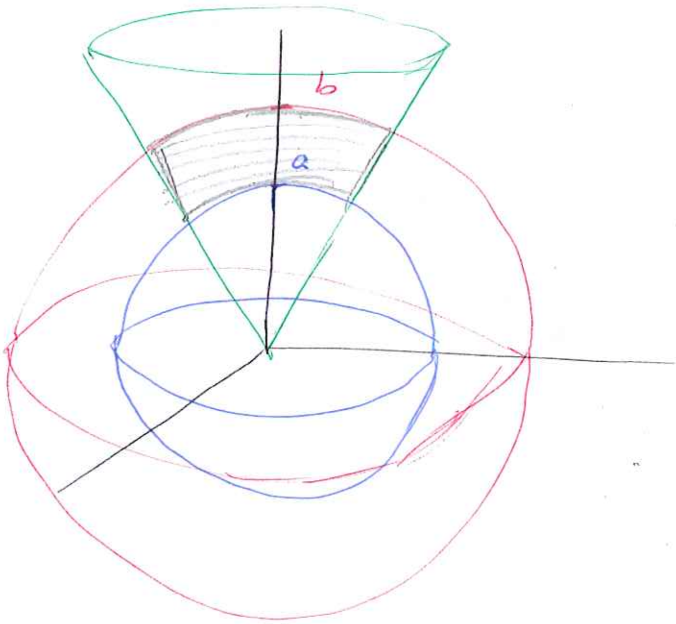
$$\xrightarrow{(0,0)} 2\underset{-4}{\frac{\partial^2 u}{\partial y\partial x}}-\overset{2}{4}+2\frac{\partial^2 v}{\partial y\partial x}=0 \Rightarrow \boxed{\frac{\partial^2 v}{\partial y\partial x}=6}$$

$$\boxed{V(x,y) \sim 0 + 2x - 2y - 3x^2 - 3y^2 + 6xy + R_2}$$

3. ARIKETA

$$W \begin{cases} x^2+y^2+z^2 = a^2 \\ x^2+y^2+z^2 = b^2 \\ x^2+y^2 \leq z^2 \end{cases}$$

$z \geq 0 \quad 0 < a < b$



BOLUMENA: $B(W) = \iiint_W 1 \, dV$

ALD-ALD: ESFERIKOAK

$x = \rho \cos\theta \sin\varphi$

$y = \rho \sin\theta \sin\varphi$

$z = \rho \cos\varphi$

$|J| = \rho^2 \sin\varphi$

$\theta \in [0, 2\pi]$

$\rho \in [a, b]$

$\varphi \in [0, \varphi_0] = [0, \frac{\pi}{4}]$

$\varphi_0 \to$ Konoan $\to x^2+y^2 = z^2 \Rightarrow$

$$\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi = \rho^2\cos^2\varphi$$

$$\sin^2\varphi = \cos^2\varphi \Rightarrow \varphi_0 = \frac{\pi}{4}$$

$$B(W) = \int_0^{2\pi}\int_0^{\pi/4}\int_a^b \rho^2 \cos\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{1}{3}\rho^3 \cos\varphi\right]_a^b d\varphi \, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\cos\varphi \left[b^3 - a^3\right] d\varphi \, d\theta = \frac{b^3-a^3}{3}\int_0^{2\pi}\left[\sin\varphi\right]_0^{\pi/4} d\theta =$$

$$= \frac{b^3-a^3}{3}\int_0^{2\pi} \frac{\sqrt{2}}{2} d\theta = \frac{b^3-a^3}{3} \cdot 2\pi \frac{\sqrt{2}}{2} \Rightarrow \boxed{B(W) = \frac{\sqrt{2}}{3}\pi(b^3-a^3)}$$

4. ARIKETA

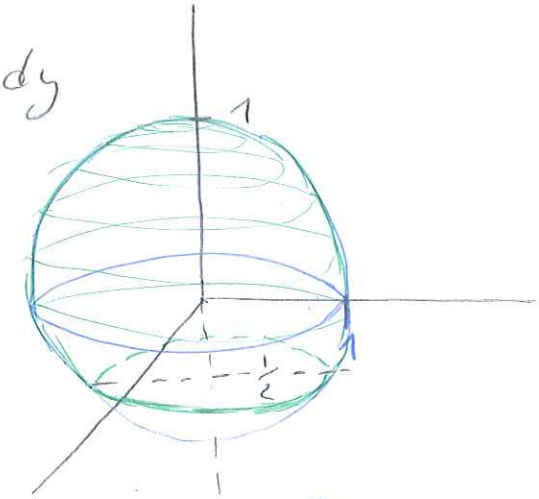
$rot\vec{F} = ?$

$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1,\ -1/2 \le z \le 1\}$

$\vec{F}(x,y,z) = (-y, x^2, z^3)$

FLUXUA $\Rightarrow \iint_S \vec{G}\, dS$ Kasu honetan $\vec{G} = rot\vec{F}$

$\iint_S rot\vec{F}\, dS = \iint_D -F_1' g_x - F_2' g_y + F_3\, dx\,dy$



$rot\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x^2 & z^3 \end{vmatrix} =$

$= (0, 0, 2x+1)$

ALD-ALD: POLARRAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$|J| = \rho$

$\rho \in [0,1]$

$\theta \in [0, 2\pi]$

$\iint_D 2x+1\, dx\, dy = \int_0^{2\pi}\int_0^1 (2\rho\cos\theta + 1)\rho\, d\rho\, d\theta =$

$= \int_0^{2\pi} \left[\frac{2}{3}\rho^3\cos\theta + \frac{1}{2}\rho^2\right]_0^1 d\theta =$

$= \int_0^{2\pi} \frac{2}{3}\cos\theta + \frac{1}{2}\, d\theta = \left[\cancel{\frac{2}{3}\sin\theta}^{\,0} + \frac{\theta}{2}\right]_0^{2\pi} = \boxed{\pi}$

STOKESEN TEOREMA $\Rightarrow \iint_S rot\vec{F}\, dS = \int_{\partial S} \vec{F}\, dS =$

$= \int_\sigma F_1 dx + F_2 dy + F_3 dz \dashv \quad \sigma(\theta) = (\cos t, \sin t, 0) \quad \theta \in [0, 2\pi]$

$= \int_0^{2\pi} \sin^2\theta + \cos^3\theta + 0\, d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2} + \cos\theta - \sin^2\theta\cos\theta\, d\theta =$

$= \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} + \sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{2\pi} = \boxed{\pi}$

7

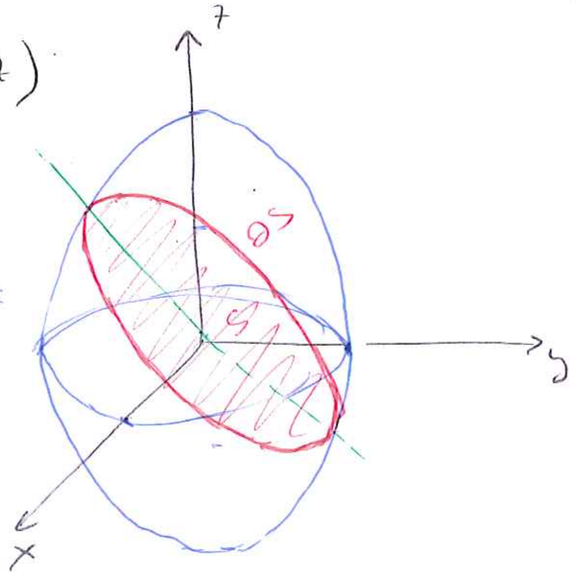
5. ARIKETA

STOKES: $\iint_S \operatorname{rot}\vec{F}\, ds = \int_{\partial S} \vec{F}\, ds$

$$\int_C (2x+y-z)dx + (2x+z)\,dy + (2x-y-z)dz$$

$$C \equiv \{ 4x^2+4y^2+z^2=4 \ \wedge \ 2x-z=0 \}$$

$$\vec{F}_{(x,y,z)} = (2x+y-z,\ 2x+z,\ 2x-y-z)$$



$$\operatorname{rot}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+y-z & 2x+z & 2x-y-z \end{vmatrix} =$$

$$= (-1-1,\ -2-1,\ 2-1) = (-2,-3,1)$$

$$z = g(x,y) = 2x \qquad g_x = 2 \qquad g_y = 0$$

$$\iint_S \operatorname{rot}\vec{F}\, dS = \iint_D -F_1' g_x - F_2' g_y + F_3\, dx\,dy =$$

$$= \iint_D -(-2)\cdot 2 - 0 + 1\, dx\,dy = 5\iint_D dx\,dy = 5A(D) \overset{\text{ELIPSEA}}{=} 5\pi ab$$

$$y=0 \to \begin{cases} 4x^2+z^2=4 \\ z=2x \end{cases} \to \cancel{4}x^2+\cancel{4}x^2=\cancel{4} \Rightarrow x = \frac{\pm 1}{\sqrt{2}} = a$$

$$x=0 \to \begin{cases} 4y^2+z^2=4 \\ z=0 \end{cases} \Rightarrow y^2=1 \Rightarrow y=\pm 1 = b$$

$$\iint_S \operatorname{rot}\vec{F}\, ds = \boxed{5\pi \frac{1}{\sqrt{2}}}$$

$$\sigma(t) = \left(\frac{1}{\sqrt{2}}\cos\theta,\ \sin\theta,\ z_0\right) = \partial S$$

$z \to$ Plenoan $\to$ $2x - z = 0$

$\sqrt{2}\cos\theta - z = 0 \Rightarrow z = \sqrt{2}\cos\theta$

$$\sigma(t) = \left(\frac{1}{\sqrt{2}}\cos\theta,\ \sin\theta,\ \sqrt{2}\cos\theta\right) \qquad \theta \in [0, 2\pi]$$

$$\int_{\partial S} (2x+y-z)\,dx + (2x+z)\,dy + (2x-y-z)\,dz =$$

$$\int_0^{2\pi} (\sqrt{2}\cos\theta + \sin\theta - \sqrt{2}\cos\theta)\left(-\frac{1}{\sqrt{2}}\sin\theta\right) + (\sqrt{2}\cos\theta + \sqrt{2}\cos\theta)\cos\theta +$$

$$+ (\sqrt{2}\cos\theta - \sin\theta - \sqrt{2}\cos\theta)(-\underset{\frac{2}{\sqrt{2}}}{\sqrt{2}}\sin\theta)\, d\theta =$$

$$= \int_0^{2\pi} \frac{-3}{\sqrt{2}}(\sqrt{2}\cos\theta\sin\theta - \sqrt{2}\cos\theta) + \frac{1}{\sqrt{2}}\sin^2\theta + 2\sqrt{2}\cos^2\theta\, d\theta =$$

$$= \int_0^{2\pi} \frac{-3}{\sqrt{2}}(\sqrt{2}\cos\theta\sin\theta - \sqrt{2}\cos\theta) + \frac{1}{\sqrt{2}}\frac{1-\cos 2\theta}{2} + 2\sqrt{2}\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \left[\frac{\theta}{2\sqrt{2}} + \theta\sqrt{2}\right]_0^{2\pi} = \frac{\pi}{\sqrt{2}} + 2\pi\sqrt{2} = \boxed{\pi\frac{5}{\sqrt{2}}}$$

2018-06-07

1. ARIKETA

MUTURRAK $f(x,y,z)=xyz$

$x+y+z-9=0 \quad \wedge \quad xy+yz+zx-24=0$

Lagrangeren funtzioa erabiliko dugu [MUTUR BALDINTZATUEN PROBLEMA]

$h(x,y,z,\lambda,\mu)=f(x,y,z)-\lambda(g_1-c_1)-\mu(g_2-c_2)$

$g_1(x,y,z)=x+y+z \qquad c_1=9$

$g_2(x,y,z)=xy+yz+zx \qquad c_2=24$

$h(\lambda,\mu,x,y,z)=xyz-\lambda(x+y+z-9)-\mu(xy+yz+zx-24)-$

MUTURRAK $\Rightarrow \nabla h=\bar{0}$

$$\nabla h = \begin{cases} \frac{\partial h}{\partial x} = yz-\lambda-\mu(y+z)=0 \\ \frac{\partial h}{\partial y} = xz-\lambda-\mu(x+z)=0 \\ \frac{\partial h}{\partial z} = xy-\lambda-\mu(y+x)=0 \\ \frac{\partial h}{\partial \lambda} = -x-y-z+9=0 \\ \frac{\partial h}{\partial \mu} = -xy-yz-zx+24=0 \end{cases}$$

$\Rightarrow x=y=z=\rho$ [irudetu, beiterik gabe]

$3\rho=9 \Rightarrow \rho=3 \qquad 3\rho^2=24 \Rightarrow \rho=\pm 2\sqrt{2}$

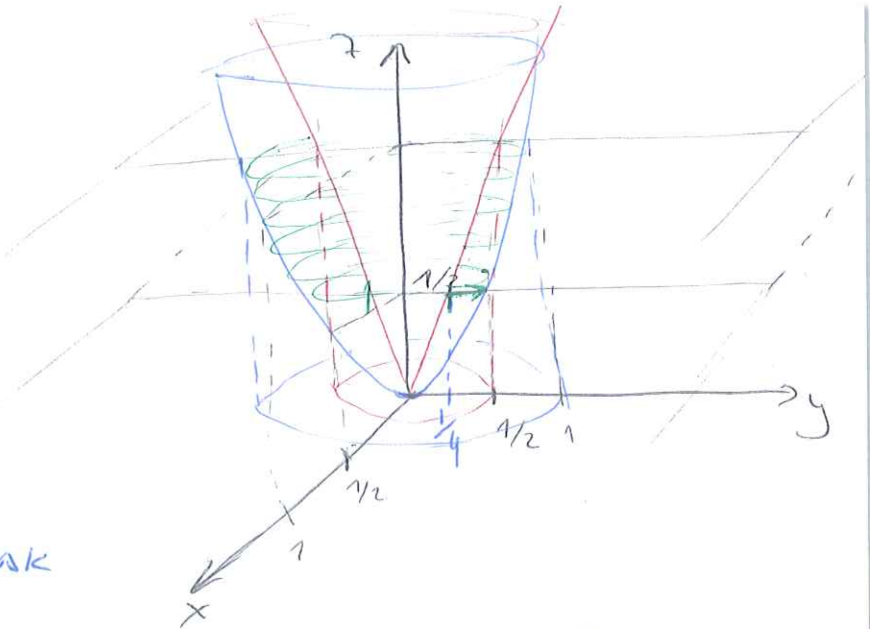
$3\neq\pm 2\sqrt{2} \Rightarrow \nexists$ dago puntunik

$\nabla g_1\neq 0? \Rightarrow \nabla g_1=(1,1,1)\neq 0$

$\nabla g_2\neq 0? \Rightarrow \nabla g_2=(y,z,x)=0 \Rightarrow y=z=x=0$

3. ARIKETA

$$W = \begin{cases} z^2 = x^2+y^2 \\ 2z = x^2+y^2 \\ z=1,\ z=\frac{1}{2} \end{cases}$$



$$B(W) = \iiint_W 1\, dx\,dy\,dz$$

ALD-ALD: ZILINDRIKOAK

$x = \rho\cos\theta$ $\qquad z \in [\frac{1}{2}, kon] \cup [parab, 1]$

$y = \rho\sin\theta$ $\qquad \theta \in [0, 2\pi]$

$z = z$

$|J| = \rho$ $\qquad \rho \in [\frac{1}{4}, \frac{1}{2}] \cup [\frac{1}{2}, 1]$

$$z: \begin{cases} z^2 = \rho^2 \\ 2z = \rho^2 \end{cases} \Rightarrow z_K = \rho \qquad z_p = \frac{\rho^2}{2}$$

$$B(W) = \int_0^{2\pi}\int_{\frac{1}{4}}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\rho} \rho\, dz\, d\rho\, d\theta + \int_0^{2\pi}\int_{\frac{1}{2}}^{1}\int_{\frac{\rho^2}{2}}^{1} \rho\, dz\, d\rho\, d\theta =$$

$$= \int_0^{2\pi}\int_{1/4}^{1/2} \rho\left(\rho - \frac{1}{2}\right) d\rho\, d\theta + \int_0^{2\pi}\int_{1/2}^{1} \rho\left(1 - \frac{\rho^2}{2}\right) d\rho\, d\theta =$$

$$= \int_0^{2\pi}\left[\frac{1}{3}\rho^3 - \frac{1}{4}\rho^2\right]_{1/4}^{1/2} d\theta + \int_0^{2\pi}\left[\frac{1}{2}\rho^2 - \frac{1}{8}\rho^4\right]_{1/2}^{1} 1\cdot d\theta =$$

$$= \int_0^{2\pi} \frac{1}{\underset{2^3}{3\cdot 8}} - \frac{1}{\underset{2^4}{4\cdot 4}} - \frac{1}{\underset{2^6}{3\cdot 4^3}} + \frac{1}{\underset{2^6}{4\cdot 4^2}}\, d\theta + \int_0^{2\pi} \frac{1}{2} - \frac{1}{\underset{2^3}{8}} - \frac{1}{\underset{2^7}{8}} + \frac{1}{\underset{2^7}{8\cdot 2^4}}\, d\theta$$

$$= \int_0^{2\pi} \frac{14}{2^6\cdot 3} d\theta + \int_0^{2\pi} \frac{33}{2^7} d\theta = \pi\cdot\left(\frac{28}{2^6\cdot 3} + \frac{33}{2^6}\right) = \boxed{\frac{127}{192}\pi}$$

$\frac{11}{24}\pi$

11

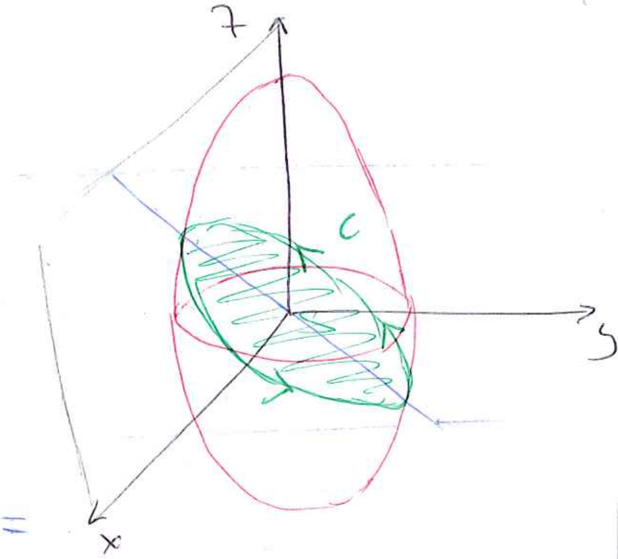
4. ARIKETA

$$\vec{F} = (2x+y-z)\hat{\imath} + (2x+z)\hat{\jmath} + (2x-y-z)\hat{k}$$

$$\begin{cases} 4x^2+4y^2+z^2=4 \\ 2x-z=0 \end{cases}$$

ZIRKULAZIOA

STOKES: $\iint_S \text{rot}\vec{F}\,dS = \int_{\partial S} \vec{F}\,dS$ ($\partial S = c$)



$$\vec{F} = (2x+y-z,\ 2x+z,\ 2x-y-z)$$

$$\text{rot}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+y-z & 2x+z & 2x-y-z \end{vmatrix} =$$

$$= (-1-1,\ -2-1,\ 2-1) = (-2,-3,1)$$

$$\iint_\phi \vec{F}\,dS = \iint_D -F_1 g_x - F_2 g_y + F_3\,dx\,dy$$

$$g(x,y) = z = 2x \qquad g_x = 2 \qquad g_y = 0$$

$$\iint_S \text{rot}\vec{F}\,dS = \iint_D 2\cdot 2 - 0 + 1\,dx\,dy = \iint_D 5\,dx\,dy = 5A(S) =$$

$A(S) = \pi a b \leftarrow$ ELIPSEA

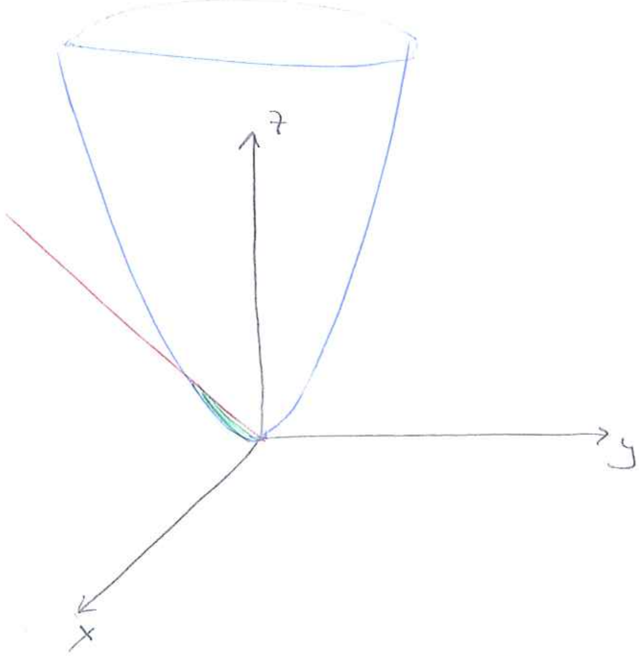
$$\begin{cases} 4x^2+4y^2+z^2=4 \Rightarrow & y=0 \rightarrow 2x^2=1 \rightarrow x = \pm 4 = b \\ 2x-z=0 \Rightarrow 2x=z \end{cases}$$

$$x=0 \rightarrow z=0 \Rightarrow y=\pm 1 = a$$

$$= 5\pi ab = 5\pi \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\int_{\partial S} \vec{F}\,dS = \frac{1}{\sqrt{2}} 5\pi}$$

5. ARIKETA



$$\vec{F}(x,y,z) = (-xz, x, y^2)$$

$$\begin{cases} z = x^2+y^2 \\ z = x \end{cases}$$

$$\iint_S rot\vec{F}\cdot\vec{n}\, dS$$

GAUSS:

$$\iint_S \vec{F}'\cdot\vec{n}dS = \iiint_V div\,\vec{F}' dV$$

Kasu honetan $\vec{F}' = rot\vec{F}$ eta $div\vec{F}' = div(rot\vec{F}) = 0$

Baina tapa jarri behar diogu:

$S = S_1 \cup S_2 \Rightarrow \iint_{S_1} rot\vec{F}\cdot\vec{n}dS + \iint_{S_2} rot\vec{F}\cdot\vec{n}\,dS = 0$

Parabo ↵ ↳ PLANOA

$$rot\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xz & x & y^2 \end{vmatrix} = (2y, -z, 1)$$

$$\iint_{S_2} rot\vec{F}\cdot\vec{n}dS = \iint_{S_2} (2y, -z, 1)\cdot\vec{n}\, dS$$

$G = x - z = 0 \qquad \nabla G(1, 0, -1) \qquad ||\nabla G|| = \sqrt{2}$

$$\vec{n} = \frac{G}{||\nabla G||} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \Rightarrow \iint_S (2y, -z, 1)\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)dS =$$

$$\begin{cases} z = x^2+y^2 \\ z = x \end{cases} \Rightarrow \left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

ALD-ALD: ZILINDRIKOAK
$x = \frac{1}{2} + \rho\cos\theta$
$y = \rho\sin\theta$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} \left(\sqrt{2}\rho\sin\theta - \frac{1}{\sqrt{2}}\right)\rho\, d\rho\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{3}\frac{1}{8}\sin\theta - \frac{1}{8\sqrt{2}}\, d\theta = \frac{-\pi}{4\sqrt{2}}$$

$$\Rightarrow \iint_S rot\vec{F}\cdot\vec{n}dS = -\iint_{S_2} rot\vec{F}\cdot\vec{n}\,dS = -\frac{-\pi}{4\sqrt{2}} = \boxed{\frac{\pi}{4\sqrt{2}}}$$

13

2. ARIKETA

$$\begin{aligned} F_1 = & \left\{ x+y+z+u-\alpha=0 \right. \\ F_2 = & \left. x^3+y^3+z^3+u^3-\beta=0 \right. \end{aligned} \qquad c^2-d^2\neq 0$$

$z = z(x,y) \qquad u = u(x,y) \qquad (x_0,y_0,z_0,u_0)=(a,b,c,d)$

1) $z \wedge u$ DEFINITUTA?

H_1: $F_1(a,b,c,d) = a+b+c+d-\alpha = 0 \Rightarrow \alpha = a+b+c+d$

$F_2(a,b,c,d) = a^3+b^3+c^3+d^3-\beta = 0 \Rightarrow \beta = a^3+b^3+c^3+d^3$

H_2: $\frac{\partial F_1}{\partial z} = 1 \qquad \frac{\partial F_2}{\partial z} = 3z^2$

$\frac{\partial F_1}{\partial u} = 1 \qquad \frac{\partial F_2}{\partial u} = 3u^2$

$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial u} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial u} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3z^2 & 3u^2 \end{vmatrix} = 3u^2-3z^2 \Big|_{\bar{x}_0} = 3(c^2-d^2)\neq 0$$

TEOR 2.2

$\Rightarrow$ (a,b,c,d)-ren ingurune batean $\exists\, g_1, g_2$ C^1 klasekoak non $z=g_1(x,y)$ eta $u=g_2(x,y)$ sistemaren soluzioak diren [$\alpha = a+b+c+d \wedge \beta = a^3+b^3+c^3+d^3$ izanik]

2) KALKULATU PLANO UKITZAILEA (a,b)-n

$x+y+z+u-\alpha=0$

$\frac{\partial}{\partial x} \rightarrow 1 + \frac{\partial z}{\partial x} + \frac{\partial u}{\partial x} = 0$

$\frac{\partial}{\partial y} \rightarrow 1 + \frac{\partial z}{\partial y} + \frac{\partial u}{\partial y}$

$$x^3+y^3+z^3+u^3-\cancel{3}=0$$

$$\frac{\partial}{\partial x}\to\ \underset{\underset{a^2}{\uparrow}}{3x^2}+\underset{\underset{c^2}{\uparrow}}{3z^2}\frac{\partial z}{\partial x}+\underset{\underset{d^2}{\uparrow}}{3u^2}\frac{\partial u}{\partial x}=0$$

$$\frac{\partial}{\partial y}\to\ \underset{\underset{b^2}{\uparrow}}{3y^2}+\underset{\underset{c^2}{\uparrow}}{3z^2}\frac{\partial z}{\partial y}+\underset{\underset{d^2}{\uparrow}}{3u^2}\frac{\partial u}{\partial y}=0$$

$$\begin{cases}1+\frac{\partial z}{\partial x}+\frac{\partial u}{\partial x}=0\Rightarrow\frac{\partial u}{\partial x}=-1-\frac{\partial z}{\partial x}\\ 3a^2+3c^2\frac{\partial z}{\partial x}+3d^2\frac{\partial u}{\partial x}=0\end{cases}$$

$$\hookrightarrow \cancel{3}a^2+\cancel{3}c^2\frac{\partial z}{\partial x}-\cancel{3}d^2-\cancel{3}d^2\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}(c^2-d^2)=d^2-a^2\Rightarrow\boxed{\frac{\partial z}{\partial x}=\frac{d^2-a^2}{c^2-d^2}}$$

$$\frac{\partial u}{\partial x}=-1-\frac{d^2-a^2}{c^2-d^2}=\frac{\cancel{d^2}-c^2-\cancel{d^2}+a^2}{c^2-d^2}\Rightarrow\boxed{\frac{\partial u}{\partial x}=\frac{a^2-c^2}{c^2-d^2}}$$

$$\begin{cases}1+\frac{\partial z}{\partial y}+\frac{\partial u}{\partial y}=0\Rightarrow\frac{\partial z}{\partial y}=-1-\frac{\partial u}{\partial y}\\ 3b^2+3c^2\frac{\partial z}{\partial y}+3d^2\frac{\partial u}{\partial y}=0\end{cases}$$

$$b^2-c^2-c^2\frac{\partial u}{\partial y}+d^2\frac{\partial u}{\partial y}=0$$

$$\frac{\partial u}{\partial y}(d^2-c^2)=c^2-b^2\Rightarrow\boxed{\frac{\partial u}{\partial y}=\frac{b^2-c^2}{c^2-d^2}}$$

$$\frac{\partial z}{\partial y}=-1-\frac{b^2-c^2}{c^2-d^2}=\frac{d^2-\cancel{c^2}-b^2+\cancel{c^2}}{c^2-d^2}\Rightarrow\boxed{\frac{\partial z}{\partial y}=\frac{d^2-b^2}{c^2-d^2}}$$

$$\boxed{\begin{aligned}z(x,y)&\approx c+\frac{d^2-a^2}{c^2-d^2}(x-a)+\frac{d^2-b^2}{c^2-d^2}(y-b)+R_1\\ u(x,y)&=d+\frac{a^2-c^2}{c^2-d^2}(x-a)+\frac{b^2-c^2}{c^2-d^2}(y-b)+R_1\end{aligned}}$$

PLANO UKITZAILEAK

15

$\boxed{2018-07-02}$

1. ARIKETA

$$\begin{cases} x e^{u+v} + 2uv = 1 \\ y e^{u-v} - \dfrac{u}{1+v} = 2x \end{cases} \rightarrow \begin{cases} x e^{u+v} + 2uv - 1 = 0 \\ y e^{u-v} - \dfrac{u}{1+v} - 2x = 0 \end{cases}$$

1) $u = u(x,y) \wedge v = v(x,y) \quad (x,y,u,v) = (1,2,0,0)$

(1) $F_i = 0$

$$F_1(1,2,0,0) = 1 + 0 - 1 = 0 \checkmark$$

$$F_2(1,2,0,0) = 2 - 0 - 2 = 0 \checkmark$$

(2) $\dfrac{\partial}{\partial i} \neq 0 \vee \Delta \neq 0$

$$\frac{\partial F_1}{\partial u} = x e^{u+v} + 2v \quad , \quad \frac{\partial F_1}{\partial v} = x e^{u+v} + 2u$$

$$\frac{\partial F_2}{\partial u} = y e^{u-v} - \frac{1}{1+v} \qquad \frac{\partial F_2}{\partial v} = -y e^{u-v} + \frac{u}{(1+v)^2}$$

$$\Delta = \begin{vmatrix} x e^{u+v} + 2v & x e^{u+v} + 2u \\ y e^{u-v} - \frac{1}{1+v} & -y e^{u-v} + \frac{u}{(1+v)^2} \end{vmatrix} =$$

$$= -yx e^{u} - 2vy e^{u-v} + x \frac{u}{(1+v)^2} e^{u+v} - yx e^{u} + x \frac{e^{u+v}}{1+v} - 2uy e^{u-v} + \frac{2u}{1+v} =$$

$$= -2yx e^{u} - \cancel{2vy e^{u-v}} - \cancel{2uy e^{u-v}} + \cancel{x \frac{u}{(1+v)^2} e^{u+v}} + x \frac{e^{u+v}}{1+v} + \cancel{\frac{2u}{1+v}}$$

$$\overset{(1,2,0,0)}{=} -2 \cdot 2 \cdot 1 + 1 = -3 \neq 0 \checkmark$$

TEOR 22

$\Rightarrow$ (1,2,0,0)-ren ingurune batean $\exists_{u,v}$ C^1

klasekoak non $u = u(x,y)$ eta $v = v(x,y)$

sistemaren soluzioak diren.

i) TAYLORREN POLINOMIOA $(1,2)$-n

$$\begin{cases} xe^{u+v}+2uv-1=0 & (1)\\ ye^{u-v}-\dfrac{u}{1+v}-2x=0 & (2)\end{cases}$$

$$\frac{\partial(1)}{\partial x} \Rightarrow e^{u+v}+e^{u+v}\frac{\partial u}{\partial x}+e^{u+v}\frac{\partial v}{\partial x}+2v\frac{\partial u}{\partial x}+2u\frac{\partial v}{\partial x}=0$$

$$\xrightarrow{(1,2)} \quad 1+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0 \Rightarrow \frac{\partial v}{\partial x}=-1-\frac{\partial u}{\partial x}$$

$$\frac{\partial(2)}{\partial x} \Rightarrow ye^{u-v}\frac{\partial u}{\partial x}-ye^{u-v}\frac{\partial v}{\partial x}-\frac{1}{1+v}\frac{\partial u}{\partial x}+\frac{u}{(1+v)^2}\frac{\partial v}{\partial x}-2=0$$

$$\xrightarrow{(1,2)} \quad 2\frac{\partial u}{\partial x}-2\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}-2=0$$

$$\frac{\partial u}{\partial x}+\cancel{2}+2\frac{\partial u}{\partial x}-\cancel{2}=0 \Rightarrow \boxed{\frac{\partial u}{\partial x}=0}$$

$$\Rightarrow \boxed{\frac{\partial v}{\partial x}=-1}$$

$$\frac{\partial(1)}{\partial y} \Rightarrow xe^{u+v}\frac{\partial u}{\partial y}+xe^{u+v}\frac{\partial v}{\partial y}+2v\frac{\partial u}{\partial y}+2u\frac{\partial v}{\partial y}=0$$

$$\xrightarrow{(1,2)} \quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=0 \Rightarrow \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial y} \Rightarrow \boxed{\frac{\partial v}{\partial y}=\frac{1}{3}}$$

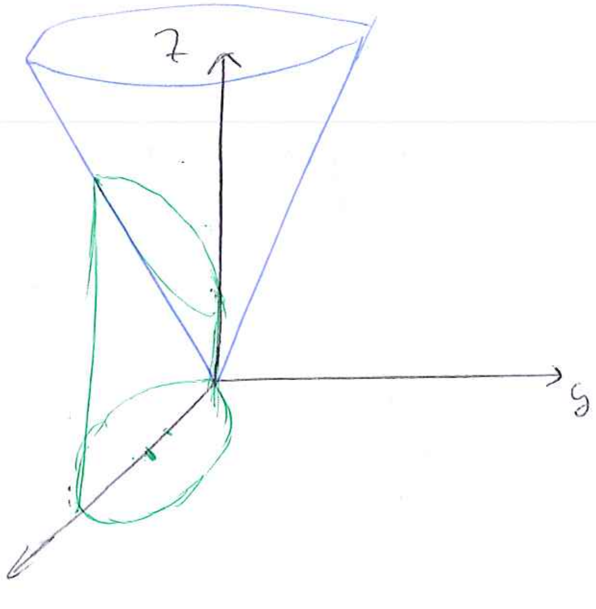
$$\frac{\partial(2)}{\partial y} \Rightarrow e^{u-v}+ye^{u-v}\frac{\partial u}{\partial y}-ye^{u-v}\frac{\partial v}{\partial y}-\frac{1}{1+v}\frac{\partial u}{\partial v}+\frac{u}{(1+v)^2}\frac{\partial v}{\partial y}=0$$

$$\xrightarrow{(1,2)} \quad 1+2\frac{\partial u}{\partial y}-2\frac{\partial v}{\partial y}-\frac{\partial u}{\partial v}=0$$

$$\Rightarrow 1+\frac{\partial u}{\partial y}+2\frac{\partial u}{\partial y}=0 \Rightarrow \boxed{\frac{\partial u}{\partial y}=-\frac{1}{3}}$$

$$\boxed{\begin{aligned} u(x,y) &= -\frac{1}{3}(y-2)+R_1 \\ v(x,y) &= -(x-1)+\frac{1}{3}(y-1)+R_1 \end{aligned}}$$

2. ARIKETA



$$\begin{cases} x^2+y^2=2x \to (x-1)^2+y^2=1 \\ z^2=x^2+y^2 \to \text{KONOA} \\ z=0 \\ z\geq 0 \end{cases}$$

$$B(W)=\iiint_W dx\,dy\,dz$$

ALD-ALD: ZILINDRIKOAK

$x = 1+\rho\cos\theta$

$y=\rho\sin\theta$ $\qquad \theta\in[0,2\pi]$

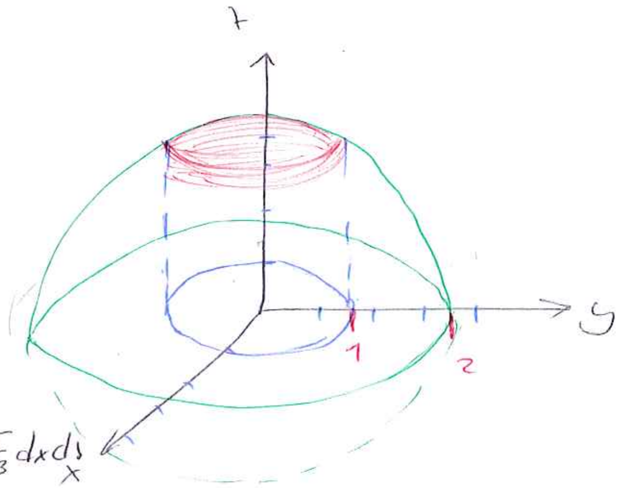
$z=z$ $\qquad \rho\in[0,1]$ → KONOA

$|J|=\rho$ $\qquad z\in[0,\rho]$

$$B(W)=\int_0^{2\pi}\int_0^1\int_0^\rho \rho\,dz\,d\rho\,d\theta=\int_0^{2\pi}\int_0^1\rho^2\,d\rho\,d\theta=$$

$$=\int_0^{2\pi}\left[\frac{1}{3}\rho^3\right]_0^1 d\theta=\int_0^{2\pi}\frac{1}{3}\,d\theta=\boxed{\frac{2}{3}\pi}$$

3. ARIKETA



$$\begin{cases} x^2+y^2=1 \\ x^2+y^2+z^2=4 \\ z\geq 0 \end{cases}$$

$\vec F(x,y,z)=(xy,yz,xz)$

a) $\displaystyle\iint_S \underbrace{\operatorname{rot}\vec F}_{\vec F'}\,dS=\iint_D -F_1'g_x-F_2'g_y+F_3\,dx\,dy$

$$\operatorname{rot}\vec F=\begin{vmatrix}\hat\imath & \hat\jmath & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & yz & xz\end{vmatrix}=(-y,-z,-x)$$

$$g_x=\frac{-2x}{2\sqrt{4-x^2-y^2}}$$

$$g(x,y)=z=\sqrt{4-x^2-y^2}$$

$$g_y=\frac{-2y}{2\sqrt{4-x^2-y^2}}$$

$$\iint_D -\frac{xy}{\sqrt{4-x^2-y^2}} - \frac{yz}{\sqrt{4-x^2-y^2}} - x \, dx\,dy =$$

ALD-ALD: POLARZAK

$$x = \rho\cos\theta \qquad \theta \in [0,2\pi]$$
$$y = \rho\sin\theta \qquad \rho \in [0,1]$$
$$|J| = \rho$$

$$= \iint_D -\frac{xy}{\sqrt{4-x^2-y^2}} - y - x \, dx\,dy =$$

$$= \int_0^{2\pi}\int_0^1 -\frac{\rho^3\cos\theta\sin\theta}{\sqrt{4-\rho^2}} - \rho^2\sin\theta - \rho^2\cos\theta \, d\rho \, d\theta =$$

$$= \int_0^1 \left[\frac{-\rho^3}{\sqrt{4-\rho^2}} \frac{1}{2}\sin^2\theta + \rho^2\cos\theta - \rho^2\sin\theta \right]_0^{2\pi} d\rho =$$

$$= \int_0^1 \rho^2 \overset{1}{\cos 2\pi} - \rho^2\cos 0 \, d\rho \; \boxed{= 0}$$

b) STOKES: $\iint_S \operatorname{rot} \vec{F} dS = \int_{\partial S} \vec{F} dS$ $\qquad \vec{F} = (xy, yz, xz)$

$\sigma(\theta) = (\cos\theta, \sin\theta, \sqrt{3})$

$$\int_\sigma F_1 dx + F_2 dy + F_3 \overset{0}{dz} = \int_0^{2\pi} -\cos\theta\sin^2\theta + \sqrt{3}\sin\theta\cos\theta \, d\theta =$$

$$= \left[-\frac{1}{3}\sin^3\theta + \frac{\sqrt{3}}{2}\sin^2\theta \right]_0^{2\pi} \boxed{= 0}$$

c) GAUSS: $\iint_{\partial\Omega} \vec{F} ds = \iint_{\partial\Omega} \vec{F}\cdot\vec{n} \, dS = \iiint_\Omega \operatorname{div}\vec{F} dV$

$\partial\Omega = S = S_1 \cup S_2$ $\qquad \operatorname{div}\operatorname{rot}\vec{F} = 0$

$$\iint_{S_1} \vec{F}\cdot\vec{n} \, dS = -\iint_{S_2} \vec{F}\cdot\vec{n} \, dS = -\iint_{S_2} (-y, -z, -x)\cdot(0,0,-1) \, dx\,dy =$$

$$= -\iint_{S_1} x \, dx\,dy = -\int_0^{2\pi}\int_0^1 \rho^2\cos\theta \, d\rho \, d\theta = \int_0^{2\pi} \frac{1}{3}\cos\theta \, d\theta = \frac{1}{3}\sin\theta \Big|_0^{2\pi} \boxed{= 0}$$

$\boxed{2017-07-03}$

1. ARIKETA

$$F \equiv x^2+y^2+z^2-2x=0 \qquad z=z(x,y)$$

a) $z=z(x,y)$-ren maximoak eta minimoak

$$\frac{\partial}{\partial x} \rightarrow 2x+2z\frac{\partial z}{\partial x}-2=0 \rightarrow 2z\frac{\partial z}{\partial x}=2-2x=0$$

$$\rightarrow x=1$$

$$\frac{\partial}{\partial y} \rightarrow 2y+2z\cdot\frac{\partial z}{\partial y}=0 \rightarrow 2z\frac{\partial z}{\partial y}=2y=0$$

$$\rightarrow y=0$$

$$(1,0) \rightarrow x^2+y^2+z^2-2x=0 \rightarrow 1+z^2-2=0 \rightarrow z=\pm 1$$

$$\frac{\partial^2}{\partial x^2} \rightarrow 2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}=0$$

$$\xrightarrow{(1,0,1)} 2+2\cdot 0+2\cdot 1\cdot\frac{\partial^2 z}{\partial x^2}=0 \Rightarrow \frac{\partial^2 z}{\partial x^2}=-1<0 \rightarrow \max$$

$$\frac{\partial^2}{\partial y^2} \rightarrow 2+2\left(\frac{\partial z}{\partial y}\right)^2+2\frac{\partial^2 z}{\partial y^2}=0$$

$$\xrightarrow{(1,0,1)} 2+2\cdot 0+2\frac{\partial^2 z}{\partial y^2}=0 \Rightarrow \frac{\partial^2 z}{\partial y^2}=-1<0 \rightarrow \max$$

$$\left.\frac{\partial^2}{\partial x^2}\right|_{(1,0,-1)} \rightarrow \frac{\partial^2 z}{\partial x^2}\geq 0 \rightarrow \min$$

$$\left.\frac{\partial^2}{\partial y^2}\right|_{(1,0,-1)} \rightarrow \frac{\partial^2 z}{\partial y^2}\geq 0 \rightarrow \min$$

$(1,0,-1)$	minimoa
$(1,0,1)$	maximoa

20

b) $P = (1,1,1)$

$d((1,1,1),(x_0,y_0,z_0)) = \sqrt{(x_0-1)^2+(y_0-1)^2+(z_0-1)^2}$

$\hookrightarrow f(x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2$

$h(\lambda,x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2-\lambda(x^2+y^2+z^2-2x)$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial x} = \cancel{2}(x-1) - \cancel{2}\lambda x + \cancel{2}\lambda = 0 \Rightarrow x-1 = \lambda\cdot(x-1) \\ \frac{\partial h}{\partial y} = \cancel{2}(y-1) - \cancel{2}\lambda y = 0 \\ \frac{\partial h}{\partial z} = \cancel{2}(z-1) - \cancel{2}\lambda z = 0 \\ \frac{\partial h}{\partial \lambda} = -x^2-y^2-z^2+2x = 0 \end{cases}$$

$\lambda = 1$

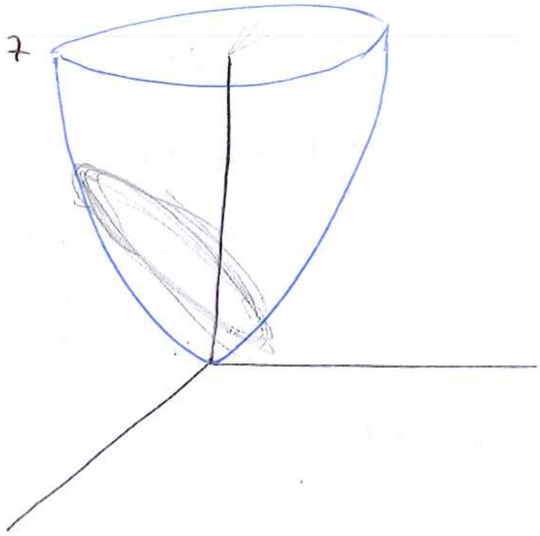
$\lambda = 0 \rightarrow x = y = z = 1 \rightarrow -3+2 \neq 0$

c) TAYLOR. 2 $(1,0,1)$

$$\boxed{z(x,y) = 1 - \frac{1}{2!}(x-1)^2 - \frac{1}{2!}y^2}$$

2. ARIKETA

$$\int_{\partial\Gamma} (y^2-xz)dx+(x^2+zy)dy+(x^2+y^2)dz$$



$$\Gamma : \begin{cases} 2x+2y+2z=1 \\ z=x^2+y^2 \end{cases}$$

$$\vec{F}(x,y,z)=(y^2-xz,\ x^2+zy,\ x^2+y^2)$$

STOKES: $\iint_S rot\,\vec{F}ds=\int_{\partial S}\vec{F}ds$

$$rot\,\vec{F}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2-xz & x^2+zy & x^2+y^2 \end{vmatrix}=(2y-y,-2x-x,\ 2x-2y)= \\ =(y,-3x,\ 2x-2y)$$

$$\iint_D \vec{F}ds=\iint_D -F_1g_x-F_2g_y+F_3\,dx\,dy$$

$$g(x,y)=z=\frac{1}{2}-x-y \qquad g_x=-1,\quad g_y=-1$$

$$\iint_\Gamma y-3x+2x-2y\,dx\,dy=\iint_\Gamma -x-y\,dx\,dy=$$

$$\begin{cases} 2x+2y+2z=1 \\ z=x^2+y^2 \end{cases} \rightarrow 2x+2y+2x^2+2y^2=1 \rightarrow (\sqrt{2}x+\tfrac{1}{\sqrt{2}})^2+(\sqrt{2}y+\tfrac{1}{\sqrt{2}})^2=2$$

ALD-ALD: POLARRAK

$$x=\frac{-1}{\sqrt{2}}+\sqrt{2}\rho\cos\theta \qquad |J|=2\rho \qquad \rho\in[0,\sqrt{2}]$$

$$y=-\frac{1}{\sqrt{2}}+\sqrt{2}\rho\sin\theta \qquad \theta\in[0,2\pi]$$

$$=\int_0^{2\pi}\int_0^{\sqrt{2}}(+\frac{1}{\sqrt{2}}-\sqrt{2}\rho\cos\theta+\frac{1}{\sqrt{2}}-\sqrt{2}\rho\sin\theta)\rho\,d\rho\,d\theta=$$

$$=\int_0^{2\pi}\int_0^{\sqrt{2}}\sqrt{2}\rho-\sqrt{2}\rho^2\cos\theta-\sqrt{2}\rho^2\sin\theta\,d\rho\,d\theta=$$

22

$$= \int_0^{2\pi} \left[\frac{\sqrt{2}}{2} \rho^2 - \frac{\sqrt{2}}{3} \rho^3 \cos\theta - \frac{\sqrt{2}}{3} \rho^3 \sin\theta \right]_0^{\sqrt{2}} d\theta =$$

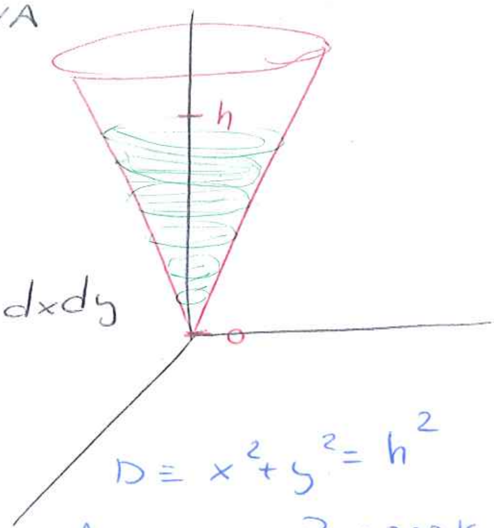
$$= \int_0^{2\pi} \sqrt{2} - \frac{4}{3}\cos\theta - \frac{4}{3}\sin\theta \, d\theta =$$

$$= \left[\sqrt{2}\,\theta - \cancel{\frac{4}{3}\sin\theta}^{\,0} + \cancel{\frac{4}{3}\cos\theta}^{\,0} \right]_0^{2\pi} = \boxed{2\sqrt{2}\pi}$$

3. ARIKETA

$\vec{F}(x,y,z) = (x^3, y^3, z^3)$ FLUXUA

$x^2 + y^2 = z^2 \qquad 0 \le z \le H$



a) ZUZENEAN

$$\text{FLUXUA} \equiv \iint_S \vec{F}\, dS = \iint_D -F_1 g_x - F_2 g_y + F_3 \, dx\,dy$$

$z = \sqrt{x^2+y^2} = g(x,y)$

$D = x^2 + y^2 = h^2$

ALD ALD: POLARRAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$g_x = \frac{x}{\sqrt{x^2+y^2}} \qquad g_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\iint_D -\frac{x^4}{\sqrt{x^2+y^2}} - \frac{y^4}{\sqrt{x^2+y^2}} + (x^2+y^2)^{3/2} \, dx\,dy =$$

$$= \int_0^{2\pi}\int_0^h \left\{ -\left[\frac{\rho^4\cos^4\theta + \rho^4\sin^4\theta}{\rho} \right] + \rho^3 \right\} \rho \, d\rho \, d\theta =$$

$$= \int_0^{2\pi}\int_0^h -\rho^4(\cos^4\theta + \sin^4\theta) + \rho^4 \, d\rho\, d\theta =$$

$$= \int_0^{2\pi} \left[\frac{1}{5}\rho^5 (1 - \cos^4\theta - \sin^4\theta) \right]_0^h d\theta =$$

$$= \int_0^{2\pi} \frac{h^5}{5}\left[1-\left(\frac{1+\cos 2\theta}{2}\right)^2-\left(\frac{1-\cos 2\theta}{2}\right)^2\right] d\theta =$$

$$= \int_0^{2\pi} \frac{h^5}{5}\left[1-\frac{1}{4}\left(1+\cancel{2\cos 2\theta}+\frac{1+\cos 4\theta}{2}\right)-\frac{1}{4}\left(1-\cancel{2\cos 2\theta}+\frac{1+\cos 4\theta}{2}\right)\right] d\theta =$$

$$= \int_0^{2\pi} \frac{h^5}{5}\left[1-\underbrace{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{4}}-\frac{1}{4}(1+\cos 4\theta)\right] d\theta =$$

$$= \left[\frac{h^5}{5}\cdot\left(\frac{\theta}{4}-\frac{\sin 4\theta}{16}\right)\right]_0^{2\pi} = \boxed{\frac{h^5}{10}\pi}$$

b) TEOREMEKIN:

$$\text{GAUSS} \equiv \iint_{\partial\Omega} \vec{F}\,dS = \iiint_{\Omega} \operatorname{div}\vec{F}\,dV$$

ALD-ALD: ZILINDRIKOAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

$\cancel{J} = \rho$

$$\operatorname{div}\vec{F} = 3x^2+3y^2+3z^2$$

$\theta \in [0, 2\pi]$

$\rho \in [0, h]$

$z \in [\sqrt{x^2+y^2}, h] = [\rho, h]$

$$\int_0^{2\pi}\int_0^h\int_\rho^h (3\rho^2+3z^2)\rho\,dz\,d\rho\,d\theta = \int_0^{2\pi}\int_0^h \left[3\rho^3 z + z^3\rho\right]_\rho^h d\rho\,d\theta =$$

$$= \int_0^{2\pi}\int_0^h 3\rho^3 h + h^3\rho - 3\rho^4 - \rho^4\,d\rho\,d\theta = \int_0^{2\pi}\int_0^h 3\rho^3 h + h^3\rho - 4\rho^4\,d\rho\,d\theta =$$

$$= \int_0^{2\pi}\left[\frac{3}{4}\rho^4 h + h^3\frac{1}{2}\rho^2 - \frac{4}{5}\rho^5\right]_0^h d\theta = \int_0^{2\pi} \overbrace{\frac{3}{4}h^5+\frac{1}{2}h^5-\frac{4}{5}h^5}^{15+10-16}\,d\theta =$$

$$= \int_0^{2\pi} \frac{9h^5}{20}\,d\theta = \frac{h^5}{10}\pi\cdot 9$$

$\boxed{2018\text{-}01\text{-}16}$

2. ARIKETA

$$F(x,y,z) = x^2+y^2+z^2+xy+2z-1$$

a) $F(x,y,z)=0 \quad z=z(x,y) \quad (0,-1,0)$-n DEFINITU?

H_1: $F(0,-1,0) = 0+1+0+0+0-1 = 0$ ✓

H_2: $\frac{\partial F}{\partial z} = 2z+2\Big|_{(0,-1,0)} = 2 \neq 0$ ✓

$\Rightarrow$ $(0,-1,0)$-ren ingurune batean $\exists z$ C^1 klasekoa non $z=z(x,y)$ ekuazioaren soluzioa den.

b) $z=z(x,y)$-ren LEHEN ORDENAKO DERIBATU PARTZIALAK

$$\frac{\partial}{\partial x} \rightarrow \quad 2x+2z\frac{\partial z}{\partial x}+y+2\frac{\partial z}{\partial x}=0$$

$$\xrightarrow{(0,-1)} \quad 0+2\cdot 0\frac{\partial z}{\partial x}-1+2\frac{\partial z}{\partial x}=0 \Rightarrow \boxed{\frac{\partial z}{\partial x}=\frac{1}{2}}$$

$$\frac{\partial}{\partial y} \rightarrow \quad 2y+2z\frac{\partial z}{\partial y}+x+2\frac{\partial z}{\partial y}=0$$

$$\xrightarrow{(0,-1)} \quad -2+2\cdot 0\frac{\partial z}{\partial y}+0+2\frac{\partial z}{\partial y}=0 \Rightarrow \boxed{\frac{\partial z}{\partial y}=1}$$

c) TAYLOR. 2.

$$\frac{\partial^2}{\partial x^2} \rightarrow \quad 2+2\left(\frac{\partial z}{\partial x}\right)^2+\cancel{2z\frac{\partial^2 z}{\partial x^2}}^{\,0}+2\frac{\partial^2 z}{\partial x^2}=0$$

$$\xrightarrow{(0,-1)} \quad 2+\overbrace{\frac{2}{4}}^{6/4}+2\frac{\partial^2 z}{\partial x^2}=0 \Rightarrow \boxed{\frac{\partial^2 z}{\partial x^2}=-\frac{3}{4}}$$

$$\frac{\partial^2}{\partial y^2} \rightarrow \quad 2+2\left(\frac{\partial z}{\partial y}\right)^2+\cancel{2z\frac{\partial^2 z}{\partial y^2}}^{\,0}+2\frac{\partial^2 z}{\partial y^2}=0$$

$$\xrightarrow{(0,-1)} \quad 2+2+2\frac{\partial^2 z}{\partial y^2}=0 \Rightarrow \boxed{\frac{\partial^2 z}{\partial y^2}=-2}$$

$$\frac{\partial}{\partial y \partial x} \rightarrow 2 \underset{1/2}{\frac{\partial z}{\partial x}} \underset{1}{\frac{\partial z}{\partial y}} + 2z \cancel{\frac{\partial^2 z}{\partial y \partial x}}^{0} + 1 + 2\frac{\partial^2 z}{\partial y \partial x} = 0$$

$$\xrightarrow{(0,-1)} \quad 1 + 1 + 2\frac{\partial^2 z}{\partial y \partial x} = 0 \Rightarrow \boxed{\frac{\partial^2 z}{\partial y \partial x} = -1}$$

$$\boxed{z(x,y) = \frac{1}{2}x + (y+1) - \frac{3}{8}x^2 - (y+1)^2 - x(y+1) + R_2}$$

3. ARIKETA

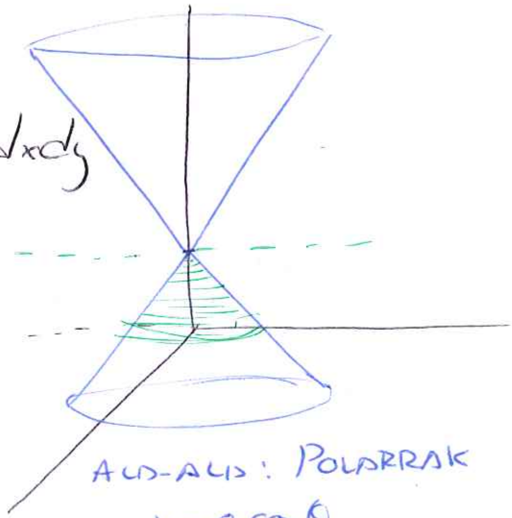
$\vec{F}(x,y,z) = (x^2, y^2, z^2)$ FLUXUA

$S = \{(x,y,z) \in \mathbb{R}^2 : z = 1 - \sqrt{x^2+y^2}$ $\quad 0 \le z \le 1$

$(z-1)^2 = -x^2 - y^2$

a) ZUZENEAN:

FLUXUA: $\iint_S \vec{F}\,dS = \iint_D -F_1 g_x - F_2 g_y + F_3 \, dx dy$



ALDAGAI ALDAKETA: POLARRAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$|J| = \rho$

$z = g(x,y) = 1 - \sqrt{x^2+y^2}$

$$g_x = \frac{-x}{\sqrt{x^2+y^2}} \qquad g_y = \frac{-y}{\sqrt{x^2+y^2}}$$

$z = 0 \rightarrow x^2 + y^2 = 1$

$$\iint_S \vec{F}ds = \iint_D + \frac{x^3}{\sqrt{x^2+y^2}} + \frac{y^3}{\sqrt{x^2+y^2}} + (1 - \sqrt{x^2+y^2})^2 \, dx dy =$$

$$= \int_0^{2\pi}\int_0^1 \frac{\rho^3(\cos^3\theta + \sin^3\theta)}{\cancel{\rho}}\rho + \overbrace{(1-\rho)^2\rho}^{(1-2\rho+\rho^2)\rho} \, d\rho \, d\theta =$$

$$= \int_0^{2\pi} \left[\frac{1}{4}\rho^4 \left[(1-\sin^2\theta)\cos\theta + (1-\cos^2\theta)\sin\theta\right] + \frac{1}{2}\rho^2 - \frac{2}{3}\rho^3 + \frac{1}{4}\rho^4\right]_0^1 d\theta =$$

$$= \int_0^{2\pi} \frac{1}{4}(\cos\theta - \sin^2\theta\cos\theta + \sin\theta - \cos^2\theta\sin\theta) + \overbrace{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}}^{\frac{6}{12} - \frac{8}{12} + \frac{3}{12}} \, d\theta =$$

$$= \left[\frac{1}{4}\left(\sin\theta - \frac{1}{3}\sin^3\theta - \cos\theta + \frac{1}{3}\cos^3\theta\right) + \frac{\theta}{12}\right]_0^{2\pi} = \boxed{\frac{\pi}{6}}$$

b) TEOREMEKIN: GAUSS

$$\iint_{\partial\Omega} \vec{F}\, dS = \iiint_{\Omega} \operatorname{div} \vec{F}\, dV$$

ALD-ALD: ZILINDRIKOAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z \qquad |J| = \rho$

$\theta \in [0, 2\pi] \qquad \rho \in [0, 1]$

$z \in [0, 1-\rho]$

$$\vec{\nabla}\cdot\vec{F} = 2x + 2y + 2z$$

$$\iiint_{\Omega} 2x+2y+2z \, dx\, dy\, dz =$$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-\rho} 2\rho^2(\cos\theta+\sin\theta) + 2\rho z \, dz\, d\rho\, d\theta =$$

$$= \int_0^{2\pi}\int_0^1 \left[2\rho^2(\cos\theta+\sin\theta)z + \rho z^2\right]_0^{1-\rho} d\rho\, d\theta =$$

$$= \int_0^{2\pi}\int_0^1 2(\rho-\rho^3)(\cos\theta+\sin\theta) + \rho - 2\rho^2 + \rho^3 \, d\rho\, d\theta =$$

$$= \int_0^{2\pi}\left[(\rho^2 - \tfrac{1}{2}\rho^4)(\cos\theta+\sin\theta) + \tfrac{1}{2}\rho^2 - \tfrac{2}{3}\rho^3 + \tfrac{1}{4}\rho^4\right]_0^1 d\theta =$$

$$= \int_0^{2\pi} \frac{1}{2}(\cos\theta+\sin\theta) + \overbrace{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}}^{6-8+3 = \frac{1}{12}}\, d\theta =$$

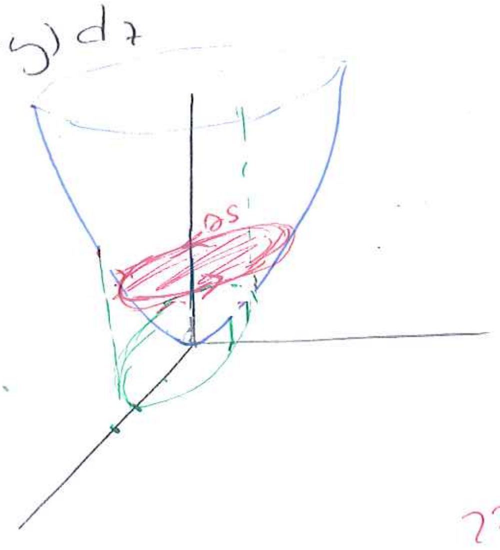
$$= \left[\frac{1}{2}(\sin\theta - \cos\theta) + \frac{\theta}{12}\right] = \boxed{\frac{\pi}{6}}$$

4. ARIKETA: STOKES

$$\int_{\gamma} (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$$

non $\gamma : \begin{cases} x^2 + 4y^2 = 1 \\ z = x^2 + y^2 \end{cases}$

$$\vec{F} = (y-z,\ z-x,\ x-y)$$



STOKES: $\displaystyle\iint_S \operatorname{rot}\vec{F}\, dS = \int_{\partial S} \vec{F}\, dS$

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} = (-1-1, -1-1, -1-1) =$$
$$= (-2,-2,-2)$$

$$\iint_S F\,dS = \iint_D -F_1 g_x - F_2 g_y + F_3 \,dx\,dy$$

$$\iint_S (-2,-2,-2)\,dS = \iint_D 4x + 4y \overset{-2}{\cancel{-2x^2 - 2y^2}}\,dx\,dy =$$

$z = g(x,y) = x^2 + y^2$ $\qquad g_x = 2x \qquad g_y = 2y$

$\left(\frac{x}{2}\right)^2 + y^2 = \frac{1}{4} \leftarrow x^2 + 4y^2 = 1$ $\qquad$ ALD-ALD: ZILINDRIKOAK

$x = 2\rho\cos\theta \qquad \theta \in [0, 2\pi]$

$y = \rho\sin\theta \qquad \rho \in [0, \frac{1}{2}]$

$|J| = 2\rho$

$$= \int_0^{2\pi}\int_0^{1/2} 8\rho^2(2\cos\theta + \sin\theta) \overset{-4\rho}{\cancel{- 4\rho^3(4\cos^2\theta + \sin^2\theta)}}\,d\rho\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{8}{3}\rho^3(2\cos\theta + \sin\theta) \overset{-2\rho^2}{\cancel{- \rho^4(4\cos^2\theta + \sin^2\theta)}}\right]_0^{1/2} d\theta =$$

$$= \int_0^{2\pi} \frac{1}{3}(2\cos\theta + \sin\theta) \overset{-\frac{1}{2}}{\cancel{- \frac{1}{16}\left(2(1+\cos 2\theta) + \frac{1-\cos 2\theta}{2}\right)}}\,d\theta =$$

$$= \left[\frac{1}{3}(2\sin\theta - \cos\theta) \overset{-\frac{1}{2}\theta}{\cancel{- \frac{1}{16}\left(2\theta + \sin 2\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)}}\right]_0^{2\pi} =$$

$$\cancel{= -\frac{1}{16}(4\pi + \pi) = -\frac{5}{16}\pi}$$

$$= -\frac{2\pi}{2} = \boxed{-\pi}$$

$$\int_{\partial S} \vec{F} ds = \int_{\gamma} F_1 dx + F_2 dy + F_3 dz$$

$$\sigma(\theta) = (\cos\theta, \tfrac{1}{2}\sin\theta, z) \qquad \vec{F} = (y-z, z-x, x-y)$$

$$z = x^2 + y^2 = \cos^2\theta + \tfrac{1}{4}\sin^2\theta$$

$$\sigma(\theta) = (\cos\theta, \tfrac{1}{2}\sin\theta, \cos^2\theta + \tfrac{1}{4}\sin^2\theta)$$

$$\sigma'(\theta) = (-\sin\theta, \tfrac{1}{2}\cos\theta, \underbrace{-2\cos\theta\sin\theta + \tfrac{1}{2}\sin\theta\cos\theta}_{-\frac{3}{2}\cos\theta\sin\theta})$$

$$\int_{\partial S} \vec{F} ds = \int_0^{2\pi} -\tfrac{1}{2}\sin^2\theta + \cancel{\cos^2\theta\sin\theta} + \tfrac{1}{4}\overset{(1-\cos^2\theta)\sin\theta}{\sin^3\theta} +$$

$$+ \cancel{\tfrac{1}{2}\overset{(1-\sin^2\theta)\cos\theta}{\cos^3\theta}} + \cancel{\tfrac{1}{8}\sin^2\theta\cos\theta} - \tfrac{1}{2}\cos^2\theta +$$

$$\cancel{-\tfrac{3}{2}\cos^2\theta\sin\theta} + \cancel{\tfrac{3}{4}\cos\theta\sin^2\theta} \, d\theta =$$

$$= \int_0^{2\pi} -\tfrac{1}{4}(1-\cos 2\theta) - \tfrac{1}{4}(1+\cos 2\theta) \, d\theta = -\pi$$

5. ARIKETA

$$\vec{F}(x,y,z) = (2xy + e^z + 1, x^2 + 3y^2z^2 + 1, 2y^3z + xe^z + 1)$$

$$C \equiv \sigma(t) = (\ln(1+t), t, \cos(\pi t)) \qquad t \in [0,1]$$

$$W = \int_C \vec{F} ds = \int_a^b \vec{F}(\sigma(t)) \cdot \sigma'(t) \, dt$$

$$\vec{F}(\sigma(t)) = (2t\ln(1+t) + e^{\cos(\pi t)} + \dots) \quad \text{ZAILA}$$

$$W = \int_C \vec{F} ds = \int_C \nabla f ds = f(\sigma(b)) - f(\sigma(a)) \quad \left[\begin{array}{l} \vec{F} \text{ IRROTAZIONALA} \\ \quad \Rightarrow \\ \exists f \text{ non } \nabla f = \vec{F} \end{array} \right.$$

$\vec{F}$ IRROTAZIONALE? $\iff$ $\text{rot}\,\vec{F} = 0$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + e^z + 1 & x^2 + 3y^2z^2 + 1 & 2y^3z + xe^z + 1 \end{vmatrix} =$$

$$= (6y^2z - 6y^2z,\ -e^z + e^z,\ 2x - 2x) = (0,0,0) \checkmark$$

$$\Rightarrow \exists f : \nabla f = \vec{F}$$

$$\vec{F} = (2xy + e^z + 1,\ x^2 + 3y^2z^2 + 1,\ 2y^3z + xe^z + 1)$$

$$\frac{\partial f}{\partial x} = 2xy + e^z + 1 \rightarrow f = x^2y + e^z x + x + h(y,z)$$

$$\frac{\partial f}{\partial y} = \cancel{x^2} + h'(y,z) = \cancel{x^2} + 3y^2z^2 + 1$$

$$\rightarrow h(y,z) = y^3z^2 + y + g(z)$$

$$\frac{\partial f}{\partial z} = \cancel{xe^z} + 2y^3z + g'(z) = 2y^3z + \cancel{xe^z} + 1$$

$$\rightarrow g(z) = z + A^{=0}$$

$$\Rightarrow f(x,y,z) = x^2y + e^z x + x + y^3z^2 + y + z$$

$$\sigma(t) = (\ln(1+t),\ t,\ \cos(\pi t)) \qquad t \in [0,1]$$

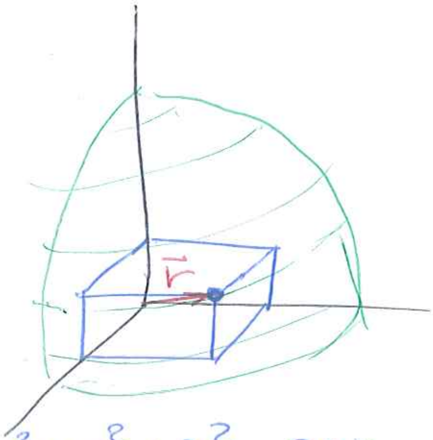
$$\sigma(0) = (0,0,1)$$

$$\sigma(1) = (\ln 2,\ 1,\ -1)$$

$$W = f(\sigma(1)) - f(\sigma(0)) = \ln^2 2 + \frac{\ln 2}{e} + \cancel{1-1} - 1 - 1$$

$$\boxed{W = \ln^2 2 + \frac{1}{e}\ln 2 - 2}$$

1. Ariketa



$$2x^2+y^2+z=1$$

$$\vec{r} = (x - \underset{0}{x_0},\ y - \underset{0}{y_0},\ z - \underset{0}{z_0})$$

$$\vec{r} = (x, y, z)$$

$$\|\vec{r}\| = \sqrt{x^2+y^2+z^2} \text{ max} \iff f = x^2+y^2+z^2 \text{ max}$$

$$g(x,y,z) = 2x^2+y^2+z \qquad c = 1$$

$$h(\lambda, x, y, z) = f(x,y,z) - \lambda(g(x,y,z) - c)$$

$$h(\lambda, x, y, z) = x^2+y^2+z^2 - \lambda(2x^2+y^2+z-1)$$

$$\nabla h = 0 \Rightarrow \begin{cases} \dfrac{\partial h}{\partial x} = \cancel{2}x - \overset{2}{\cancel{4}}\lambda x = 0 \\ \dfrac{\partial h}{\partial y} = \cancel{2}y - \cancel{2}\lambda y = 0 \\ \dfrac{\partial h}{\partial z} = 2z - \lambda = 0 \\ \dfrac{\partial h}{\partial \lambda} = -2x^2 - y^2 - z + 1 = 0 \end{cases}$$

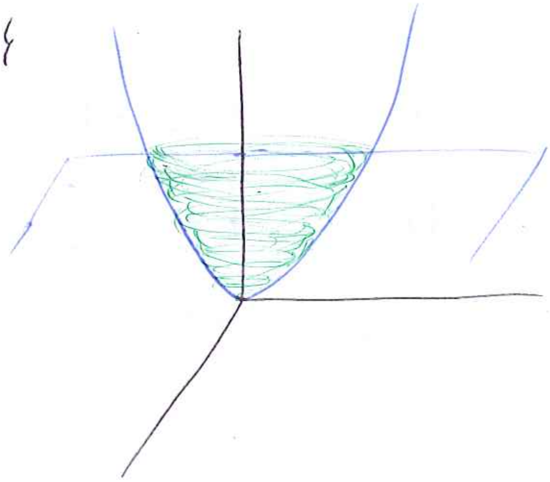
$$\lambda = 1 \longrightarrow z = \frac{1}{2}$$

2017-05-31

1. ARIKETA   MUTUR ABSOLUTUAK

$f(x,y,z) = xy + z$

$M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 1\}$



- M-ren BARRUAN

$\nabla f = \bar{0}$ beher

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = y = 0 \\ \frac{\partial f}{\partial y} = x = 0 \\ \frac{\partial f}{\partial z} = 1 \neq 0 \rightarrow \text{EZIN} \end{cases}$$

- M-ren MUGAN

  - $x^2 + y^2 = z$

$h(\lambda, x, y, z) = f(x,y,z) - \lambda(g_1 - c_1)$

$g_1(x,y,z) = z - x^2 - y^2 \quad g = 0$

$h(\lambda, x, y, z) = xy + z - \lambda(z - x^2 - y^2)$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial x} = y + 2\lambda x = 0 \rightarrow y^2 + 2\lambda xy = 0 \\ \frac{\partial h}{\partial y} = x + 2\lambda y = 0 \rightarrow x^2 + 2\lambda xy = 0 \\ \qquad\qquad x^2 = y^2 \\ \frac{\partial h}{\partial z} = 1 - \lambda = 0 \Rightarrow \lambda = 1 \\ \frac{\partial h}{\partial \lambda} = -z + x^2 + y^2 = 0 \Rightarrow z = x^2 + y^2 = 2x^2 \end{cases}$$

$\Rightarrow (0,0,0)$

$\nabla g_1 \neq 0$? $\nabla g_1 = (-2x, 2y, 1) \neq \bar{0}$

  - $z = 1$

$h(\lambda, x, y, z) = f(x,y,z) - \lambda(g_2 - c_2)$

$g_2 = z \qquad c_2 = 1$

$$h(\lambda,x,y,z) = xy + z - \lambda(z-1)$$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial h}{\partial x} = y = 0 \\ \dfrac{\partial h}{\partial y} = x = 0 \\ \dfrac{\partial h}{\partial z} = 1 - \lambda = 0 \\ \dfrac{\partial h}{\partial \lambda} = -z + 1 = 0 \end{cases} \Rightarrow (0,0,1)$$

$$\nabla g_2 \neq 0 \, ? \Rightarrow \nabla g_2 = (0,0,1) \neq \bar{0}$$

- EBAKIDURAN

$$h(\lambda,\mu,x,y,z) = xy + z - \lambda(z - x^2 - y^2) - \mu(z-1)$$

$$\nabla \vec{h} = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial h}{\partial x} = y + 2x\lambda = 0 \rightarrow y^2 + 2xy\lambda = 0 \\ \dfrac{\partial h}{\partial y} = x + 2y\lambda = 0 \rightarrow x^2 + 2xy\lambda = 0 \\ \qquad\qquad\qquad\qquad\qquad x^2 = y^2 \\ \dfrac{\partial h}{\partial z} = 1 - \lambda - \mu = 0 \\ \dfrac{\partial h}{\partial \lambda} = -z + x^2 + y^2 = 0 \Rightarrow x^2 + y^2 = 1 \Rightarrow 2x^2 = 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \dfrac{\partial h}{\partial \mu} = -z + 1 = 0 \Rightarrow z = 1 \qquad\qquad x = \dfrac{\pm 1}{\sqrt{2}} = y \end{cases}$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}, 1\right)$$

- EBALUATU

$f(0,0,0) = 0 \qquad\qquad f(0,0,1) = 1$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) = \frac{3}{2} \qquad\qquad f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1\right) = \frac{1}{2}$

$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1\right) = \frac{3}{2} \qquad\qquad f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) = \frac{1}{2}$

$\Rightarrow$ $(0,0,0)$ minimo absolutua

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) \wedge \left(-\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1\right)$ MAXIMO ABSOLUTUAK

2. ARIKETA

$$F(x,y,z) = \sin(\pi(x+y+z)) + \ln z^2$$

a) $F(x,y,z)=0$ $\quad z = f(x,y)$ $\quad (-1,-1,1)$-en DEFINITZEN DU

$H_1)$ $F(-1,-1,1) = \sin(\pi\cdot(-1)) + \ln 1 = 0$ ✓

$$H_2 = \frac{\partial F}{\partial z} = \cos(\pi(x+y+z))\cdot\pi + \frac{1}{z^2}2z\Big|_{(-1,-1,1)} = -\pi + 2 \neq 0$$

$\Rightarrow$ $(-1,-1,1)$-en inguruan badean $\exists f$ C^1 klasekoa non $z=f(x,y)$ soluzioa den

b) DERIBATU PARTZIALAK

$$\sin[\pi(x+y+z)] + \ln z^2 = 0$$

$$\frac{\partial}{\partial x} \rightarrow \cos[\pi(x+y+z)]\pi + \cos[\pi(x+y+z)]\pi\frac{\partial z}{\partial x} + \frac{1}{z^2}2z\frac{\partial z}{\partial x} = 0$$

$$\xrightarrow{(-1,-1)} \quad +\pi + \pi\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-\pi}{2+\pi}}$$

$$\frac{\partial}{\partial y} \rightarrow \cos[\pi(x+y+z)]\pi + \cos[\pi(x+y+z)]\pi\frac{\partial z}{\partial y} + \frac{1}{z^2}2z\frac{\partial z}{\partial y} = 0$$

$$\xrightarrow{(-1,-1)} \quad +\pi + \pi\frac{\partial z}{\partial y} + 2\frac{\partial z}{\partial y} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-\pi}{2+\pi}}$$

$$\frac{\partial^2}{\partial x^2} \rightarrow -\pi^2\sin[\pi(x+y+z)] - \pi^2\sin[\pi(x+y+z)]\frac{\partial z}{\partial x} +$$

$$-\pi^2\sin[\pi(x+y+z)]\frac{\partial z}{\partial x} - \pi^2\sin[\pi(x+y+z)]\left(\frac{\partial z}{\partial x}\right)^2 + \cos[\pi(x+y+z)]\pi\frac{\partial^2 z}{\partial x^2} +$$

$$-\frac{2}{z^3}2z\left(\frac{\partial z}{\partial x}\right)^2 + \frac{2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{1}{z^2}2z\frac{\partial^2 z}{\partial x^2} = 0$$

$$\xrightarrow{(-1,-1)} \quad \pi\frac{\partial^2 z}{\partial x^2} - 4\left(\frac{\partial z}{\partial x}\right)^2 + 2\left(\frac{\partial z}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial x^2} = 0$$

$$(\pi+2)\frac{\partial^2 z}{\partial x^2} = 2\left(\frac{\partial z}{\partial x}\right)^2 = 2\cdot\frac{\pi^2}{(2+\pi)^2} \Rightarrow \boxed{\frac{\partial^2 z}{\partial x^2} = 2\frac{\pi^2}{(2+\pi)^3}}$$

$$\frac{\partial^2}{\partial y^2} = -\pi^2 \sin[\pi(x+y+z)] - \pi^2 \sin[\pi(x+y+z)]\frac{\partial z}{\partial y} - \pi^2 [\sin[\pi(x+y+z)]] \frac{\partial z}{\partial y} +$$

$$-\pi^2 \sin[\pi(x+y+z)]\left(\frac{\partial z}{\partial y}\right)^2 + \pi \cos[\pi(x+y+z)] \frac{\partial^2 z}{\partial y^2} - \frac{2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 +$$

$$+ \frac{2}{z}\frac{\partial^2 z}{\partial y^2} = 0$$

$$\xrightarrow{(-1,-1)} \pi \frac{\partial^2 z}{\partial y^2} - 2\left(\frac{\partial z}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial y^2} = 0$$

$$(\pi+2)\frac{\partial^2 z}{\partial y^2} = 2\left(\frac{\partial z}{\partial y}\right)^2 = 2 \cdot \frac{\pi^2}{(2+\pi)^2} \Rightarrow \boxed{\frac{\partial^2 z}{\partial y^2} = 2\frac{\pi^2}{(2+\pi)^3}}$$

$$\frac{\partial^2}{\partial x \partial y} = -\pi^2 \sin[\pi(x+y+z)] - \pi^2 \sin[\pi(x+y+z)]\frac{\partial z}{\partial x} - \pi^2 \sin[\pi(x+y+z)]\frac{\partial z}{\partial y}$$

$$-\pi^2 \sin[\pi(x+y+z)]\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + \pi\cos[\pi(x+y+z)]\frac{\partial^2 z}{\partial x\partial y} - \frac{2}{z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} +$$

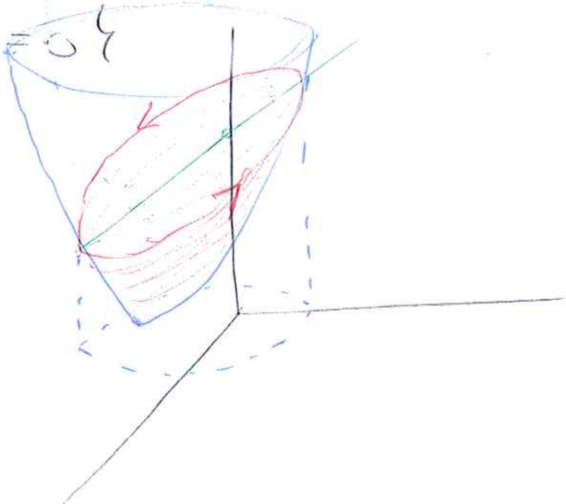
$$+ \frac{2}{z}\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\xrightarrow{(-1,-1)} \pi \frac{\partial^2 z}{\partial x\partial y} - 2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + 2\frac{\partial^2 z}{\partial x\partial y} = 0$$

$$(2+\pi)\frac{\partial^2 z}{\partial x \partial y} = 2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} = 2\frac{\pi^2}{(2+\pi)^2} \rightarrow \boxed{\frac{\partial^2 z}{\partial x\partial y} = 2\frac{\pi^2}{(2+\pi)^3}}$$

3. ARIKETA

$$W = \{ x^2 + (y+1)^2 = 2z \ \wedge \ y - z + 5 = 0 \}$$



a) W-ren BOLUMENA

$$\begin{cases} x^2 + (y+1)^2 = 2z = 10 + 2y \\ y - z + 5 = 0 \rightarrow z = 5 + y \end{cases}$$

$$x^2 + y^2 + 2y + 1 = 10 + 2y$$

$$x^2 + y^2 = 9$$

$B(W') = \iiint_W 1\,dx\,dy\,dz =$

ALD-ALD: ZILINDRIKOAK

$x = \rho\cos\theta \qquad \theta \in [0, 2\pi]$

$y = \rho\sin\theta \qquad \rho \in [0, 3]$

$z = z \qquad z \in [\text{parabo}, \text{plano}]$

$|J| = 1$

$z_{pr} = \frac{1}{2}(x^2 + (y+1)^2) =$

$= \frac{1}{2}(\rho^2\cos^2\theta + \rho^2\sin^2\theta + 2\rho\sin\theta + 1) =$

$= \frac{1}{2}(\rho^2 + 2\rho\sin\theta + 1)$

$z_{pl} = 5 + y = 5 + \rho\sin\theta$

$$= \int_0^{2\pi}\int_0^3\int_{\frac{1}{2}(\rho^2+2\rho\sin\theta+1)}^{5+\rho\sin\theta} \rho\,dz\,d\rho\,d\theta = \int_0^{2\pi}\int_0^3 5\rho + \cancel{\rho^2\sin\theta} - \frac{1}{2}(\rho^3 + \cancel{2\rho^2\sin\theta} + \rho)\,d\rho\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{5}{2}\rho^2 - \frac{1}{8}\rho^4 - \frac{1}{4}\rho^2\right]_0^3 d\theta = \int_0^{2\pi}\frac{45}{2} - \frac{81}{8} - \frac{9}{4}\,d\theta =$$

$$= \int_0^{2\pi}\frac{180-81-18}{8}\,d\theta = \boxed{\frac{81}{4}\pi}$$

b) $\vec{F} = (xy, xz, yz)$ ZIRKULAZIOA

STOKES: $\iint_S \operatorname{rot}\vec{F}\,ds = \int_{\partial S}\vec{F}\,ds$

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix} = (z-x, 0, z-y)$$

$g(x,y) = y + 5$

$\iint_\phi \vec{F}\,ds = \iint_D -F_1 g_x - F_2 g_y + F_3\,dx\,dy \qquad g_x = 0 \quad g_y = 1$

$$I = \iint_D (x-z)\cdot 0 - 0\cdot 1 + z - y\,dx\,dy = \iint_D z-y\,dx\,dy = \iint_D \cancel{y} + 5 - \cancel{y}\,dx\,dy =$$

$$= \iint_D 5\,dx\,dy = 5\cdot A(S) = 5\cdot\pi r^2 = \boxed{45\pi}$$

$$\int_\sigma \vec{F} ds = \int_\sigma F_1 dx + F_2 dy + F_3 dz = \qquad z = y + 5$$

$$\sigma(\theta) = (3\cos\theta, 3\sin\theta, -5 + 3\sin\theta)$$

$$\sigma(0) = (3, 0, 5)$$

$\Rightarrow$ ORIENTAZIOA MANTENDU

$$\sigma(\pi/2) = (0, 3, 7)$$

$$= \int_0^{2\pi} 9\cos\theta\sin\theta \overbrace{(-3\sin\theta)}^{dx} + 9\cos^2\theta \overbrace{(5 + 3\sin\theta)}^{dy} + 9\sin\theta \overbrace{\cos\theta}^{dz} (5 + 3\sin\theta) d\theta$$

$$= \int_0^{2\pi} -27\sin^2\theta\cos\theta + 45 \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} + 27\cos^2\theta\sin\theta + 45\sin\theta\cos\theta + 27\sin^2\theta\cos\theta \, d\theta$$

$$= \left[-9\sin^3\theta + \frac{45}{2}\theta + \frac{45}{4}\sin 2\theta - 9\cos^3\theta + \frac{45}{2}\sin^2\theta + 9\sin^3\theta \right]_0^{2\pi} =$$

$$= \frac{45}{2} 2\pi = \boxed{45\pi}$$

4. ARIKETA

$$\vec{F}(x, y, z) = (yz(2x + y), xz(x + 2y), xy(x + y))$$

a) $\vec{F}$ KONTSERBAKORRA $\Rightarrow \text{rot}\,\vec{F} = 0$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz(2x+y) & xz(x+2y) & xy(x+y) \end{vmatrix} =$$

$$= (x^2 + 2xy - x^2 - 2xy, -2xy - y^2 + 2xy + y^2, 2zx + 2zy - 2zx - 2zy)$$

$$= (0, 0, 0) \Rightarrow \vec{F} \text{ IRROTAZIONALA} \Leftrightarrow \vec{F} \text{ KONTSERBAKORRA}$$

b) $\int_c \vec{F}\, d\sigma$

$c: \sigma(t): (1+t, 1+2t^2, 1+3t^3) \quad t \in [0,1]$

$\vec{F}$ KONTS $\Rightarrow \exists f$ non $\nabla f = \vec{F}$

$$\frac{\partial f}{\partial x} = yz(2x+y) = yzx^2 + y^2zx + h(y,z)$$

$$\frac{\partial f}{\partial y} = zx^2 + 2zxy + h'(y,z) = xz(x+2y) \Rightarrow h'(y,z) = 0$$

$$\Rightarrow h(y,z) = g(z) + k^{=0}$$

$$\frac{\partial f}{\partial z} = yx^2 + y^2x + g'(z) = xy(x+y) \Rightarrow g'(z) = 0$$

$$\Rightarrow g(z) = A^{=0}$$

$$\Rightarrow f(x,y,z) = yzx^2 + y^2zx$$

$$\int_\sigma \vec{F} ds = \int_\sigma \nabla f ds = f(\sigma(b)) - f(\sigma(a))$$

$\sigma(1) = (2,3,4)$

$\sigma(0) = (1,1,1)$

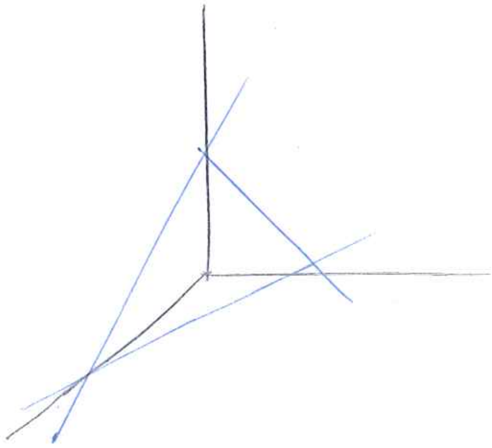
$f(\sigma(1)) = 3 \cdot 4 \cdot 4 + 9 \cdot 4 \cdot 2 = 120$

$f(\sigma(0)) = 2$

$$\int_\sigma \vec{F} ds = 120 - 2 = \boxed{118}$$

5. ARIKETA: GAUSS

$$\iint_S x^3\,dy\,dz + y^3\,dx\,dz + x^3\,dx\,dy$$



$S \equiv x + y + z = a \qquad a > 0$

$\vec{F} = (x^3, y^3, z^3)$

GAUSS: $\displaystyle\iint_{\partial\Omega} \vec{F}\,dS = \iiint_{\Omega} \operatorname{div}\vec{F}\,dV$

$\operatorname{div}\vec{F} = 3x^2 + 3y^2 + 3z^2$

$$I = \int_0^a \int_0^{a-x} \int_0^{a-x-y} 3x^2 + 3y^2 + 3z^2\,dz\,dy\,dx =$$

$$= \int_0^a \int_0^{a-x} \Big[z(3x^2 + 3y^2) + z^3 \Big]_0^{a-x-y} dy\,dx =$$

$$= \int_0^a \int_0^{a-x} a(3x^2 + 3y^2) - 3x^3 - 3y^2x - 3x^2y - 3y^3 + (a-x-y)^3\,dy\,dx =$$

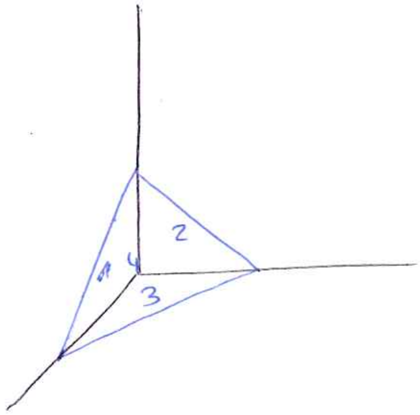
$$= \int_0^a \Big[a(3x^2y + y^3) - 3x^3y - y^3x - \frac{3}{2}x^2y^2 - \frac{3}{4}y^4 - \cancel{\frac{1}{4}(a-x-y)^4} \Big]_0^{a-x} dx =$$

$$= \int_0^a a\left(3x^2(a-x) + (a-x)^3\right) - 3x^3a + 3x^4 - (a-x)^3x - \frac{3}{2}x^2(a-x)^2 - \frac{3}{4}(a-x)\,dx$$

$$= \Big[a^2x^3 - \frac{3}{4}ax^5 - \frac{a}{4}(a-x)^4 - \frac{3}{4}x^4a + \frac{3}{5}x^5 - \frac{1}{2} \ldots$$

$$\iint_{\partial\Omega} \vec{F}\, dS = \iint_{\partial\Omega_1} \vec{F}\, dS + \iint_{\partial\Omega_2} \vec{F}\, dS + \iint_{\partial\Omega_3} \vec{F}\, dS + \iint_{\partial\Omega_4} \vec{F}\, dS$$

$$\iint_{\partial\Omega} \vec{F}\, ds = \iint \vec{F} \cdot \vec{n}\, dS$$



$$\iint_{\partial\Omega_1} (x^3, y^3, z^3) \cdot (0,-1,0)\, dS \overset{y=0}{=} 0$$

$$\iint_{\partial\Omega_2} (x^3, y^3, z^3) \cdot (-1,0,0)\, dS \overset{x=0}{=} 0$$

$$\iint_{\partial\Omega_3} (x^3, y^3, z^3) \cdot (0,0,-1)\, dS \overset{z=0}{=} 0$$

$$\iint_{\partial\Omega_4} \vec{F}\, dS = \iint_D -F_1 g_x - F_2 g_y + F_3\, dx\, dy =$$

$g(x,y) = a - x - y \qquad g_x = -1 \qquad g_y = -1$

$$= \int_0^a \int_0^{a-x} x^3 + y^3 + (a-x-y)^3\, dy\, dx =$$

$$= \int_0^a \left[x^3 y + \frac{1}{4} y^4 - \frac{1}{4}(a-x-y)^4 \right]_0^{a-x} dx =$$

$$= \int_0^a x^3(a-x) + \frac{1}{4}(a-x)^4 - \cancel{\frac{1}{4}(a-x-a+x)}^{\,0} + \frac{1}{4}(a-x)^4\, dx$$

$$= \left[\frac{1}{4} a x^4 - \frac{1}{5} x^5 - \frac{1}{20}(a-x)^5 + \frac{1}{20}(a-x)^5 \right]_0^a =$$

$$= \frac{1}{4} a^5 - \frac{1}{5} a^5 + \cancel{\frac{1}{20} a^5 - \frac{1}{20} a^5} = \boxed{\frac{a^5}{20}}$$

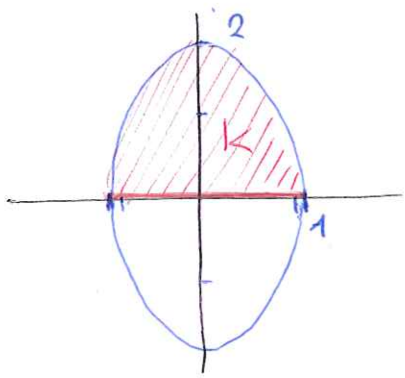
40

2016-06-06

1. ARIKETA

$$f(x,y) = 4x^2 + y^2 - 4x - 3y$$

$$\begin{cases} y \geq 0 \\ 4x^2 + y^2 \leq 4 \end{cases}$$



- K-ren BARRUAN

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 8x - 4 = 0 \Rightarrow x = \frac{1}{2} \\ \frac{\partial f}{\partial y} = 2y - 3 = 0 \Rightarrow y = \frac{3}{2} \end{cases}$$

$$\Rightarrow \left(\frac{1}{2}, \frac{3}{2}\right)$$

- K-ren MUGAN

  - $y = 0$

$$h(\lambda, x, y) = 4x^2 + y^2 - 4x - 3y - \lambda y$$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial x} = 8x - 4 = 0 \Rightarrow x = \frac{1}{2} \\ \frac{\partial h}{\partial y} = 2y - 3 - \lambda = 0 \\ \frac{\partial h}{\partial \lambda} = -y = 0 \end{cases}$$

$$\Rightarrow \left(\frac{1}{2}, 0\right) \qquad \nabla g \neq 0? \Rightarrow (0,1) \neq (0,0)$$

  - $4x^2 + y^2 = 4$

$$h(\lambda, x, y) = 4x^2 + y^2 - 4x - 3y - \lambda(4x^2 + y^2 - 4)$$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial x} = 8x - 4 - \lambda 8x = 0 \\ \frac{\partial h}{\partial y} = 2y - 3 - 2\lambda y = 0 \\ \frac{\partial h}{\partial \lambda} = -4x^2 - y^2 + 4 = 0 \end{cases}$$

$$\begin{cases} 8x(1-\lambda)-4=0 \\ 2y(1-\lambda)-3=0 \end{cases} \Rightarrow \quad x=\frac{1}{2(1-\lambda)} \qquad y=\frac{3}{2(1-\lambda)}$$

$$4x^2+y^2=4 \Rightarrow \frac{4}{4(1-\lambda)^2}+\frac{9}{4(1-\lambda)^2}=4 \Rightarrow 13=16(1-\lambda)^2$$

$$(1-\lambda)=\pm\frac{\sqrt{13}}{4}$$

$$x=\frac{1}{2(1-\lambda)}=\frac{\pm 4}{2\sqrt{13}} \Rightarrow x=\frac{\pm 2}{\sqrt{13}}$$

$$y=\frac{1}{2(1-\lambda)}=\frac{\pm 4\cdot 3}{2\sqrt{13}} \Rightarrow y=\pm\frac{6}{\sqrt{13}}$$

$$\Rightarrow \underline{\left(\frac{2}{\sqrt{13}},\frac{6}{\sqrt{13}}\right),\left(-\frac{2}{\sqrt{13}},-\frac{6}{\sqrt{13}}\right)} \qquad \nabla g \neq 0? \quad \nabla g=(8x,2y)=(0,0) \to (0,0)$$

- EBAKIDURA

$$h(\lambda,\mu,x,y)=4x^2+y^2-4x-3y-\lambda y-\mu(4x^2+y^2-4)$$

$$\nabla h=\bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial x}=8x-4-8\mu x=0 \Rightarrow \\ \frac{\partial h}{\partial y}=2y-3-\lambda-2\mu y=0 \\ \frac{\partial h}{\partial \lambda}=-y=0 \\ \frac{\partial h}{\partial \mu}=-4x^2-y^2+4=0 \Rightarrow 4x^2=4 \Rightarrow x=\pm 1 \end{cases}$$

$$\Rightarrow (1,0),(-1,0)$$

- EBALUATU

$$f\left(\tfrac{1}{2},\tfrac{3}{2}\right)=-\frac{13}{4} \qquad f\left(\tfrac{1}{2},0\right)=-1$$

$$f(1,0)=0 \qquad f(-1,0)=8$$

$$f\left(\tfrac{2}{\sqrt{13}},\tfrac{6}{\sqrt{13}}\right)=4-2\sqrt{13}=-3,21 \qquad f\left(\tfrac{-2}{\sqrt{13}},\tfrac{-6}{\sqrt{13}}\right)=4+2\sqrt{13}=11,21$$

$$\boxed{\begin{array}{l}\left(\frac{1}{2},\frac{3}{2}\right) \Rightarrow \text{minimo absolutua} \\ \left(\frac{-2}{\sqrt{13}},\frac{-6}{\sqrt{13}}\right) \Rightarrow \text{MAXIMO ABSOLUTUA}\end{array}}$$

2. ARIKETA

$$\begin{cases} x^2+y^2+z^2=3 \\ x^2+3xy+2z=0 \end{cases} \qquad \begin{array}{l} y=y(x) \\ z=z(x) \end{array} \qquad \begin{array}{l} y(1)=-1 \\ z(1)=1 \end{array}$$

$$F_1 = x^2+y^2+z^2-3=0$$

$$F_2 = x^2+3xy+2z=0$$

$$F_1(1,-1,1) = 1+1+1-3=0 \checkmark$$

$$\vec{F}_2(1,-1,1) = 1-3+2=0 \checkmark$$

$$\frac{\partial F_1}{\partial y} = 2y \qquad \frac{\partial F_2}{\partial y} = 3x$$

$$\frac{\partial F_1}{\partial z} = 2z \qquad \frac{\partial F_2}{\partial z} = 2 \qquad \Rightarrow \Delta = \begin{vmatrix} 2y & 2z \\ 3x & 2 \end{vmatrix}_{(1,-1,1)} = -4-6 \neq 0$$

$\Rightarrow (1,-1,1)$-en ingurune batean $\exists z, y$ C^1 klasekoak non $y=y(x)$ $\wedge$ $z=z(x)$ sistemaren soluzioak diren.

$$\lim_{x\to 1}\frac{y(x)+z(x)}{x-1} = \frac{0}{0} \to \text{L'HOPITAL} = \lim_{x\to 1}\frac{\frac{3}{5}-\frac{2}{5}}{1} = \boxed{\frac{1}{5}}$$

$$(1)\ x^2+y^2+z^2-3=0 \quad \wedge \quad (2)\ x^2+3xy+2z=0$$

$$\frac{\partial(1)}{\partial x} \to 2x+2y\frac{\partial y}{\partial x}+2z\frac{\partial z}{\partial x}=0$$

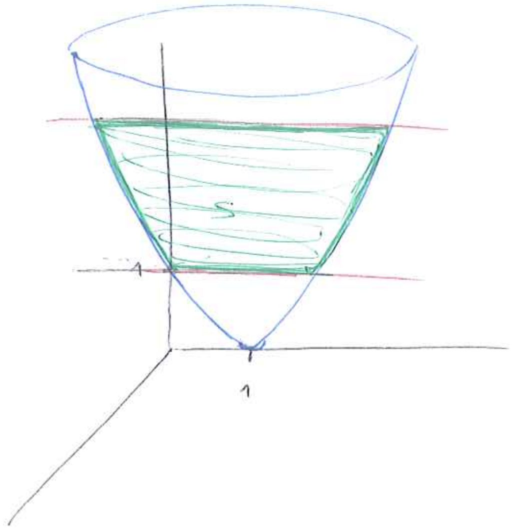
$$(1,-1,1) \to 2-2\frac{\partial y}{\partial x}+2\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2}{5}$$

$$\frac{\partial(2)}{\partial x} \to 2x+3y+3x\frac{\partial y}{\partial x}+2\frac{\partial z}{\partial x}=0$$

$$(1,-1,1) \to 2-3+3\frac{\partial y}{\partial x}+2\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial y}{\partial x} = \frac{3}{5}$$

3. ARIKETA

$$\begin{cases} z = 1 \\ z = 4 \\ z = x^2+(y-1)^2 \end{cases}$$



$$z = g(x,y) = x^2+(y-1)^2$$

$$g_x = 2x \qquad g_y = 2(y-1)$$

$$A(S) = \iint_S f\,dx\,dy = \iint_D f(x,y,g(x,y)) \cdot \sqrt{1+g_x^2+g_y^2}\,dx\,dy =$$

$$f(x,y,z) = 1$$

ALD: ALD: POLARRAK

$$x = \rho\cos\theta$$

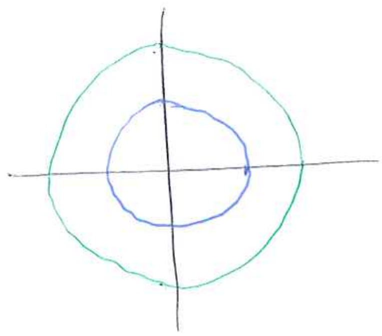
$$y = 1+\rho\sin\theta$$

$$|J| = \rho$$

$$= \iint_D \sqrt{1+4x^2+4(y-1)^2}\,dx\,dy =$$

$$z=4 \Rightarrow 4 = x^2+(y-1)^2 \Rightarrow r=2$$

$$z=1 \Rightarrow 1=x^2+(y-1)^2 \Rightarrow r=1$$



$$\theta \in [0, 2\pi] \qquad \rho \in [1,2]$$

$$= \int_0^{2\pi}\int_1^2 \sqrt{1+4\rho^2}\,\rho\,d\rho\,d\theta =$$

$$= \int_0^{2\pi} \frac{1}{3}\cdot\frac{1}{4}\left(1+4\rho^2\right)^{3/2}\Big]_1^2 d\theta = \int_0^{2\pi} \frac{1}{12}\left(17^{3/2}-5^{3/2}\right)d\theta =$$

$$= \boxed{\frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)}$$

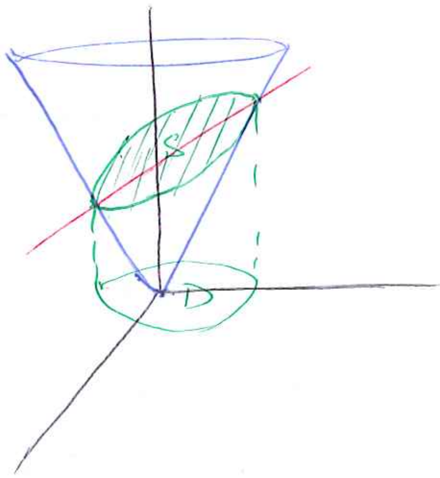
5. ARIKETA STOKES

$$\int_{\Gamma} (y-1)dx + z^2 dy + y\,dz \quad : \Gamma : \begin{cases} x^2+y^2 = \frac{z^2}{2} \\ z = y+1 \end{cases}$$

STOKES: $\iint_S \text{rot}\vec{F}\,ds = \int_{\partial S} \vec{F}\,ds$

$$\text{rot}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-1 & z^2 & y \end{vmatrix} = (1-2z, 0, -1)$$

$$\iint_S \vec{F}\,ds = \iint_D -F_1 g_x - F_2 g_y + F_3\,dx\,dy =$$



$$\begin{cases} x^2+y^2 = \frac{z^2}{2} \\ z = y+1 \end{cases} \rightarrow x^2+y^2 = \frac{(y+1)^2}{2}$$

$$2x^2 + 2y^2 = y^2+2y+1$$

$$x^2 + \frac{1}{2}y^2 - y = \frac{1}{2} \rightarrow x^2 + \left(\frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}\right)^2 = 1$$

$$z = g(x,y) = y+1 \qquad g_x = 0 \quad g_y = 1$$

$$= \iint_D -1\,dx\,dy = -\iint_D 1\,dx\,dy = -A(S) = -\pi ab$$

$$a = \frac{1}{\sqrt{2}} \qquad b = \sqrt{2}+1$$

$$y=0 \Rightarrow x^2 + \frac{1}{2} = 1 \Rightarrow x = \sqrt{\frac{1}{2}}$$

$$x=0 \Rightarrow \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}} = 1 \Rightarrow \frac{1}{\sqrt{2}}y = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}}$$

$$\Rightarrow \int_{\partial S} \vec{F}\,ds = \boxed{-\pi\left(1+\frac{1}{\sqrt{2}}\right)}$$

2016-01-12

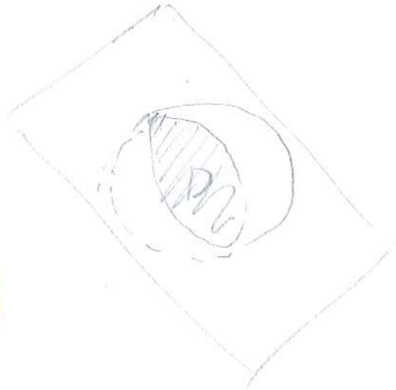
1. ARIKETA

$$f(x,y,z) = xy + z^2$$

$$D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 4 \wedge y = x\}$$

1) D-REN BARRUAN

MUTUR BALDINTZATUEN PROBLEMA



$g(x,y,z) = y - x$

$c = 0$

$h(\lambda,x,y,z) = f(x,y,z) - \lambda(g(x,y,z) - c)$

$h(\lambda,x,y,z) = xy + z^2 - \lambda(y - x)$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = y + \lambda = 0 \Rightarrow y = -\lambda \\ \dfrac{\partial f}{\partial y} = x - \lambda = 0 \Rightarrow x = \lambda \\ \dfrac{\partial f}{\partial z} = 2z = 0 \Rightarrow z = 0 \\ \dfrac{\partial f}{\partial \lambda} = -y + x = 0 \Rightarrow y = x \end{cases} \Rightarrow \lambda = x = y = z = 0$$

$\nabla g \neq \bar{0}$? $\nabla g = (-1,1,0) \neq \bar{0} \Rightarrow$ Ez dago punturik

$\Rightarrow (0,0,0)$ PUNTU KRITIKOA

2) D-REN MUGAN (∂D) — EBAKIDURA —

$g_1(x,y,z) = y - x \qquad c_1 = 0$

$g_2(x,y,z) = x^2+y^2+z^2 \qquad c_2 = 4$

$h(\lambda,\mu,x,y,z) = f(x,y,z) - \lambda(g_1 - c_1) - \mu(g_2 - c_2)$

$h(\lambda,\mu,x,y,z) = xy + z^2 - \lambda(x^2+y^2+z^2-4) - \mu(y-x)$

$\nabla h = \bar{0}$

$$\frac{\partial h}{\partial x} = y - 2\lambda x + \mu = 0 \Rightarrow y\cdot(1-2\lambda)+\mu = 0$$

$$\frac{\partial h}{\partial y} = x - 2\lambda y - \mu = 0 \Rightarrow x\cdot(1-2\lambda)-\mu = 0$$

$$\frac{\partial h}{\partial z} = 2z - 2\lambda z = 0 \Rightarrow z\cdot(1-\lambda) = 0$$

$$\frac{\partial h}{\partial \lambda} = -x^2-y^2-z^2+4 = 0$$

$$\frac{\partial h}{\partial \mu} = -y+x = 0 \Rightarrow y = x$$

$\lambda = 1 \Rightarrow y\cdot(-1)+\mu = 0 = -y-\mu \Rightarrow \mu = y = x = 0$

$\hookrightarrow z = \pm 2$.

$z = 0 \Rightarrow x^2+y^2 = 4 \Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$

$\Rightarrow (0,0,2), (0,0,-2), (\sqrt{2},\sqrt{2},0), (-\sqrt{2},-\sqrt{2},0)$ PUNTU KRITIKOAK

$\nabla g_2 \neq \bar{0}$? $\nabla g_2 = (2x, 2y, 2z) = \bar{0} \Rightarrow (0,0,0)$

- EBALUATU

$f(0,0,0) = 0$

$f(0,0,2) = 4$

$f(0,0,-2) = 4$

$f(\sqrt{2},\sqrt{2},0) = 2$

$f(-\sqrt{2},-\sqrt{2},0) = 2$

$\Rightarrow (0,0,0)$ MINIMO ABSOLUTUA

$(0,0,2) \wedge (0,0,-2)$ MAXIMO ABSOLUTUAK

2. ARIKETA

$$F = xy - 2yz + 3xz^5 + 1 = 0$$

$z = z(x,y) \qquad (x_0, y_0, z_0) = (1,-1,0)$

a)

$H_1: F(1,-1,0) = -1+1 = 0$ ✓

$H_2: \frac{\partial f}{\partial z} = -2y + 15xz^4 \Big|_{(1,-1,0)} = 2 \neq 0$ ✓

$\overset{\text{TEOR 2.1}}{\Longrightarrow}$ $(1,-1,0)$ ingurune batean $\exists g$ funtzio bakar bat non $F(x,y,g(x,y)) = 0$

b) TAYLOR 2. MAILAKOA $(1,-1)$

$$\frac{\partial}{\partial x} \to y + 3z^5 + 15xz^4\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial x} = 0$$

$$\xrightarrow{(1,-1)} -1+0+0+2\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{1}{2}$$

$$\frac{\partial}{\partial y} \to x - 2z - 2y\frac{\partial z}{\partial y} + 15xz^4\frac{\partial z}{\partial y} = 0$$

$$\xrightarrow{(1,-1)} 1 + 2\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{1}{2}$$

$$\frac{\partial^2}{\partial x^2} \to -2y\frac{\partial^2 z}{\partial x^2} + 15z^4\frac{\partial z}{\partial x} + 60xz^3\left(\frac{\partial z}{\partial x}\right)^2 + 15xz^4\frac{\partial^2 z}{\partial x^2} = 0$$

$$\xrightarrow{(1,-1)} 2\frac{\partial^2 z}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2}{\partial y^2} \to -2\overbrace{\frac{\partial z}{\partial y}}^{-\frac{1}{2}} - 2\frac{\partial z}{\partial y} - 2y\frac{\partial^2 z}{\partial y^2} + 60xz^3\left(\frac{\partial z}{\partial y}\right)^2 + 15xz^4\frac{\partial^2 z}{\partial y^2} = 0$$

$$\xrightarrow{(1,-1)} -2\cdot\frac{-1}{2} - 2\cdot\frac{-1}{2} + 2\frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 z}{\partial y^2} = -1$$

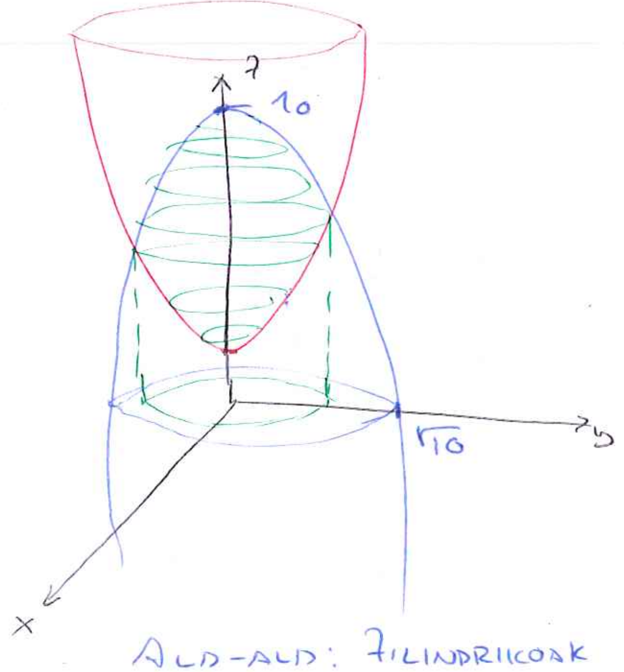
$$\frac{\partial^2}{\partial x\partial y} \to 1 - 2\frac{\partial z}{\partial x} - 2y\frac{\partial^2 z}{\partial x\partial y} + 15z^4\frac{\partial z}{\partial y} + 60xz^3\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + 15xz^4\frac{\partial^2 z}{\partial x\partial y} = 0$$

$$\xrightarrow{(1,-1)} 1 - 2\cdot\frac{1}{2} + 2\frac{\partial^2 z}{\partial x\partial y} = 0 \Rightarrow \frac{\partial^2 z}{\partial x\partial y} = 0$$

$$\boxed{z(x,y) \sim \frac{1}{2}(x-1) - \frac{1}{2}(y+1) - \frac{1}{2!}(y+1)^2 + R_2}$$

3. ARIKETA

DIBERGENTZIAREN TEOREMA



$$\Omega \begin{cases} z = 10 - x^2 - y^2 \\ z = 2 + x^2 + y^2 \end{cases}$$

$$\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\iint_{\partial\Omega} \vec{F}\,ds = \iint_{\partial\Omega} \vec{F}\cdot\vec{n}\,ds = \iiint_{\Omega} \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

ALD-ALD: ZILINDRIKOAK

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

$|J| = \rho$

$$10 - \rho^2 = 2 + \rho^2 \Rightarrow 8 = 2\rho^2$$

$$\Rightarrow \rho^2 = 4 \Rightarrow \rho = \pm 2 \Rightarrow \rho \in [0,2]$$

$$z_1 = 2 + \rho^2 \qquad z_2 = 10 - \rho^2$$

$$\iiint_{\Omega} \operatorname{div}\vec{F}\,dV = \int_0^{2\pi}\int_0^2\int_{2+\rho^2}^{10-\rho^2} 3\rho\,dz\,d\rho\,d\theta =$$

$$= \int_0^{2\pi}\int_0^2 3\rho\,(10-\rho^2-2-\rho^2)\,d\rho\,d\theta = \int_0^{2\pi}\int_0^2 3\rho\,(8-2\rho^2)\,d\rho\,d\theta =$$

$$= \int_0^{2\pi} 3\cdot\left[4\rho^2 - \frac{1}{2}\rho^4\right]_0^2 d\theta = 3\int_0^{2\pi} 16 - 8\,d\theta = 3\cdot 2\pi\cdot 8 = \boxed{48\pi}$$

$$\iint_{\partial\Omega} \vec{F}\,ds = \iint_D -F_1 g_x - F_2 g_y + F_3\,dx\,dy$$

$$g_1(x,y) = 10 - x^2 - y^2 \qquad g_2(x,y) = 2 + x^2 + y^2$$

$$\iint_D -x(-2x) - y(-2y) + 10 - x^2 - y^2\,dx\,dy + \iint_D -x\,2x - y\,2y + 2 + x^2 + y^2\,dx\,dy$$

$$= \iint_D 10 + x^2 + y^2\,dx\,dy + \iint_D 2 - x^2 - y^2\,dx\,dy = \int_0^{2\pi}\int_0^2 12\rho\,d\rho\,d\theta = \boxed{48\pi}$$

10

4. ARIKETA

$$F(x,y,z) = (axy - z^3, (a-2)x^2, (1-a)xz^2)$$

- Bilatu a non $\exists f \Rightarrow \nabla f = F$

$\exists f$ non $\nabla f = F \Leftrightarrow F$ KONTS $\Leftrightarrow \mathrm{rot}\,\vec{F} = 0$

$$\mathrm{rot}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} =$$

$$= (0, -(1-a)z^2 - 3z^2, 2(a-2)x - ax) = \bar{0}$$

$$(1-a)z^2 + 3z^2 = (1-a+3)z^2 = 0 \Rightarrow a = 4$$

$$2(a-2)x - ax = (2a-4-a)x = 0 \Rightarrow a = 4$$

- $\sigma(t) = (2\cos t, 2\sin t, t) \qquad (2,0,0) \rightarrow (\sqrt{2}, \sqrt{2}, \pi/4)$

$$\vec{F}(x,y,z) = (4xy - z^2, 2x^2, -3xz^2)$$

$$W = \int_\sigma \vec{F}ds = \int_a^b \vec{F}(\sigma(t)) \cdot \sigma'(t)dt = f(\sigma(b)) - f(\sigma(a))$$

$$\frac{\partial f}{\partial x} = 4xy - z^3 \Rightarrow f = 2x^2y - z^3x + h(y,z)$$

$$\frac{\partial f}{\partial y} = 2x^2 + h'(y,z) = 2x^2 \Rightarrow h'(y,z) = 0$$

$$\Rightarrow h(y,z) = g(z) + \overset{\text{0}}{k}$$

$$\frac{\partial f}{\partial z} = -\cancel{3}\overset{2}{z}x + g'(z) = -3xz^2 \Rightarrow g'(z) = 0$$

$$\Rightarrow g(z) = ;K = 0 \qquad \Rightarrow f(x,y,z) = 2x^2y - xz^3$$

$$W = 2(\sqrt{2})^2 \cdot \sqrt{2} - \sqrt{2} \cdot \frac{\pi^3}{4^3} \Rightarrow \boxed{W = 4\sqrt{2} - \frac{\pi^3}{4^3}\sqrt{2}}$$

8T

ALD-ALD: POLARRAK
−2 DSBRD:

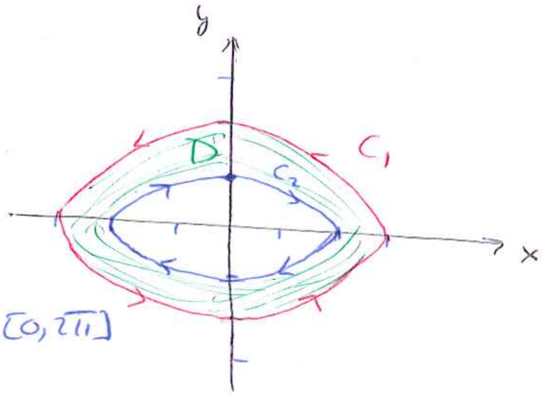
5. ARIKETA

GREENEN TMA: $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\rightarrow \iint_D (x-1)\,dx\,dy \qquad D = \{(x,y) : \frac{x^2}{4} + y^2 \geq 1,\ \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$

$\frac{\partial Q}{\partial x} = x \qquad \frac{\partial P}{\partial y} = 1$

$Q = \frac{1}{2}x^2 \qquad P = y$



$C_1 : \ \sigma_1(\theta) = (3\cos\theta, 2\sin\theta) \quad \theta \in [0, 2\pi]$

$\sigma_1(0) = (3, 0)$

$\sigma_1(\frac{\pi}{2}) = (0, 2)$ $\Rightarrow$ ORIENTAZIOA MANTENDU

$C_2 : \ \sigma_2(\theta) = (2\cos\theta, \sin\theta) \quad \theta \in [0, 2\pi]$

$\sigma_2(0) = (2, 0)$

$\sigma_2(\frac{\pi}{2}) = (0, 1)$ $\Rightarrow$ ORIENTAZIOA ALDATU

- $\int_C P\,dx + Q\,dy = \int_C y\,dx + \frac{1}{2}x^2\,dy =$

$= \int_0^{2\pi} -6\sin^2\theta + 9\cos^2\theta\cos\theta \,\cancel{d\theta} - \int_0^{2\pi} -2\sin^2\theta + 2\cos^2\theta\cos\theta\,d\theta =$

$= \int_0^{2\pi} -4\,\frac{1-\cos 2\theta}{2} + 7\cos\theta - 7\sin^2\theta\cos\theta\,d\theta = \boxed{-4\pi}$

- $\iint_D (x-1)\,dx\,dy = \int_0^{2\pi}\int_0^1 (3\rho\cos\theta - 1)6\rho\,d\rho\,d\theta - \int_0^{2\pi}\int_0^1 (2\rho\cos\theta - 1)2\rho\,d\rho\,d\theta =$

$= \int_0^{2\pi} \left[6\rho^3\cos\theta - 3\rho^2\right]_0^1 d\rho\,d\theta - \int_0^{2\pi} \left[\frac{4}{3}\rho^3\cos\theta - \rho^2\right]_0^1 d\theta =$

$= \int_0^{2\pi} 6\cos\theta - 3\,d\theta - \int_0^{2\pi} \frac{4}{3}\cos\theta - 1\,d\theta = \left[\cancel{6\sin\theta} - 3\theta\right]_0^{2\pi} - \left[\cancel{\frac{4}{3}\sin\theta} - \theta\right]_0^{2\pi} = \boxed{-4\pi}$

52

2016-07-11

1. ARIKETA

a) MAXIMO ETA MINIMOAK

$$x^2+y^2+z^2-2x=0 \qquad z=z(x,y)$$

$$\frac{\partial}{\partial x} \rightarrow 2x+2z\frac{\partial z}{\partial x}-2=0$$

$$z\frac{\partial z}{\partial x}=1-x=0 \Rightarrow x=1$$

$$\frac{\partial}{\partial y} \rightarrow 2y+2z\frac{\partial z}{\partial x}=0$$

$$z\frac{\partial z}{\partial x}=-y=0 \Rightarrow y=0$$

$$\frac{\partial^2}{\partial x^2}=2+\cancel{2\left(\frac{\partial z}{\partial x}\right)^2}^{0}+2\frac{\partial^2 z}{\partial x^2}=0$$

$$\frac{\partial^2 z}{\partial x^2}=-1<0 \rightarrow MAX$$

$\rightarrow$ $\boxed{(1,0)\text{-n MAXIMOA}}$

b) TAYLOR 2. (0,1)

$(0,1) \rightarrow 1+z^2=0 \rightarrow$ EZIN DA ???

c) JATORRITIK DISTANTZIA

$$h(\lambda,x,y,z)=x^2+y^2+z^2-\lambda(x^2+y^2+z^2-2x)$$

$$\nabla h=\bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial x}=2x-2\lambda x-2=0 \rightarrow x(1-\lambda)=1 \\ \frac{\partial h}{\partial y}=2y-2\lambda y=0 \rightarrow y\cdot(1-\lambda)=0 \rightarrow y=0 \\ \frac{\partial h}{\partial z}=2z-2\lambda z=0 \rightarrow z\cdot(1-\lambda)=0 \rightarrow z=0 \\ \frac{\partial h}{\partial \lambda}=-x^2-y^2-z^2+2x=0 \rightarrow x\cdot(2-x)=0 \end{cases}$$

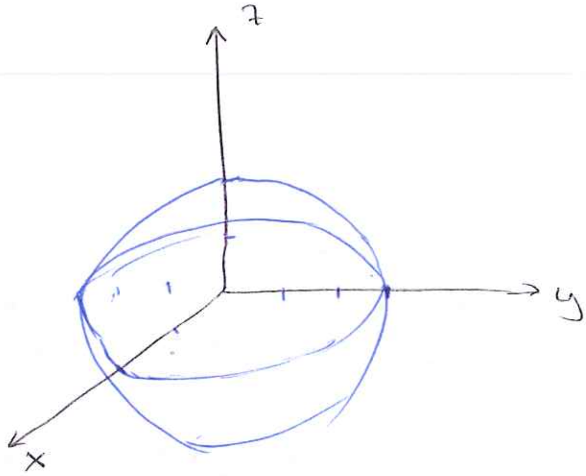
$\boxed{(2,0,0)\text{-n MINIMOA}}$

53

3. ARIKETA

$$9x^2+4y^2+9z^2=36 \xrightarrow{y=0} x^2+z^2=4$$

$$\vec{F}=xz^2\hat{\imath}+x^2y\hat{\jmath}+y^2z\hat{k}$$



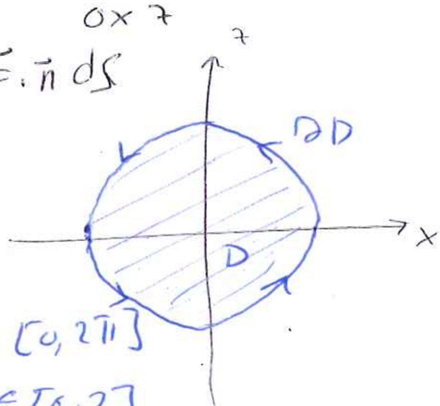
a) FLUXUA

STOKES: $\iint_S rot\vec{F}ds=\int_{\partial S}\vec{F}ds$

$$rot\vec{F}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xz^2 & x^2y & y^2z\end{vmatrix}=(2yz,\ 2zx,\ 2xy)$$

$$\iint_\phi \vec{F}ds=\iint_D -F_1g_x-F_2g_y+F_3dxdy=\iint_S \vec{F}\cdot\vec{n}\,ds$$

OXZ



$\vec{n}=(0,-1,0)$

$$\iint_D -2zx\,dzdx=$$

ALD-ALD: POLARRAK
$x=\rho\cos\theta$ $\quad\theta\in[0,2\pi]$
$z=\rho\sin\theta$ $\quad\rho\in[0,2]$
$|J|=\rho$

$$=\int_0^{2\pi}\int_0^2-2\rho^3\cos\theta\sin\theta\,d\rho\,d\theta=\int_0^{2\pi}-\frac{1}{2}\rho^4\cos\theta\sin\theta\,d\theta=$$

$$=\int_0^{2\pi}-8\cos\theta\sin\theta\,d\theta=-4\cos^2\theta\Big|_0^{2\pi}\boxed{=0}$$

$$\int_\partial \vec{F}ds=\int_\sigma F_1dx+F_2dy+F_3dz=$$

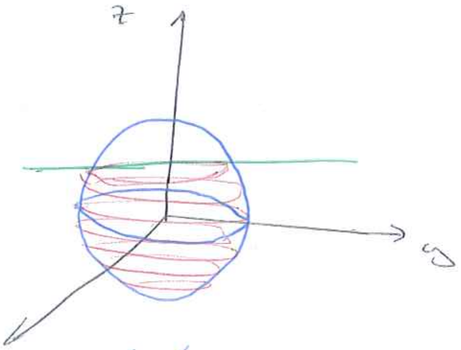
$$\sigma(\theta)=(2\cos\theta,0,2\sin\theta)$$

$$\sigma'(\theta)=(-2\sin\theta,0,2\cos\theta)$$

$$=\int_0^{2\pi}0+0+0\quad\boxed{=0}$$

$\boxed{2015-01-13}$

1. ARIKETA MUTUR ABSOLUTUAK



$$f_{(x,y,z)} = x^2+y^2+z^2+x+y+z$$

$$K \equiv \begin{cases} x^2+y^2+z^2=4 \\ z=1 \end{cases}$$

f diferentziagarria ∧ jarraitua ⇒ ∃ mutur absolutuak

- K-ren BARRUAN

$$\nabla f = \bar{0} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 2x+1=0 \Rightarrow x=-\frac{1}{2} \\ \frac{\partial f}{\partial y} = 2y+1=0 \Rightarrow y=-\frac{1}{2} \\ \frac{\partial f}{\partial z} = 2z+1=0 \Rightarrow z=-\frac{1}{2} \end{cases}$$

$$\Rightarrow \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

- K-ren INGURUAN

  - $x^2+y^2+z^2=4$

$$h(\lambda,x,y,z) = x^2+y^2+z^2+x+y+z-\lambda(x^2+y^2+z^2-4)$$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \frac{\partial h}{\partial x} = 2x+1-2\lambda x=0 \Rightarrow x\cdot(1-\lambda) = -\frac{1}{2} \\ \frac{\partial h}{\partial y} = 2y+1-2\lambda y=0 \Rightarrow y\cdot(1-\lambda) = -\frac{1}{2} \\ \frac{\partial h}{\partial z} = 2z+1-2\lambda z=0 \Rightarrow z\cdot(1-\lambda) = -\frac{1}{2} \\ \frac{\partial h}{\partial \lambda} = -x^2-y^2-z^2+4=0 \end{cases}$$

$$x^2+y^2+z^2=4 \Rightarrow \frac{1}{4(1-\lambda)^2}+\frac{1}{4(1-\lambda)^2}+\frac{1}{4(1-\lambda)^2}=4=\frac{3}{4(1-\lambda)^2}$$

$$\Rightarrow (1-\lambda)^2 = \frac{3}{16} \Rightarrow 1-\lambda = \frac{\pm\sqrt{3}}{4}$$

$$= \left(\frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$$

$\nabla g \neq 0 \; ? \Rightarrow g(x,y,z) = x^2+y^2+z^2 \Rightarrow \nabla g = (2x, 2y, 2z) = \bar{0}$

$\Rightarrow (0,0,0)$

- $z = 1$

$h(\lambda, x, y, z) = x^2+y^2+z^2+x+y+z-\lambda(z-1)$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial h}{\partial x} = 2x+1 = 0 \Rightarrow x = -\dfrac{1}{2} \\ \dfrac{\partial h}{\partial y} = 2y+1 = 0 \Rightarrow y = -\dfrac{1}{2} \\ \dfrac{\partial h}{\partial z} = 2z+1-\lambda = 0 \\ \dfrac{\partial h}{\partial \lambda} = -z+1 = 0 \Rightarrow z = 1 \end{cases}$$

$\Rightarrow \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$

$\nabla g \neq 0 \; ? \Rightarrow g(x,y,z) = z \Rightarrow \nabla g = (0,0,1) \neq \bar{0}$

- EBAKIDURAN

$h(\lambda, \mu, x, y, z) = x^2+y^2+z^2+x+y+z-\lambda(x^2+y^2+z^2-4)-\mu(z-1)$

$$\nabla h = \bar{0} \Rightarrow \begin{cases} \dfrac{\partial h}{\partial x} = 2x+1-2\lambda x = 0 \Rightarrow x(1-\lambda) = -\dfrac{1}{2} \\ \dfrac{\partial h}{\partial y} = 2y+1-2\lambda y = 0 \Rightarrow y\cdot(1-\lambda) = -\dfrac{1}{2} \\ \dfrac{\partial h}{\partial z} = 2z+1-2\lambda z-\mu = 0 \\ \dfrac{\partial h}{\partial \lambda} = -x^2-y^2-z^2+4 = 0 \Rightarrow x^2+y^2 = 3 \\ \dfrac{\partial h}{\partial \mu} = -z+1 = 0 \Rightarrow z = 1 \end{cases}$$

$x^2+y^2 = 3 \Rightarrow \dfrac{1}{4(1-\lambda)^2} + \dfrac{1}{4(1-\lambda)^2} = 3 \Rightarrow \dfrac{1}{2(1-\lambda)^2} = 3 \Rightarrow 1-\lambda = \pm\dfrac{1}{\sqrt{6}}$

$\Rightarrow \left(\dfrac{-1}{2\sqrt{6}}, \dfrac{1}{2\sqrt{6}}, 1\right), \left(\dfrac{-1}{2\sqrt{6}}, \dfrac{-1}{2\sqrt{6}}, 1\right)$

• EBALUATU

$f(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) = -\frac{3}{4}$ $\qquad f(0,0,0)=0$

$f(\frac{-2}{\sqrt{3}},\frac{-2}{\sqrt{3}},\frac{-2}{\sqrt{3}}) = 4-2\sqrt{3}$ $\qquad f(\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}) = 4+2\sqrt{3}$

$f(-\frac{1}{2},-\frac{1}{2},1) = \frac{3}{2}$ $\qquad f(\frac{1}{2\sqrt{6}},\frac{1}{2\sqrt{6}},1) = \frac{13}{12}+\frac{\sqrt{6}+6}{6}$

$f(\frac{-1}{2\sqrt{6}},\frac{-1}{2\sqrt{6}},1) = \frac{13}{2}+\frac{6-\sqrt{6}}{6}$

$\Rightarrow \boxed{\begin{array}{ll} (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) & \text{minimo absolutua} \\ (\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}) & \text{MAXIMO ABSOLUTUA} \end{array}}$

2. ARIKETA

$xy+yz+4xz^5+2=0 \qquad z=z(x,y) \quad (1,-2,0)$

$H_1)\ F(1,-2,0) = -2+0+0+2=0\ ✓$

$H_2)\ \frac{\partial F}{\partial z} = y+20xz^4 \Big|_{(1,-2,0)} = -2 \neq 0$

$\Rightarrow (1,-2,0)$-ren inguruan bakarra $\exists z$ C^1 klasekoa

non $z=z(x,y)$ ekuazioaren soluzioa den.

$\frac{\partial}{\partial x} \rightarrow y+y\frac{\partial z}{\partial x}+4z^5+20xz^4\frac{\partial z}{\partial x}=0$

$\xrightarrow{(1,-2,0)} -2-2\frac{\partial z}{\partial x}+0+0=0 \Rightarrow \frac{\partial z}{\partial x}=-1$

$\frac{\partial}{\partial y\partial x} \rightarrow 1+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial y\partial x}+\cancel{20z^4\frac{\partial z}{\partial y}}+\cancel{80xz^3\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}+\cancel{20xz^4\frac{\partial^2 z}{\partial y\partial x}}$

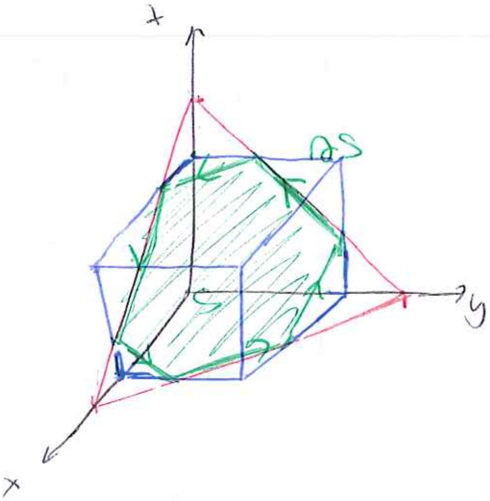
$\xrightarrow{(1,-2,0)} 1-1-2\frac{\partial^2 z}{\partial y\partial x} \Rightarrow \boxed{\frac{\partial^2 z}{\partial y\partial x}=0}$

3. Ariketa

$$\vec{F}(x,y,z) = (y^2 - z^2,\ z^2 - x^2,\ x^2 - y^2)$$

$$x + y + z = \frac{3}{2}$$

$$\varphi = \{0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$$



Stokes: $\displaystyle\iint_S \operatorname{rot}\vec{F}\,ds = \int_{\partial S} \vec{F}\,ds$

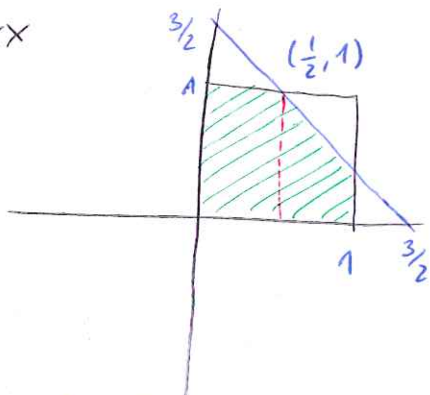
$$\operatorname{rot}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z,\ -2x - 2z,\ -2x - 2y)$$

$$\operatorname{rot}\vec{F} = -2\,(y+z,\ x+z,\ x+y)$$

$$\iint_S \vec{F}\,dS = \iint_D -F_1 g_x - F_2 g_y + F_3\,dx\,dy$$

$$g(x,y) = z = \frac{3}{2} - x - y \qquad g_x = -1 \qquad g_y = -1$$

OXX



$$x \in [0, \tfrac{1}{2}] \cup [\tfrac{1}{2}, 1]$$

$$y \in [0, 1] \cup [0, \tfrac{3}{2} - x]$$

$$I = -2\iint_D y + z + x + z + x + y\,dx\,dy =$$

$$= -2\iint_D \cancel{y} + \frac{3}{2} \cancel{- x - y + x} + \frac{3}{2} \cancel{- x - y + x + y}\,dx\,dy =$$

$$= -6\iint_D dx\,dy = -6\left[\int_0^{1/2}\int_0^1 dy\,dx + \int_{1/2}^1 \int_{3/2 - x}^1 dy\,dx\right] =$$

$$= -6\left[\int_0^{1/2} 1\,dx + \int_{1/2}^1 1 - \frac{3}{2} + x\,dx\right] = -6 \cdot \left[\frac{1}{2} + \left[-\frac{1}{2}x + \frac{1}{2}x^2\right]_{1/2}^{1}\right] =$$

$$-\frac{1}{2} + x$$

58

$$= -\int_{\pi/4}^{5\pi/4} \frac{1-\cos 2\theta}{2} + \sin\theta\cos\theta - \cos\theta \, d\theta +$$

$$+ \int_{\pi/4}^{5\pi/4} \frac{1+\cos 2\theta}{2} - \cos\theta\sin\theta + \sin\theta \, d\theta =$$

$$= \left[-\cancel{\frac{\theta}{2}} + \frac{\sin 2\theta}{4} - \frac{1}{2}\sin^2\theta + \sin\theta + \cancel{\frac{\theta}{2}} + \frac{\sin 2\theta}{4} + \frac{1}{2}\cos^2\theta - \cos\theta \right]_{\pi/4}^{5\pi/4} =$$

$$= \left[-\frac{1}{4} - \cancel{\frac{1}{4}} - \cancel{\frac{\sqrt{2}}{2}} - \frac{1}{4} + \cancel{\frac{1}{4}} + \cancel{\frac{\sqrt{2}}{2}} - \frac{1}{4} - \frac{1}{4} \right] = \boxed{-1}$$

$$I = \iint_D -\sin x \, dx\,dy \Rightarrow$$

$$x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$$

$$y \in [\cos x, \sin x] \cup [0, \sin x]$$

$$I_1 = \int_{\pi/4}^{\pi/2} \int_{\cos x}^{\sin x} -\sin x \, dy\,dx = \int_{\pi/4}^{\pi/2} -\sin x(\sin x - \cos x)\,dx =$$

$$= \int_{\pi/4}^{\pi/2} -\frac{1-\cos 2x}{2} + \sin x\cos x \, dx = \left[-\frac{x}{2} + \frac{\sin 2x}{4} + \frac{1}{2}\sin^2 x \right]_{\pi/4}^{\pi/2} =$$

$$= -\frac{\pi}{4} + 0 + \cancel{\frac{1}{2}} + \frac{\pi}{8} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} = -\frac{\pi}{8}$$

$$I_2 = \int_{\pi/2}^{\pi} \int_0^{\sin x} -\sin x \, dy\,dx = \int_{\pi/2}^{\pi} -\sin^2 x \, dx = \int_{\pi/2}^{\pi} -\frac{1-\cos 2x}{2} \, dx =$$

$$= \left[-\frac{x}{2} + \frac{\sin 2x}{4} \right]_{\pi/2}^{\pi} = -\frac{\pi}{2} + \frac{\pi}{4} = -\frac{\pi}{4}$$

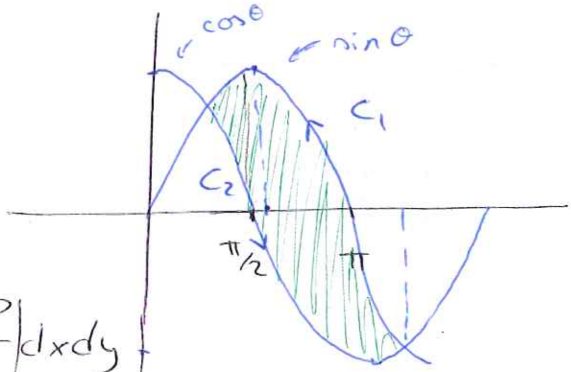
$$I = 2\cdot(I_1 + I_2) = \boxed{-\frac{3\pi}{4}}$$

$$= -6\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{4} - \frac{1}{8}\right] = -6 \cdot \frac{5}{8} = \boxed{-\frac{15}{4}}$$

4. ARIKETA -GREEN-

$$\int_C y\sin x\,dx + (y-1)\,dy$$

$$C \equiv \begin{cases} y = \sin x \\ y = \cos x \end{cases}$$



GREEN: $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\sin x = \cos x$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$P = y\sin x \qquad Q = y - 1$

$\frac{\partial P}{\partial y} = \sin x \qquad \frac{\partial Q}{\partial x} = 0$

$$\int_\sigma \vec{F}\,ds = \int_\sigma F_1\,dx + F_2\,dy$$

$C_1 = \sigma(\theta) = (\theta, \sin\theta) \qquad \theta \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\sigma\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$\Rightarrow$ ORIENTAZIOA ALDATU

$\sigma(\pi) = (\pi, 0)$

$C_2 = \sigma(\theta) = (\theta, \cos\theta)$

$\sigma\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$\Rightarrow$ ORIENTAZIOA MANTENDU

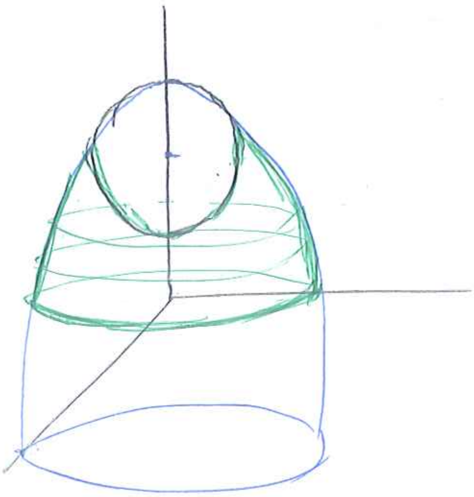
$\sigma(\pi) = (\pi, -1)$

$$I = -\int_{\pi/4}^{5\pi/4} \sin\theta\sin\theta\cdot 1 + (\sin\theta - 1)\cos\theta\,d\theta +$$

$$+ \int_{\pi/4}^{5\pi/4} \cos\theta\cos\theta\cdot 1 - (\cos\theta - 1)\sin\theta\,d\theta =$$

5. ARIKETA

$$W = \{ 0 \le z \le 3 - x^2 - y^2 ,\ x^2+y^2+z^2 \ge 4z-3 \}$$



$$\vec{F}(x,y,z) = (y,x,z)$$

$$x^2+y^2+z^2-4z = -3$$

$$x^2+y^2+(z-2)^2 = 1$$

$$\text{FLUXUA} = \iint_S \vec{F}\,dS = \iiint_\Omega \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = 0$$

$$\iint_{S_1} \vec{F}\,dS + \iint_{S_2} \vec{F}\,dS = 0$$

$S_2 \rightarrow$ Zirkunf

$S_1 \rightarrow$ Parabo

$$\iint_S \vec{F}\,dS = \iint_S -F_1 g_x - F_2 g_y + F_3 \,dx\,dy$$

$$z = g(x,y) = \sqrt{1-x^2-y^2}+2$$

$$g_x = \frac{-x}{\sqrt{1-x^2-y^2}} \qquad g_y = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\iint_{S_1} \vec{F}\,dS = \iint_D \frac{xy}{\sqrt{1-x^2-y^2}} + \frac{yx}{\sqrt{1-x^2-y^2}} + \sqrt{1-x^2-y^2}+2 \,dx\,dy =$$

ALD-ALD: POLARRAK

$x = \rho\cos\theta \qquad \rho\in[0,1] \qquad J=\rho$

$y = \rho\sin\theta \qquad \theta\in[0,2\pi]$

$$= \int_0^{2\pi}\int_0^1 \cancel{\frac{\rho^3}{\sqrt{1-\rho^2}}\cos\theta\sin\theta} + \sqrt{1-\rho^2}\,\rho + 2\rho \,d\rho\,d\theta =$$

$$\int_0^1 2\pi \cdot \left(\sqrt{1-\rho^2}\,\rho + 2\rho\right) d\rho =$$

$$= \left[2\pi\left(\frac{-1}{3}(1-\rho^2)^{3/2} + \rho^2\right)\right]_0^1 =$$

$$= 2\pi \cdot \left(1 + \frac{1}{3}\right) = \boxed{\frac{8\pi}{3}}$$