11. Gaia

Integrazio konplexua eta Cauchyren teoremak

11.1 Aldagai errealeko funtzio konplexuak

Definizioa. Izan bitez $I \subset \mathbb{R}$ eta $u, v \colon I \to \mathbb{R}$ aldagai errealeko funtzio errealak.

$$\begin{aligned} h \colon I &\to \mathbb{C} \\ t &\rightsquigarrow h(t) = u(t) + iv(t). \end{aligned}$$

aldagai errealeko funtzio konplexua da.

h-ren deribatua eta integrala definitzeko, u eta v-renak erabiltzen dira:

$$\begin{aligned} h'(t) &= u'(t) + iv'(t), \\ \int_a^b h(t)\,dt &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt. \end{aligned}$$

Horretarako, u eta v deribgarriak edo integragarriak izatea beharko dugu. Zenbait propietate aldagai errealeko funtzioen propietateetatik atera daitezke. Batzuetan, hala ere, konplexuak erabili behar dira.

Teorema 11.1 (Katearen erregela). *$f \colon D \subset \mathbb{C} \to \mathbb{C}$ eta $h \colon (a, b) \to \mathbb{C}$ funtzio deribagarriak badira eta $h(a, b) \subset D$ betetzen bada, $f \circ h$ deribagarria da eta*

$$\frac{d}{dt} f(h(t)) = f'(h(t))h'(t).$$

Deribatu konplexua.

Proposizioa 11.2. *Izan bedi $h \colon (a, b) \to \mathbb{C}$ jarraitua. Orduan*

$$\left| \int_a^b h(t)\,dt \right| \leq \int_a^b |h(t)|\,dt.$$

11.2 Kurbak plano konplexuan

Definizioa. Izan bitez $a, b \in \mathbb{R}$, $a < b$. Esaten dugu $\gamma\colon [a,b] \to \mathbb{C}$ aplikazio jarraitua *kurba parametrizatua* dela.

(Como LOS XVI)

- $\gamma(a)$ kurbaren *hasierako puntua* eta $\gamma(b)$ *amaierako puntua* dira.
- γ injektiboa bada, esaten dugu *kurba sinplea* dela. (kurbak bere burua ebakitzen badu, ez da sinplea izango)
- $\gamma(a) = \gamma(b)$ bada, esaten da *kurba itxia* dela.
- γ itxia bada eta $[a, b)$ tartean injektiboa, orduan, esango dugu *kurba itxi sinplea* edo *Jordanen kurba* dela.

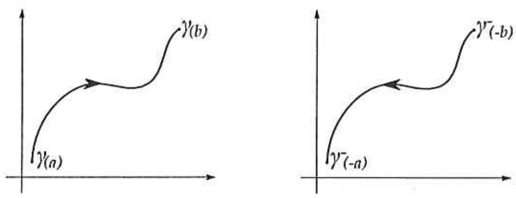
Definizioa. Izan bitez $I = [a,b]$, $J = [\alpha, \beta]$ eta $h\colon J \to I$ funtzio jarraitu bijektiboa. Izan bitez $\gamma\colon [a,b] \to \mathbb{C}$ eta $\tilde{\gamma} = \gamma \circ h$. Esaten da $\tilde{\gamma}$ γ-ren *birparametrizazioa* dela.

h gorakorra baldin bada, $\tilde{\gamma}(\alpha) = \gamma(a)$ eta $\tilde{\gamma}(\beta) = \gamma(b)$ dira eta esaten dugu birparametrizazioak orientazioa mantentzen duela. Aldiz, h beherakorra denean, $\tilde{\gamma}(\alpha) = \gamma(b)$ eta $\tilde{\gamma}(\beta) = \gamma(a)$ dira eta esaten da birparametrizazioak orientazioa aldatzen duela.

Definizioa. γ kurba itxi sinplea bada, γ-k orientazio positiboa duela diogu baldin eta erlojuaren orratzen kontrako orientazioa badu.

Definizioa. Izan bedi $\gamma\colon [a,b] \to \mathbb{C}$ kurba. γ-ren aurkako kurba hurrengoa da:

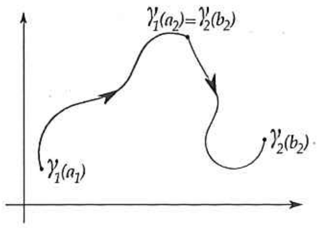
$$\begin{aligned} \gamma^{-}\colon [-b, -a] &\to \mathbb{C} \\ t &\to \gamma(-t). \end{aligned}$$



γ^{-} orientazioa aldatzen duen γ-ren birparametrizazioa da. Muturrak trukatzen dira: γ-ren hasierako puntua γ^{-}-en amaierakoa da eta alderantziz.

Definizioa. Izan bitez $\gamma_1\colon [a_1, b_1] \to \mathbb{C}$, $\gamma_2\colon [a_2, b_2] \to \mathbb{C}$ bi kurba eta demagun $\gamma_1(b_1) = \gamma_2(a_2)$ dela. Orduan

$$\begin{aligned} \gamma_1 + \gamma_2\colon [a_1, b_1 + b_2 - a_2] &\to \mathbb{C} \\ t &\to \begin{cases} \gamma_1(t), & t \in [a_1, b_1] \\ \gamma_2(t - b_1 + a_2), & t \in [b_1, b_1 + b_2 - a_2] \end{cases} \end{aligned}$$



Definizioa. Izan bedi $\gamma\colon [a,b] \to \mathbb{C}$ kurba parametrizatua. γ C^1 klasekoa bada, esaten dugu *kurba leuna* dela. Halaber, γ C^1 zatika bada, esaten dugu *kurba zatika leuna* dela. Hau da, γ zatika leuna da jarraitua bada eta $a = t_0 < t_1 < \cdots < t_n = b$ existitzen badira non γ deribagarria den (t_i, t_{i+1}) tarteetan*, $i = 0, \ldots, n-1$, deribatua jarraitua izanik. Kasu horretan, esaten da ere γ bidea dela.

*t_i puntuetan ez da zertan C^1 klasekoa izan, baina puntu kopuru horrek FINITUA izan behar du.

Adibideak.

(i) Izan bitez $a, b \in \mathbb{C}$. a eta b puntuak lotzen dituen zuzenkia hurrengoa da:

↳ Lehen mailako polinomioa:

$$\begin{aligned} \gamma\colon [0,1] &\to C \\ t &\to (1-t)a + tb. \end{aligned}$$

$\gamma = u + tv$

$\gamma(0) = a \longrightarrow u = a$

$\gamma(1) = b \longrightarrow v = b - a$

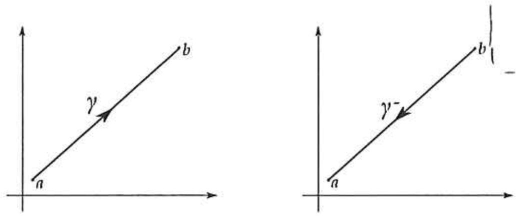
Antzera, b eta a puntuak lotzen dituen zuzenkia;

$$\begin{aligned} \gamma^-\colon [0,1] &\to C \\ t &\to ta + (1-t)b. \end{aligned}$$

Definizioa erabiliz:

$\gamma^-\colon [-1,0] \longrightarrow \mathbb{C}$

$t \mapsto (1+t)a - tb$



(ii) Izan bitez $z_0 \in \mathbb{C}$, $R > 0$. z_0 zentroko eta R erradiodun zirkunferentzia, orientazio positiboarekin, honela parametrizatzen da:

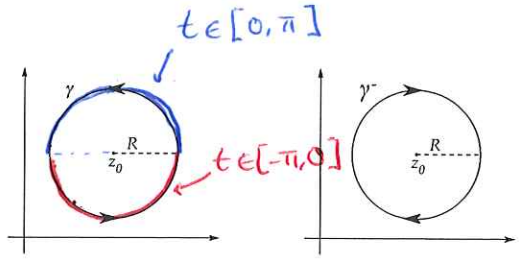
↳ $\Omega = \{z \in \mathbb{C} : |z - z_0| = R\}$

$z \in \mathbb{C} \to z - z_0 = R e^{it}$

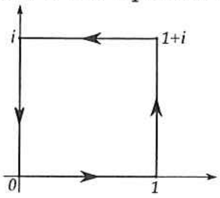
$$\begin{aligned} \gamma\colon [-\pi,\pi] &\to \mathbb{C} \\ t &\to z_0 + Re^{it}. \end{aligned}$$

Zirkunferentzia bera baina kontrako noranzkoan, zuzenean definiziotik.

$$\begin{aligned} \gamma^-\colon [-\pi,\pi] &\to \mathbb{C} \\ t &\to z_0 + Re^{-it}. \end{aligned}$$



(iii) Erpinak 0, 1, $1+i$ eta i puntuetan dituen karratua, orientazio positiboarekin:



$$\gamma(t) = \begin{cases} t, & t \in [0,1], \\ 1+(t-1)i, & t \in [1,2], \\ (3-t)+i, & t \in [2,3], \\ (4-t)i, & t \in [3,4]. \end{cases}$$

Edo, $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ moduan deskonposatuz, non

$$\begin{aligned} \gamma_1(t) &= t, && t \in [0,1], \\ \gamma_2(t) &= 1 + ti, && t \in [0,1], \\ \gamma_3(t) &= (1-t)+i, && t \in [0,1], \\ \gamma_4(t) &= (1-t)i, && t \in [0,1]. \end{aligned}$$

Edo, $\gamma = \gamma_1 + \gamma_2 - \tilde{\gamma_3} - \tilde{\gamma_4}$ moduan deskonposatuz, non γ_1 eta γ_2 goiko kurbak diren eta

$$\begin{aligned} \tilde{\gamma_3}(t) &= t + i, && t \in [0,1], \\ \tilde{\gamma_4}(t) &= ti, && t \in [0,1]. \end{aligned}$$

11.3 Aldagai konplexuko funtzioen integrazioa

Definizioa. Izan bitez $\gamma\colon [a,b] \to \mathbb{C}$ bidea eta f γ-ren ingurune batean definitutako funtzio jarraitua. Orduan, f-ren γ bidearen gaineko integrala honela definitzen da:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

$f(z) = u(x,y) + iv(x,y)$ baldin bada, eta $\gamma(t) = x(t) + iy(t)$, orduan

$$\begin{aligned} \int_\gamma f(z)\,dz &= \int_a^b \big(u(x(t),y(t)) + iv(x(t),y(t))\big)\big(x'(t) + iy'(t)\big)\,dt \\ &= \int_a^b \big(u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\big)\,dt \\ &\quad + i\int_a^b \big(u(x(t),y(t))y'(t) + v(x(t),y(t))x'(t)\big)\,dt. \end{aligned}$$

Oharra. Izan bitez (P, Q) planoko bektore-eremu jarraitua eta $\gamma(t) = (x(t), y(t))$, $a \leq t \leq b$, kurba zatika leuna. Honela definitzen da (P, Q)-ren integrala γ-ren gainean:

$$\int_\gamma Pdx + Qdy = \int_a^b [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)]\, dt.$$

Horren arabera,

$$\boxed{\int_\gamma f(z)\, dz = \int_\gamma (udx - vdy) + i \int_\gamma (udy + vdx).}$$

Teorema 11.3. *Izan bitez γ bidea, $\tilde{\gamma}$ bere birparametrizazioa eta f γ-ren ingurune batean jarraitua. Orduan*

(i) $\tilde{\gamma}$-k orientazioa mantentzen baldin badu, $\displaystyle\int_{\tilde{\gamma}} f(z)\, dz = \int_\gamma f(z)\, dz.$

(ii) $\tilde{\gamma}$-k orientazioa aldatzen baldin badu, $\displaystyle\int_{\tilde{\gamma}} f(z)\, dz = -\int_\gamma f(z)\, dz.$

Froga. Izan bedi $h\colon [\alpha, \beta] \to [a, b]$, non $\tilde{\gamma} = \gamma \circ h$ den. Definizioaren arabera,

$$\int_{\tilde{\gamma}} f(z)\, dz = \int_\alpha^\beta f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\, dt = \int_\alpha^\beta f(\gamma(h(t)))\gamma'(h(t))h'(t)\, dt.$$

Egin dezagun $\xi = h(t)$ aldagai-aldaketa azken integralean. Orduan,

$$\int_{\tilde{\gamma}} f(z)\, dz = \int_{h(\alpha)}^{h(\beta)} f(\gamma(\xi))\gamma'(\xi)\, d\xi.$$

$\tilde{\gamma}$-k orientazioa mantentzen badu, $h(\alpha) = a$ eta $h(\beta) = b$, beraz

$$\int_{\tilde{\gamma}} f(z)\, dz = \int_a^b f(\gamma(\xi))\gamma'(\xi)\, d\xi = \int_\gamma f(z)\, dz.$$

Aldiz, $\tilde{\gamma}$-k orientazioa aldatzen badu, $h(\alpha) = b$ eta $h(\beta) = a$, eta ondorioz

$$\int_{\tilde{\gamma}} f(z)\, dz = \int_b^a f(\gamma(\xi))\gamma'(\xi)\, d\xi = -\int_a^b f(\gamma(\xi))\gamma'(\xi)\, d\xi = -\int_\gamma f(z)\, dz. \qquad \square$$

Definizioa. Izan bitez $\gamma\colon [a, b] \to \mathbb{C}$ bidea eta f γ-ren ingurune batean definitutako funtzio jarraitua.

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|\, dt.$$

Bereziki,

$$l(\gamma) = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

γ kurbaren luzera da.

Proposizioa 11.4 (Integral konplexuaren propietateak). *Izan bitez γ, γ_1, γ_2 bideak, f eta g kurben ingurune batean definitutako funtzio jarraituak eta $\alpha, \beta \in \mathbb{C}$.*

(i) $\displaystyle\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$

(ii) $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.$

(iii) $\displaystyle\int_{\gamma^-} f(z)\, dz = -\int_\gamma f(z)\, dz.$

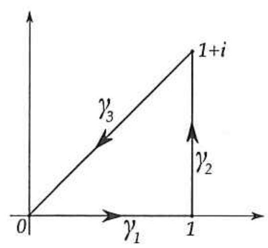
(iv) $\displaystyle\left|\int_\gamma f(z)\, dz\right| \le \int_\gamma |f(z)|\, |dz|.$

(v) $\displaystyle\left|\int_\gamma f(z)\, dz\right| \le \max_{z\in\gamma} |f(z)|\, l(\gamma).$

Adibideak.

(i) $\displaystyle\int_\gamma z^2\, dz$ non γ erpinak 0, 1 eta $1+i$ puntuetan dituen triangelua den, erlojuaren orratzen kontrako noranzkoan. γ hiru zatitan banatuko dugu, $\gamma = \gamma_1 + \gamma_2 - \gamma_3$, non

$$\begin{aligned}
\gamma_1 \colon [0,1] &\to \mathbb{C} \\
t &\to t \\
\gamma_2 \colon [0,1] &\to \mathbb{C} \\
t &\to 1+ti \\
\gamma_3 \colon [0,1] &\to \mathbb{C} \\
t &\to (1+i)t.
\end{aligned}$$



Orduan,

$$\begin{aligned}
\int_\gamma z^2\, dz &= \int_{\gamma_1} z^2\, dz + \int_{\gamma_2} z^2\, dz - \int_{\gamma_3} z^2\, dz \\
&= \int_0^1 t^2\, dt + \int_0^1 (1+ti)^2\, i\, dt - \int_0^1 ((1+i)t)^2(1+i)\, dt \\
&= \int_0^1 t^2\, dt + i\int_0^1 (1 - t^2 + 2it)\, dt - (1+i)^3 \int_0^1 t^2\, dt \\
&= (1 - i - 1 - 3i - 3i^2 - i^3) \int_0^1 t^2\, dt + i\int_0^1 dt - 2\int_0^1 t\, dt \\
&= (3-3i)\frac{t^3}{3}\Big|_0^1 + i - t^2\Big|_0^1 \\
&= 1 - i + i - 1 = 0.
\end{aligned}$$

(ii) $\displaystyle\int_{|z|=R} z\,dz$, non $|z| = R$ jatorrian zentratutako R erradioko zirkunferentzia den, erlojuaren orratzen kontrako norazkoan hartuta. $\gamma(t) = Re^{it}$, $t \in [-\pi, \pi]$ moduan parametrizatuko dugu. Orduan,

$$\begin{aligned}\int_{|z|=R} z\,dz &= \int_{-\pi}^{\pi} Re^{it} Rie^{it}\,dt = iR^2 \int_{-\pi}^{\pi} (\cos(2t) + i\sin(2t))\,dt \\ &= iR^2 \left(\frac{\sin 2t}{2} - i\frac{\cos 2t}{2}\right)\Bigg|_{-\pi}^{\pi} = 0.\end{aligned}$$

(iii) $\displaystyle\int_{|z|=R} \overline{z}\,dz = \int_{-\pi}^{\pi} Re^{-it} Rie^{it}\,dt = iR^2 \int_{-\pi}^{\pi} dt = 2\pi R^2 i.$

(iv) $\displaystyle\int_{|z|=R} \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{it}} Rie^{it}\,dt = i\int_{-\pi}^{\pi} dt = 2\pi i.$

(v) $\displaystyle\int_{|z|=R} \sqrt{z}\,dz$, non $\sqrt{z}$ $\mathbb{C} \setminus (-\infty, 0]$ multzoan definitutako erro karratuaren adar nagusia den.

$$\begin{aligned}\int_{|z|=R} \sqrt{z}\,dz &= \int_{-\pi}^{\pi} \sqrt{R}e^{i\frac{t}{2}} Rie^{it}\,dt = iR\sqrt{R}\int_{-\pi}^{\pi} e^{\frac{3t}{2}i}\,dt \\ &= iR\sqrt{R}\,\frac{2}{3i}e^{\frac{3t}{2}i}\Bigg|_{-\pi}^{\pi} = \frac{2R\sqrt{R}}{3}\left(e^{\frac{3\pi}{2}i} - e^{-\frac{3\pi}{2}i}\right) = -\frac{4R\sqrt{R}}{3}i.\end{aligned}$$

(vi) $\displaystyle\int_{|z|=R} \sqrt{z}\,dz$, non $\sqrt{z}$ $\mathbb{C} \setminus [0, +\infty)$ multzoan definitutako adar nagusia den, hots, $\sqrt{-1} = i$. Hemen, parametrizazioaren definizio-tartea $\sqrt{z}$-ren definizio-eremura egokitu behar dugu, $t \in [0, 2\pi]$. Beraz,

$$\begin{aligned}\int_{|z|=R} \sqrt{z}\,dz &= \int_{0}^{2\pi} \sqrt{R}e^{i\frac{t}{2}} Rie^{it}\,dt = iR\sqrt{R}\int_{0}^{2\pi} e^{\frac{3t}{2}i}\,dt \\ &= iR\sqrt{R}\,\frac{2}{3i}e^{\frac{3t}{2}i}\Bigg|_{0}^{2\pi} = \frac{2R\sqrt{R}}{3}\left(e^{3\pi i} - e^{0i}\right) = -\frac{4R\sqrt{R}}{3}.\end{aligned}$$

11.4 Kalkulu integralaren oinarrizko teorema

Teorema 11.5 (Kalkulu integralaren oinarrizko teorema). *Izan bitez $\gamma\colon [a, b] \to \mathbb{C}$ bidea eta f jarraitua γ-ren irudiaren puntuetan. Demagun F holomorfoa existitzen dela non $F' = f$ den. Orduan*

$$\int_{\gamma} f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

$F' = f$ bada, esaten dugu F f-ren jatorrizkoa dela.

Froga. Izan bedi $g(t) = F(\gamma(t))$. Katearen erregelaren arabera, $g'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$. Orduan

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b g'(t)\,dt = g(b) - g(a) = F(\gamma(b)) - F(\gamma(a)). \ \square$$

Adibidea. $\displaystyle\int_{|z|=1} z^n\,dz$ kalkulatuko dugu, $n \in \mathbb{Z}$ guztietarako.

- $n \geq 0$ bada, $f(z) = z^n$ funtzioaren jatorrizkoa $F(z) = \dfrac{z^{n+1}}{n+1}$ funtzioa da $\mathbb{C}$ osoan. Beraz,
$$\int_{|z|=1} z^n\,dz = 0, \quad \forall n \geq 0.$$
- $n \leq -2$ bada, $f(z) = \dfrac{1}{z^{-n}}$ ez da jarraitua $z = 0$ puntuan, eta jatorrizkoa ere ez da definituta egongo puntu horretan, baina $F(z) = \dfrac{1}{(1+n)z^{-n-1}}$ f-ren jatorrizkoa da $\mathbb{C} - \{0\}$ multzoan. Ondorioz,
$$\int_{|z|=1} z^n\,dz = 0, \quad \forall n \leq -2.$$
- Azkenik, $n = -1$ bada, $\dfrac{1}{z}$ funtzioaren jatorrizkoa logaritmoaren edozein adar da, baina logaritmoaren adarrak holomorfoak izan daitezen, jatorritik infinitura doan kurba bat kendu behar da $\mathbb{C}$-n, eta kurba horrek $|z| = 1$ zirkunferentzia ebakitzen du; beraz, ezin da aurkitu $\dfrac{1}{z}$ funtzioaren jatorrizko holomorforik $|z| = 1$ kurbaren ingurune batean.

  Ikusi dugun bezala, $\displaystyle\int_{|z|=1} \frac{dz}{z} = 2\pi i \neq 0.$

Teorema 11.6. *Izan bitez $\Omega \subset \mathbb{C}$ multzo irekia eta f jarraitua Ω-n. Baliokideak dira:*

(i) f-k jatorrizko funtzioa du Ω-n.

(ii) f-ren integrala Ω-ko kurba itxietan 0 da.

(iii) Ω-ko γ_1 eta γ_2 kurben hasierako puntuak eta amaierako puntuak berdinak badira, $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ da.

Froga. (i) $\Rightarrow$ (ii): aurreko teoremaren ondorioa da.

(ii) $\Rightarrow$ (iii): $\gamma_1 + \gamma_2^-$ kurba itxia da. Orduan,

$$0 = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2^-} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz.$$

(iii) $\Rightarrow$ (i): izan bedi $a \in \Omega$ eta $\gamma(a,z)$, a puntuan hasi eta z puntuan amaitzen den bide bat Ω-n. Defini dezagun

$$F(z) = \int_{\gamma(a,z)} f(w)\,dw.$$

Ondo definituta dago (iii)-ren arabera.

Frogatuko dugu $F'(z_0) = f(z_0)$ dela $z_0 \in \Omega$ guztietarako, hau da,

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = 0.$$

Izan bedi $\epsilon > 0$ edozein. Ω irekia eta f jarraitua direnez, existitzen da $\delta > 0$ non, $|z - z_0| < \delta$ bada, $z \in \Omega$ eta $|f(z) - f(z_0)| < \epsilon$ diren. Gainera, (iii)-ren arabera, z horietarako,

$$\int_{\gamma(a,z)} f(w)\,dw = \int_{\gamma(a,z_0)} f(w)\,dw + \int_{[z_0,z]} f(w)\,dw,$$

non $[z_0, z]$ notazioak z_0-tik z-ra doan zuzenkia adierazten duen (zuzenkia Ω-ren parte da). Beraz,

$$F(z) = F(z_0) + \int_0^1 f(z_0 + t(z - z_0))(z - z_0)\,dt.$$

(Zuzenkia $z_0 + t(z - z_0)$, $0 \le t \le 1$, parametrizatu dugu.) Orduan,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \int_0^1 [f(z_0 + t(z - z_0)) - f(z_0)]\,dt.$$

Baldin $|z - z_0| < \delta$ bada, $w \in [z_0, z]$ guztietarako $|w - z_0| < \delta$ beteko da eta, hortaz, $|f(w) - f(z_0)| < \epsilon$ izango da. Ondorioz,

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \le \int_0^1 |f(z_0 + t(z - z_0)) - f(z_0)|\,dt < \epsilon. \qquad \square$$

11.5 Cauchyren teorema integrala

Teorema 11.7 (Cauchyren teorema). *Izan bitez $\Omega \subset \mathbb{C}$ non Greenen teorema aplika daitekeen, f Ω-ren ingurune batean holomorfoa, f' jarraitua izanik eta $\gamma = \partial\Omega$ Ω-ren muga. Orduan*

$$\int_\gamma f(z)\,dz = 0.$$

Froga. Demagun γ-k orientazio positiboa duela. Ikusi dugun bezala,

$$\int_\gamma f(z)\,dz = \int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx).$$

f' jarraitua denez, $u, v \in C^1$ eta aplika daiteke Greenen teorema bi integral hauetan, beraz

$$\int_\gamma f(z)\,dz = \iint_\Omega \Big(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\,dxdy + i\iint_\Omega \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big)\,dxdy.$$

f holomorfoa denez, Cauchy-Riemannen baldintzak betetzen ditu, hau da, $u_x = v_y$ eta $u_y = -v_x$, goiko bi integral bikoitzak anulatzen direlarik, eta ondorioz,

$$\int_\gamma f(z)\,dz = 0.$$

□

Teorema 11.8 (Cauchy-Goursaten teorema). *Izan bitez $\Omega \subset \mathbb{C}$ eremu sinpleki konexua, f holomorfoa Ω-n eta γ kurba itxi sinple zatika leuna Ω-n. Orduan,*

$$\int_\gamma f(z)\,dz = 0.$$

Korolarioa 11.9. *Izan bitez $\Omega \subset \mathbb{C}$ multzo irekia eta sinpleki konexua eta f holomorfoa Ω-n. Orduan*

(i) *γ_1 eta γ_2 kurben hasierako puntuak eta amaierako puntuak berdinak badira, $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ da.*

(ii) *f-k jatorrizko funtzioa du.*

Froga. (i) $\gamma = \gamma_1 + \gamma_2^-$ kurba itxia da, beraz, Cauchy-Goursaten teoremaren arabera, $\displaystyle\int_\gamma f(z)\,dz = 0$. Hots,

$$0 = \int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2^-} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$$

eta, ondorioz, $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

(ii) Izan bedi $a \in \Omega$, non $[a, z]$ a eta z puntuak lotzen dituen zuzenkia Ω-ren barrualdean geratzen den. $F(z) = \int_a^z f(z)\,dz = \int_{[a,z]} f(z\,dz$ bada, frogatuko dugu $f'(z) = f(z)$ dela $z \in \Omega$ guztietarako.

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\Big(\int_z^{z_0} f(w)\,dw - \int_z^{z_0} f(z_0)\,dw.$$

Izan bedi $\epsilon > 0$ edozein. f holomorfoa denez, bereziki jarraitua da z_0-n eta ondorioz, existitzen da $\delta > 0$ non $|z - z_0| < \delta$ denean $|f(z) - f(z_0)| < \epsilon$ den. Orduan, $|z - z_0| < \delta$ bada, $w \in [z, z_0]$ guztietarako, $|w - z_0| < \delta$ ere eta beraz

$$\Big|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\Big| = \frac{1}{|z - z_0|}\int_z^{z_0} |f(w) - f(z_0)|\,|dz| < \frac{1}{|z - z_0|}\epsilon|z - z_0| = \epsilon,$$

hau da,

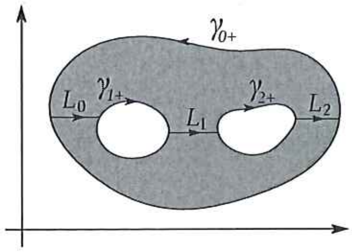
$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0). \qquad \square.$$

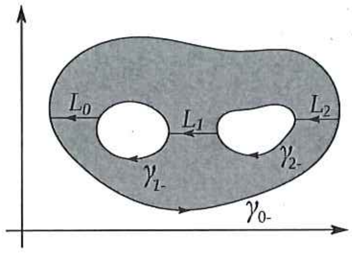
Teorema 11.10 (Cauchyren teoremaren forma orokortua). *Izan bitez $\gamma_0, \gamma_1, \ldots, \gamma_n$ bide itxi sinpleak eta Ω_i γ_i kurbak mugatzen duen eremua, $i = 0, \ldots, n$ guztietarako. Demagun $\Omega_i \subset \Omega_0$ dela $i = 1, \ldots, n$ guztietarako eta $\Omega_i \cap \Omega_j = \emptyset$ dela $i \neq j$ bada. Izan bedi $\Omega = \Omega_0 - \bigcup_{i=1}^{n} \Omega_i$ eta aukera dezagun γ_i kurban orientazioa Ω ezkerraldean gera dadin. f funtzio holomorfoa bada Ω-ren ingurune batean, orduan*

$$\sum_{i=0}^{n} \int_{\gamma_i} f(z)\, dz = 0.$$

Froga. Demagun $n = 2$ dela, eta defini ditzagun hurrengo bideak:

$$\begin{aligned} \alpha =& \gamma_{0+} + L_0 + \gamma_{1+} + L_1 + \gamma_{2+} + L_2, \\ \beta =& \gamma_{0-} - L_2 + \gamma_{2-} - L_1 + \gamma_{1-} - L_0. \end{aligned}$$





α eta β multzo sinpleki konexuen barruan geratzen dira, beraz, f holomorfoa denez, Cauchy-Goursat-en teoremaren arábera,

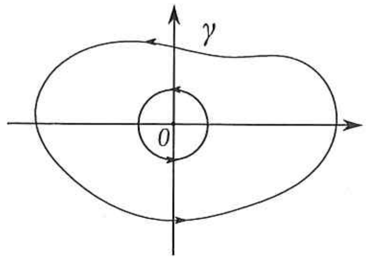
$$\int_{\alpha} f(z)\, dz = \int_{\beta} f(z)\, dz = 0.$$

Orduan,

$$\begin{aligned} 0 &= \int_{\alpha} f(z)\, dz + \int_{\beta} f(z)\, dz \\ &= \int_{\gamma_{0+}} f(z)dz + \int_{L_0} f(z)dz + \int_{\gamma_{1+}} f(z)dz + \int_{L_1} f(z)dz + \int_{\gamma_{2+}} f(z)dz + \int_{L_2} f(z)dz \\ &\quad + \int_{\gamma_{0-}} f(z)dz - \int_{L_2} f(z)dz + \int_{\gamma_{2-}} f(z)dz - \int_{L_1} f(z)dz + \int_{\gamma_{1-}} f(z)dz - \int_{L_0} f(z)dz \\ &= \int_{\gamma_0} f(z)\, dz + \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz. \end{aligned}$$

Beraz, $\int_{\gamma_0} f(z)\, dz + \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz = 0.$ $\square$

Adibidea. Izan bedi γ jatorria barneratzen duen edozein kurba. Orduan, $|z| = R$ zirkunferentzia γ-k mugatzen duen eremuaren barruan geratzen bada,



$$\int_\gamma \frac{dz}{z} = \int_{|z|=R} \frac{dz}{z} = 2\pi i.$$

Oharra. Aurreko teorema honela ere ulertu daiteke: $\partial\Omega$, Ω-ren muga, kurba konposatutzat hartzen da, γ_j, $0 \leq j \leq k$, kurba guztiez osatua. $\partial\Omega$-ren gainean noranzko bat definitzeko, Ω multzoa uzten da ezker aldean. Orduan, f-ren integrala $\partial\Omega$-ren gainean 0 da.

11.6 Cauchyren formula integrala

Teorema 11.11 (Cauchyren formula integrala). *Izan bitez γ bide itxi sinplea, erlojuaren orratzen kontrako noranzkoarekin hartuta, $\Omega \subset \mathbb{C}$ γ-ren barrualdea, f holomorfoa $\overline{\Omega}$-n eta $z_0 \in \Omega$. Orduan,*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Froga. Baldin $C_r = \{z : |z - z_0| = r\}$ zirkunferentzia Ω-n badago, Cauchyren teoremaren forma orokortuagatik,

$$\begin{aligned}
\int_\gamma \frac{f(z)}{z - z_0} dz &= \int_{C_r} \frac{f(z)}{z - z_0} dz \\
&= \int_{C_r} \frac{f(z_0) + f(z) - f(z_0)}{z - z_0} dz \\
&= f(z_0) \int_{C_r} \frac{dz}{z - z_0} + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz.
\end{aligned}$$

C_r, $\gamma(t) = z_0 + re^{it}$, $t \in [-\pi, \pi]$ aplikazioaren bidez parametrizatuz,

$$\int_{C_r} \frac{dz}{z - z_0} = \int_{-\pi}^{\pi} \frac{rie^{it}}{re^{it}} dt = 2\pi i.$$

Bestalde, f jarraitua denez z_0-n, $\lim_{z \to z_0} f(z) = f(z_0)$, hau da, edozein $\epsilon > 0$ emanda, $\delta > 0$ existitzen da non, $|z - z_0| < \delta$ bada, $|f(z) - f(z_0)| < \epsilon$ den. Har dezagun $r < \delta$. Orduan,

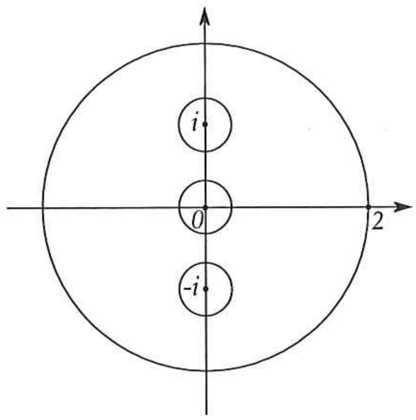
$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{\epsilon\, l(C_r)}{r} = 2\pi\epsilon,$$

hau da, $\lim_{r\to 0}\int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz=0$ eta ondorioz, $\int_\gamma \frac{f(z)}{z-z_0}\,dz=2\pi i f(z_0)$. □

Adibidea. Kalkula dezagun $\int_\gamma \frac{\cos z}{z^3+z}\,dz$, γ desberdinetarako.

- $|z|=2$ zirkunferentzia.

  $\frac{\cos z}{z^3+z}$ funtzioa ez da holomorfoa 0, i eta $-i$ puntuetan. Bere integrala γ-n zehar kalkulatzeko erabil dezakegu Cauchyren teorema eremua sinpleki konexua ez denean. Orduan, $r<1/2$ hartuz,



$$
\begin{aligned}
\int_{|z|=2}\frac{\cos z}{z^3+z}\,dz &= \int_{|z-i|=r}\frac{\cos z}{z(z-i)(z+i)}\,dz+\int_{|z|=r}\frac{\cos z}{z(z-i)(z+i)}\,dz\\
&+\int_{|z+i|=r}\frac{\cos z}{z(z-i)(z+i)}\,dz\\
&=2\pi i\left.\frac{\cos z}{z(z+i)}\right|_{z=i}+2\pi i\left.\frac{\cos z}{(z-i)(z+i)}\right|_{z=0}+2\pi i\left.\frac{\cos z}{z(z-i)}\right|_{z=-i}\\
&=2\pi i\frac{\cos i}{i2i}+2\pi i\frac{\cos 0}{(-i)i}+2\pi i\frac{\cos(-i)}{(-i)(-2i)}\\
&=-\pi i\frac{e^{-1}+e}{2}+2\pi i-\pi i\frac{e+e^{-1}}{2}\\
&=2\pi i(1-\cosh 1).
\end{aligned}
$$

Beste aukera bat da $\frac{1}{z^3+z}$ frakzio sinpleetan deskonposatzea, honela,

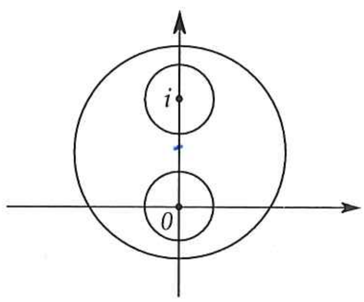
$$\frac{\cos z}{z^3+z}=\frac{\cos z}{z}-\frac{1}{2}\frac{\cos z}{z+i}-\frac{1}{2}\frac{\cos z}{z-i},$$

eta ondorioz,

$$\begin{aligned}\int_{|z|=2} \frac{\cos z}{z^3+z}\,dz &= \int_{|z|=2} \frac{\cos z}{z}\,dz - \frac{1}{2}\int_{|z|=2} \frac{\cos z}{z+i}\,dz - \frac{1}{2}\int_{|z|=2} \frac{\cos z}{z-i}\,dz \\ &= 2\pi i \cos 0 - \frac{1}{2} 2\pi i \cos(-i) - \frac{1}{2} 2\pi i \cos i \\ &= 2\pi i (1 - \cosh 1).\end{aligned}$$

- $\left|z - \dfrac{i}{2}\right| = 1$ zirkunferentzia. Lehen bezala, bi modutan egin daiteke.

  $r < 1/2$ hartuz,



$$\begin{aligned}\int_{|z-\frac{i}{2}|=r} \frac{\cos z}{z^3+z}\,dz &= \int_{|z-i|=r} \frac{\cos z}{z(z-i)(z+i)}\,dz + \int_{|z|=r} \frac{\cos z}{z(z-i)(z+i)}\,dz \\ &= 2\pi i \left.\frac{\cos z}{z(z+i)}\right|_{z=i} + 2\pi i \left.\frac{\cos z}{(z-i)(z+i)}\right|_{z=0} \\ &= 2\pi i \frac{\cos i}{i2i} + 2\pi i \frac{\cos 0}{(-i)i} \\ &= 2\pi i \Big(1 - \frac{\cosh 1}{2}\Big).\end{aligned}$$

  Edo, frakzio sinpleetako deskonposaketa erabiliz,

$$\begin{aligned}\int_{|z-\frac{i}{2}|=1} \frac{\cos z}{z^3+z}\,dz &= \int_{|z-\frac{i}{2}|=1} \frac{\cos z}{z}\,dz - \frac{1}{2}\int_{|z-\frac{i}{2}|=1} \frac{\cos z}{z+i}\,dz - \frac{1}{2}\int_{|z-\frac{i}{2}|=1} \frac{\cos z}{z-i}\,dz \\ &= 2\pi i \cos 0 - \frac{1}{2}0 - \frac{1}{2}2\pi i \cos i \\ &= 2\pi i \Big(1 - \frac{\cosh 1}{2}\Big).\end{aligned}$$

- $|z| = 1/2$ zirkunferentzia. Kasu honetan, etengune bakar bat geratzen da kurbaren barruan; beraz,

$$\int_{|z|=1/2} \frac{\cos z}{z^3+z}\,dz = \int_{|z|=1/2} \frac{\cos z}{z(z^2+1)}\,dz = 2\pi i \left.\frac{\cos z}{z^2+1}\right|_{z=0} = 2\pi i \cos 0 = 2\pi i.$$

Teorema 11.12 (Funtzio holomorfoen deribagarritasuna). *Izan bedi γ kurba itxi sinplea, erlojuaren orratzen kontrako noranzkoarekin hartuta, eta Ω, γ-ren barrualdea. f holomorfoa bada $\overline{\Omega}$-n, edozein ordenatako deribatuak ditu bertan. Gainera, $z_0 \in \Omega$ bada,*

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz.}$$

Froga. Indukzioz frogatzen da. $n = 0$ kasua Cauchyren formula integrala da. $f^{(n-1)}$-erako balio duela onartuta, $f^{(n)}$-rako balio duela ikusi behar da. Horretarako,

$$\lim_{h\to 0} \int_\gamma f(z) \left[\frac{1}{h} \left(\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n} \right) - \frac{n}{(z-z_0)^{n+1}} \right] dz = 0$$

ikusi behar da, zeren eta

$$f^{(n)}(z_0) = \lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h}$$

eta, indukzioaren hipotesia erabiliz,

$$\begin{aligned} f^{(n-1)}(z_0+h) &= \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0-h)^n}\, dz, \\ f^{(n-1)}(z_0) &= \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^n}\, dz. \end{aligned}$$

Ikus dezagun $n = 1$ kasua:

$$\begin{aligned} &\frac{1}{h}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0} \right) - \frac{1}{(z-z_0)^2} \\ &\quad = \frac{1}{h} \frac{(z-z_0)^2 - (z-z_0-h)(z-z_0) - h(z-z_0-h)}{(z-z_0-h)(z-z_0)^2} \\ &\quad = \frac{1}{h} \frac{(z-z_0)^2 - (z-z_0)^2 + h(z-z_0) - h(z-z_0) + h^2}{(z-z_0-h)(z-z_0)^2} \\ &\quad = \frac{h}{(z-z_0)^2(z-z_0-h)} \end{aligned}$$

da. Izan bitez d z_0-tik γ-rainoko distantzia eta $M = \max_{z\in\gamma} |f(z)|$. Orduan,

$$\begin{aligned} &\left| \int_\gamma f(z) \left[\frac{1}{h}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0} \right) - \frac{1}{(z-z_0)^2} \right] dz \right| \\ &\quad = \left| f(z) \frac{h}{(z-z_0)^2(z-z_0-h)}\, dz \right| \\ &\quad \le \int_\gamma |f(z)| \frac{|h|}{|z-z_0-h|\,|z-z_0|} |dz| \le \frac{M\, l(\gamma)\, |h|}{d^2(d-|h|)} \end{aligned}$$

dugu eta limitea 0 da. □

Adibidea. Kalkulatuko dugu $\int_\gamma \frac{cosz}{z^2(z-1)}\,dz$, γ kurba batzuetarako.

- $|z| = 2$ zirkunferentzia. Deskonposa dezakegu $1/(z^2(z-1))$ frakzio sinpleetan. Honela,
$$\frac{\cos z}{z^2(z-1)} = \frac{\cos z}{z-1} - \frac{\cos z}{z} - \frac{\cos z}{z^2},$$
beraz,
$$\begin{aligned}\int_{|z|=2} \frac{\cos z}{z^2(z-1)}\,dz &= \int_{|z|=2} \frac{\cos z}{z-1}\,dz - \int_{|z|=2} \frac{\cos z}{z}\,dz - \int_{|z|=2} \frac{\cos z}{z^2}\,dz\\ &=2\pi i\cos 1 - 2\pi i \cos 0 - \frac{2\pi i}{1!}(\cos z)'\big|_{z=0}\\ &=2\pi i(\cos 1 - \cos 0 + \sin 0) = 2\pi i(\cos 1 - 1).\end{aligned}$$
Edo, Cauchyren teorema eremua sinpleki konexua ez denean erabiliz, $r < 1/2$ hartuta,
$$\begin{aligned}\int_{|z|=2} \frac{\cos z}{z^2(z-1)}\,dz &= \int_{|z|=r} \frac{\cos z}{z^2(z-1)}\,dz + \int_{|z-1|=r} \frac{\cos z}{z^2(z-1)}\,dz\\ &=\frac{2\pi i}{1!}\Big(\frac{\cos z}{z-1}\Big)'\Big|_{z=0} + 2\pi i\frac{\cos z}{z^2}\Big|_{z=1}\\ &=2\pi i\Big(\frac{-\sin 0(0-1) - \cos 0}{(0-1)^2} + \frac{\cos 1}{1^2}\Big) = 2\pi i(\cos 1 - 1).\end{aligned}$$

- $|z| = 1/3$ zirkunferentzia. Kasu honetan $z_0 = 0$ da kurbaren barruan geratzen den etengune bakarra, beraz,
$$\int_{|z|=1/3} \frac{\cos z}{z^2(z-1)}\,dz = \frac{2\pi i}{1!}\Big(\frac{\cos z}{z-1}\Big)'\Big|_{z=0} = 2\pi i(-\cos 0) = -2\pi i.$$

- $|z-1| = 1/3$ zirkunferentzia. Berriro, etengune bakarra dugu kurbaren barruan, orain $z_0 = 1$. Orduan,
$$\int_{|z-1|=1/3} \frac{\cos z}{z^2(z-1)}\,dz = 2\pi i\frac{\cos z}{z^2}\Big|_{z=1} = 2\pi i\cos 1.$$

Teorema 11.13 (Moreraren teorema). *Izan bitez $\Omega \subset \mathbb{C}$ irekia eta sinpleki konexua eta $f\colon \Omega \to \mathbb{C}$ jarraitua. $\int_\gamma f(z)\,dz = 0$ bada γ bide itxi sinple guztietarako, orduan f holomorfoa da Ω-n.*

Froga. Cauchyren teoremaren korolario modura ikusi dugu baldintza horietan existitzen dela f-ren jatorrizko funtzioa, F, holomorfoa D-n. Orduan, $F' = f$ ere holomorfoa da. □

Teorema hori Cauchyren teorema integralaren alderantzizkoa da. Han esaten genuen funtzio holomorfoen integrala kurba itxietan 0 dela; hemen, integrala 0 bada, funtzioa holomorfoa dela.

Teorema 11.14 (Liouvilleren teorema). *$f\colon \mathbb{C} \to \mathbb{C}$ funtzio osoa eta bornatua baldin bada orduan konstantea da.*

Froga. f bornatua denez, existitzen da $M > 0$ non $|f(z)| \leq M$ den, $\forall z \in \mathbb{C}$. Orduan, $\forall z_0 \in \mathbb{C}$, eta $\forall R > 0$,

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\,dz\right| \leq \frac{M}{2\pi}\int_{|z-z_0|=R}\frac{1}{|z-z_0|^2}\,|dz| = \frac{2\pi RM}{2\pi R^2} = \frac{M}{R}.$$

Limiteak hartuz $R \to \infty$ denean, $|f'(z_0)| = 0$, $\forall z_0 \in \mathbb{C}$, beraz f konstantea da. □

Teorema 11.15 (Batezbestekoaren propietatea). *f holomorfoa bada $|z - z_0| \leq r$ zirkuluan,*

$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt.}$$

Hau da, f-ren batezbestekoa $|z - z_0| = r$ zirkunferentzian $f(z_0)$ da.

Froga. Cauchyren formula integralaren arabera, $C_r = \{z : |z - z_0| = r\}$ bada,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z - z_0}\,dz.$$

$\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$ parametrizazioa hartuz,

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0 + re^{it})}{re^{it}}\,rie^{it}dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt. \qquad \square$$

Teorema 11.16 (Modulu maximoaren printzipioa). *Izan bitez $\Omega \subset \mathbb{C}$ irekia eta konexua eta f Ω-n holomorfoa eta ez konstantea. Orduan, $|f(z)|$-k ezin du maximoa lortu Ω-n. Bereziki, f jarraitua bada $\Omega \cup \partial\Omega$ multzoan, orduan, $|f|$-k bere maximoa $\partial\Omega$-n lortzen du.*

Teorema 11.17 (Aljebraren oinarrizko teorema). *Koefiziente konplexuetako polinomio ez konstante orok, gutxienez, erro bat dauka plano konplexuan.*

Froga. Izan bedi P polinomio ez-konstantea. Baldin $P(z) \neq 0$ bada, z guztietarako, $1/P$ osoa da. Gainera, $\lim_{z\to\infty} 1/P(z) = 0$ denez, $1/P$ bornatua da. Liouvilleren teorema erabiliz, $1/P$ konstantea da eta, beraz, P ere bai. Baina esan dugunez P ez dela konstantea, kontraesan batera heldu gara eta $P(z)$ ezin da ez-nulua izan puntu guztietan, existitu behar da z_0 non $P(z_0) = 0$ den. □

Denak !

ANALISI BEKTORIALA ETA KONPLEXUA

11. Gaia: INTEGRAZIO KONPLEXUA ETA CAUCHYREN TEOREMAK

Ariketak

1. Kalkula itzazu hurrengo integralak

$$I_1 = \int x\, dz \qquad I_2 = \int y\, dz \qquad I_3 = \int \overline{z}\, dz$$

ondorengo bideetan,

(i) 0 eta $1-i$ puntuak lotzen dituen segmentua *Em.:* $I_1 = \frac{1-i}{2}$; $I_2 = \frac{-1+i}{2}$; $I_3 = 1$.

(ii) $|z| = 1$ zirkunferentzia *Em.:* $I_1 = \pi i$; $I_2 = -\pi$; $I_3 = 2\pi i$.

(iii) $|z-a| = R$ zirkunferentzia, $a \in \mathbb{C}$, $R > 0$ *Em.:* $I_1 = \pi R^2 i$; $I_2 = -R^2\pi$; $I_3 = 2\pi R^2 i$.

2. Izan bedi C unitate zirkunferentziaren goiko erdia, 1-etik -1-eraino. Kalkulatu

(i) $\displaystyle\int_C (z^2 + z\overline{z})\, dz$ *Em.:* $-8/3$

(ii) $\displaystyle\int_C z\,\mathrm{Im}(z^2)\, dz$ *Em.:* $-\pi/2$

3. Izan bedi $D = \{z : 1 < |z| < 2, \mathrm{Im}\, z > 0\}$ eta C, D-ren muga noranzko positiboan. Kalkulatu integral hauek:

(i) $\displaystyle\int_C |z|\overline{z}\, dz$ *Em.:* $7\pi i$

(ii) $\displaystyle\int_C \frac{z}{\overline{z}}\, dz$ *Em.:* $4/3$

(iii) $\displaystyle\int_C |z|\, dz$ *Em.:* -3

4. Kalkulatu integral hauek:

(i) $\displaystyle\int_{1-i}^{2+i} (3z^2 + 2z)\, dz$ *Em.:* $7 + 19i$.

(ii) $\int_i^{i/2} e^{\pi z}\, dz$ *Em.:* $\frac{1+i}{\pi}$.

(iii) $\displaystyle\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz$ *Em.:* $\frac{e^2+1}{e}$.

5. Kalkulatu $\displaystyle\int_\gamma z^{-1/2}\, dz$, γ unitate zirkunferentziaren goiko erdia izanik, 1-etik -1-eraino, eta $z^{1/2}$ definitzeko $\sqrt{1} = -1$ ematen duen adarra hartuz.

Em.: $2-2i$

6. Izan bedi $\gamma_R = \{z : |z| = R\}$. Frogatu

$$\left|\int_{\gamma_R} \frac{\log z}{z^2}\, dz\right| \le \frac{2\pi(\ln R + \pi)}{R}.$$

Ondorioz, erabaki integralak 0-rantz jotzen duela R infiniturantz doanean.

7. Egiaztatu z^{-1} funtzioak integral berbera duela jatorrian zentraturiko elipse guztietarako. Ondorioz, atera

$$\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2 \sin^2 t}\, dt = \frac{2\pi}{ab}\ , \quad (a>0, b>0).$$

8. Froga ezazu $\displaystyle\int_{|z|=1} \frac{e^{kz}}{z}\, dz = 2\pi i$ dela eta ondoriozta ezazu hurrengo formula

$$\int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta = 2\pi.$$

9. Kalkula ezazu $\displaystyle\int_\gamma \frac{e^{z^2}}{z^2-6z}\, dz$, γ zirkunferentzia hauetariko bakoitzerako, orientazio positiboarekin hartuta:

(i) $|z-2|=1$ *Em.:* 0

(ii) $|z-2|=3$ *Em.:* $-\pi i/3$

(iii) $|z-2|=5$ *Em.:* $\dfrac{\pi i}{3}(e^{36}-1)$

10. Kalkulatu hurrengo integralak Cauchyren formula erabiliz. Hartu zirkunferentzien orientazio positiboa.

(i) $\displaystyle\int_{|z-1|=1/2} \frac{e^{1/z}}{z^2+z}\, dz.$ *Em.:* 0

(ii) $\displaystyle\int_{|z-2|=2} \frac{\cosh z}{z^4-1}\, dz.$ *Em.:* $\dfrac{\pi i(e^2+1)}{4e}$

(iii) $\displaystyle\int_{|z-1-i|=1} \frac{\sin\pi(z-1)}{z^2-2z+2}\, dz.$ *Em.:* $i\pi\sinh\pi$

(iv) $\displaystyle\int_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}\, dz.$ *Em.:* $\dfrac{2\pi i}{3}\cosh\pi$

(v) $\displaystyle\int_{|z|=4} \frac{1}{(z^2+9)(z+9)}\, dz.$ *Em.:* $-\dfrac{\pi}{45}i$

(vi) $\displaystyle\int_{|z|=2} \frac{\sin z\sin(z-1)}{z^2-z}\, dz.$ *Em.:* 0

11. Hurrengo integralak kalkulatu deribatuetarako formula integrala erabiliz. Hartu zirkunferentzien orientazio positiboa.

(i) $\displaystyle\int_{|z-1|=1} \frac{\sin\pi z}{(z^2-1)^2}\, dz.$ *Em.:* $-\pi^2 i/2$

(ii) $\displaystyle\int_{|z|=2} \frac{\cosh z}{(z+1)^3(z-1)}\, dz.$ *Em.:* $-\dfrac{i\pi}{2e}$

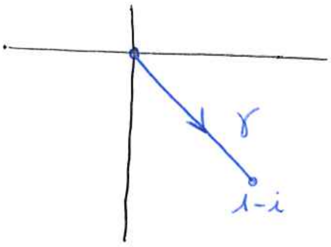
(iii) $\displaystyle\int_{|z|=2} \frac{z\sinh z}{(z^2-1)^2}\, dz.$ *Em.:* 0

(iv) $\displaystyle\int_{|z-3|=6} \frac{z}{(z-2)^3(z+4)}\, dz.$ *Em.:* $-\pi i/27$

(v) $\displaystyle\int_{|z+i|=2} \frac{e^{1/(z+2)}}{(z^2+4)^2}\, dz.$ *Em.:* $-\dfrac{5\pi}{64}e^{1/4}(\cos\frac{1}{4}+i\sin\frac{1}{4})$

1. ARIKETA

i)



$\gamma : [0,1] \longrightarrow \mathbb{C}$

$\gamma(t) = (1-i)t, \quad t \in [0,1]$

$\gamma'(t) = 1-i$

$$I_1 = \int_\gamma Re(z)\,dz = \int_0^1 Re(\gamma(t))\gamma'(t)\,dt = \int_0^1 t\cdot(1-i)\,dt =$$

$$= (1-i)\left.\frac{t^2}{2}\right|_0^1 = \boxed{\frac{1-i}{2}}$$

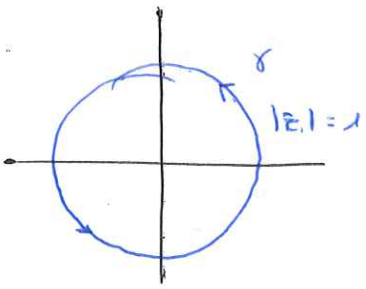
$$I_2 = \int_\gamma Im(z)\,dz = \int_\gamma Im(\gamma(t))\gamma'(t)\,dt = \int_0^1 -t\cdot(1-i)\,dt =$$

$$= -(1-i)\left.\frac{t^2}{2}\right|_0^1 = \boxed{\frac{-1+i}{2}}$$

$$I_3 = \int_\gamma \bar{z}\,dz = \int_\gamma \overline{\gamma(t)}\,\gamma'(t)\,dt = \int_0^1 (1+i)t\cdot(1-i)\,dt =$$

$$= 2\cdot\left.\frac{t^2}{2}\right|_0^1 = \boxed{1}$$

ii)



$\gamma : [-\pi,\pi] \longrightarrow \mathbb{C}$

$\gamma(\varphi) = e^{i\cdot\varphi}, \quad \varphi \in [-\pi,\pi]$

$\gamma'(\varphi) = ie^{i\varphi}$

$$I_1 = \int_\gamma Re(z)\,dz = \int_\gamma Re(\gamma(\varphi))\gamma'(\varphi)\,d\varphi = \int_{-\pi}^{\pi} \cos\varphi\cdot ie^{i\varphi}\,d\varphi =$$

$$= i\int_{-\pi}^{\pi} \cos\varphi(\cos\varphi + i\sin\varphi)\,d\varphi =$$

$$= i\int_{-\pi}^{\pi} \cos^2\varphi \, d\varphi - \int_{-\pi}^{\pi} \cos\varphi \sin\varphi \, d\varphi =$$

$$= i\int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{\cos 2\varphi}{2}\right) d\varphi - \int_{-\pi}^{\pi} 2 \sin 2\varphi \, d\varphi =$$

$$= i\left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_{-\pi}^{\pi} - \left[-\cos 2\varphi\right]_{-\pi}^{\pi} = \boxed{\pi i}$$

$$\boxed{I_2} = \int_\gamma \text{Im}(z)\,dz = \int_\gamma \text{Im}(\gamma(\varphi)) \ '(\varphi)\,d\varphi = \int_{-\pi}^{\pi} \sin\varphi \cdot i e^{i\varphi} \, d\varphi =$$

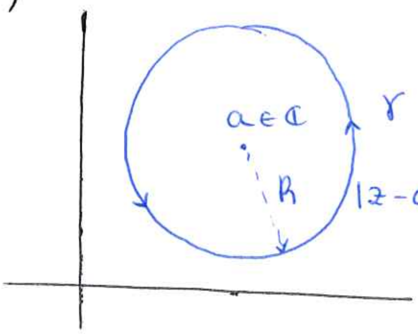
$$= i\int_{-\pi}^{\pi} \sin\varphi \left(\cos\varphi + i \sin\varphi\right) d\varphi = i\int_{-\pi}^{\pi} 2 \sin 2\varphi \, d\varphi - \int_{-\pi}^{\pi} \sin^2\varphi \, d\varphi =$$

$$= i \cdot \left[-\cos 2\varphi\right]_{-\pi}^{\pi} - \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\right]_{-\pi}^{\pi} = \boxed{-\pi}$$

$$\boxed{I_3} = \int_\gamma \bar{z}\,dz = \int_\gamma \bar{\gamma}(\varphi) \cdot \gamma'(\varphi)\,d\varphi = \int_{-\pi}^{\pi} e^{-i\varphi} \cdot i e^{i\varphi} \, d\varphi =$$

$$= i\int_{-\pi}^{\pi} d\varphi = \boxed{2\pi i}$$

iii)



$\gamma : [-\pi, \pi] \longrightarrow \mathbb{C}$

$\gamma(\varphi) = a + Re^{i\varphi}, \quad \varphi \in (-\pi, \pi]$

$\gamma'(\varphi) = Rie^{i\varphi}$

$$\boxed{I_1} = \int_\gamma \text{Re}(z)\,dz = \int_\gamma \text{Re}(\gamma(\varphi))\,\gamma'(\varphi)\,d\varphi = \int_{-\pi}^{\pi} R\cos\varphi \, Rie^{i\varphi} \, d\varphi =$$

$$= R^2 i \int_{-\pi}^{\pi} \cos\varphi \, e^{i\varphi} \, d\varphi = \boxed{\pi R^2 i}$$

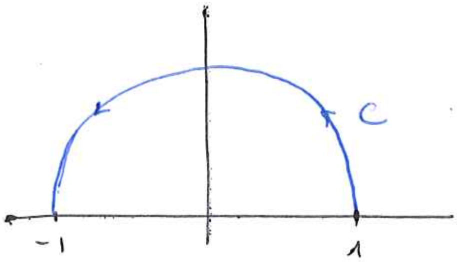
$$\boxed{J_2 = \int_\gamma \mathrm{Im}(z)\,dz = \int_\gamma \mathrm{Im}(\gamma(\varphi))\ '(\varphi)\,d\varphi = \int_{-\pi}^{\pi} R\sin\varphi\, R i e^{i\varphi}\,d\varphi = R^2 i\int_{-\pi}^{\pi} \sin e^{i\varphi}\,d\varphi = -R^2\pi}$$

$$\boxed{J_3 = \int_\gamma \bar{z}\,dz = \int_\gamma \bar{\gamma}(\varphi)\gamma'(\varphi)\,d\varphi = \int_{\pi}^{\pi} (a + R e^{-i\varphi}) R i e^{i\varphi}\,d\varphi =}$$

$$= Ri\int_{-\pi}^{\pi}(a\cos\varphi + a i \sin\varphi)\,d\varphi + Ri\int_{-\pi}^{\pi} R\,d\varphi =$$

$$= \cancel{Ria[\sin\varphi]_{-\pi}^{\pi}} - \cancel{Ra[-\cos\varphi]_{-\pi}^{\pi}} + R^2 i\int_{-\pi}^{\pi} d\varphi = \boxed{2\pi R^2 i}$$

2. ARIKETA



$$\gamma : [0,\pi] \longrightarrow \mathbb{C}$$

$$\gamma(\varphi) = e^{i\varphi}, \quad \varphi \in [0,\pi]$$

$$\gamma'(\varphi) = i e^{i\varphi}$$

i)
$$\int_C (z^2 + z\cdot\bar{z})\,dz = \int_\gamma [\gamma^2(\varphi) + \gamma(\varphi)\bar{\gamma}(\varphi)]\gamma'(\varphi)\,d\varphi =$$

$$= \int_\gamma [\gamma^2(\varphi) + |\gamma(\varphi)|^2]\gamma'(\varphi)\,d\varphi = \int_0^{\pi} (e^{2i\varphi} + 1) i e^{i\varphi}\,d\varphi$$

$$= i\int_0^{\pi} (e^{3i\varphi} + e^{i\varphi})\,d\varphi = i\left[\frac{e^{3i\varphi}}{3i} + \frac{e^{i\varphi}}{i}\right]_0^{\pi} =$$

$$= \frac{e^{3i\pi}}{3} + \frac{e^{i\pi}}{1} - \frac{e^0}{3} - e^0 = -\frac{1}{3} - 1 - \frac{1}{3} - 1 = -8/3$$

ii) $\int_C z\cdot \mathrm{Im}(z^2)\,dz = \int_\gamma \gamma(\varphi)\cdot \mathrm{Im}(\gamma^2(\varphi))\,\gamma'(\varphi)\,d\varphi =$

$= \int_0^{\pi} e^{i\varphi}\cdot \mathrm{Im}(e^{2i\varphi})\cdot i e^{i\varphi}\,d\varphi = i\int_0^{\pi} e^{2i\varphi}\cdot \sin(2\varphi)\,d\varphi =$

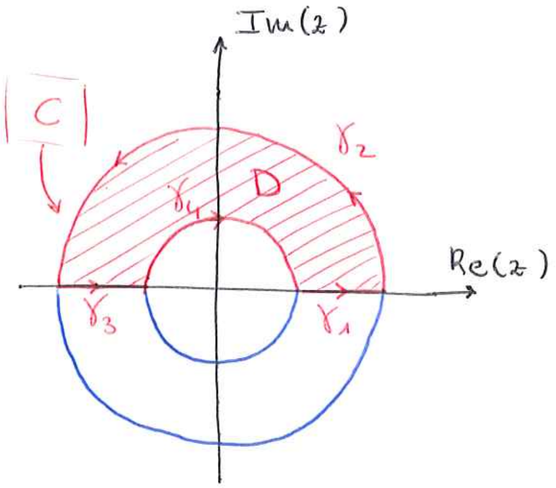
$2\varphi = x$
$2d\varphi = dx$

$= \frac{i}{2}\int_0^{2\pi} e^{ix}\sin x\,dx = \frac{i}{2}\int_0^{2\pi}(\cos x + i\sin x)\sin x\,dx =$

$= \frac{i}{2}\int_0^{2\pi}\frac{1}{2}\sin 2x\,dx + \frac{i}{2}\int_0^{2\pi} i\sin^2 x\,dx =$

$= \cancel{\frac{i}{4}\left[-\frac{\cos 2x}{2}\right]_0^{2\pi}} - \frac{1}{2}\left[\frac{x}{2} - \cancel{\frac{\sin 2x}{4}}\right]_0^{2\pi} = -\pi/2$

3. ARIKETA



$D = \{z : 1 < |z| < 2,\ \mathrm{Im}\,z > 0\}$

$C \equiv \partial D$

$\gamma_1 : [0,1] \longrightarrow \mathbb{C}$

$\gamma_1(t) = 1+t, \quad t\in[0,1]$

$\gamma_2 : [0,\pi] \longrightarrow \mathbb{C}$

$\gamma_2(\varphi) = 2e^{i\varphi}, \quad \varphi\in[0,\pi]$

$\gamma_3 : [0,1] \longrightarrow \mathbb{C}$

$\gamma_3(t) = -2+t, \quad t\in[0,1]$

$\gamma_4 : [0,\pi] \longrightarrow \mathbb{C}$

$\gamma_4(\varphi) = e^{i\varphi}, \quad \varphi\in[0,\pi]$

Parametrizazio guztiek norabidea mantentzen dute γ_4-k izan ezik.

Horrela, $\int_C f(z)\,dz = \sum_{i=1}^{4}\int_{\gamma_i} f(z)\,dz$.

i) $$\int_C |z|\,\bar{z}\,dz = \sum_{i=1}^{4} \int_{\gamma_i} |\gamma_i(t)| \cdot \bar{\gamma}_i(t) \cdot \gamma_i'(t)\,dt =$$

$$= \int_0^1 (1+t)(1+t)\cdot 1 \cdot dt + \int_0^{\pi} 2 \cdot 2e^{-i\varphi} \cdot 2ie^{i\varphi}\,d\varphi +$$

$$+ \int_0^1 \underbrace{(2-t)}_{\bar{\gamma}\,!!}(-2+t)\cdot 1\,dt - \int_0^{\pi} 1 \cdot e^{-i\varphi} \cdot ie^{i\varphi}\,d\varphi =$$

$$= \int_0^1 (t^2+2t+1)\,dt + 8i\int_0^{\pi} d\varphi - \int_0^1 (t^2-4t+4)\,dt - i\int_0^{\pi} d\varphi =$$

$$= \left|\frac{t^3}{3} + t^2 + t\right|_0^1 + 8i\pi - \left|\frac{t^3}{3} - 2t^2 + 4t\right|_0^1 - i\pi =$$

$$= \frac{1}{3} + 1 + 1 - \frac{1}{3} + 2 - 4 + 7i\pi = \boxed{7\pi i}$$

ii) $$\int_C \frac{z}{\bar{z}}\,dz = \sum_{i=1}^{4} \int_{\gamma_i} \frac{\gamma(t)}{\bar{\gamma}(t)} \cdot \gamma'(t)\,dt =$$

$$= \int_0^1 \frac{1+t}{1+t} \cdot 1\,dt + \int_0^{\pi} \frac{2e^{i\varphi}}{2e^{-i\varphi}} \cdot 2ie^{i\varphi}\,d\varphi + \int_0^1 \frac{-2+t}{-2+t} \cdot 1\,dt -$$

$$- \int_0^{\pi} \frac{e^{i\varphi}}{e^{-i\varphi}} \cdot ie^{i\varphi}\,d\varphi = t\Big|_0^1 + t\Big|_0^1 + 2i\int_0^{\pi} e^{3i\varphi}\,d\varphi - i\int_0^{\pi} e^{3i\varphi}\,d\varphi =$$

$$= 2 + 2i \cdot \frac{e^{3i\varphi}}{3i}\Big|_0^{\pi} - i\,\frac{e^{3i\varphi}}{3i}\Big|_0^{\pi} = 2 + \frac{2}{3}(e^{3\pi i} - 1) - \frac{1}{3}(e^{3\pi i} - 1) =$$

$$= 2 - \frac{4}{3} + \frac{2}{3} = 4/3$$

iii) $$\int_C |z|\,dz = \sum_{i=1}^{4} \int_{\gamma_i} |\gamma(t)| \cdot \gamma'(t)\,dt = \int_0^1 (1+t)\cdot 1\,dt + \int_0^{\pi} 2 \cdot 2ie^{i\varphi}\,d\varphi +$$

$$+ \int_0^1 (2-t)\cdot 1\,dt - \int_0^{\pi} 1 \cdot ie^{i\varphi}\,d\varphi =$$

$$= t + \frac{t^2}{2}\Big|_0^1 + 4i \cdot \frac{e^{ia}}{i}\Big|_0^{\pi} + \left(2t - \frac{t^2}{2}\right)\Big|_0^1 - i \cdot \frac{e^{ia}}{i}\Big|_0^{\pi} =$$

$$= 1 + 1/2 + 4(-1-1) + (2 - 1/2) - (-1-1) = \boxed{-3}$$

4. ARIKETA

i) $$\int_{1-i}^{2+i} (3z^2 + 2z)\, dz = z^3 + z^2 \Big|_{1-i}^{2+i} = (2+i)^3 + (2+i)^2 - (1-i)^3 - (1-i)^2 =$$

$$= (4+4i-1)(2+i) + (4+4i-1) - (1-2i-1) - (1-2i-1)(1-i) =$$

$$= (3+4i)(2+i) + (3+4i) + 2i + 2i(1-i) =$$

$$= 6 + 3i + 8i - 4 + 3 + 4i + 2i + 2i + 2 = \boxed{7 + 19i}$$

ii) $$\int_i^{i/2} e^{\pi z}\, dz = \frac{e^{\pi z}}{\pi}\Big|_i^{i/2} = \frac{1}{\pi}\left(e^{\frac{\pi}{2} i} - e^{\pi i}\right) = \boxed{\frac{1+i}{\pi}}$$

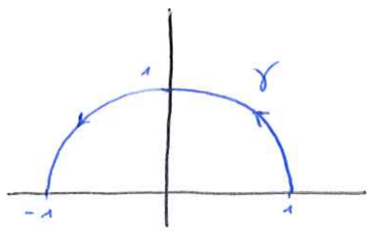
iii) $$\int_0^{\pi + 2i} \cos(z/2)\, dz = 2 \sin\left(\frac{z}{2}\right)\Big|_0^{\pi+2i} = 2 \sin\left(\frac{\pi}{2} + i\right) - \cancel{2 \sin 0}^{\,0} =$$

$$= 2 \cdot \sin\frac{\pi}{2} \cos i + \cancel{2 \sin i \cos\frac{\pi}{2}}^{\,0} = 2 \cos i = 2 \cosh 1 =$$

$$= 2 \cdot \frac{e + e^{-1}}{2} = \boxed{\frac{e^2 + 1}{e}}$$

5. ARIKETA

$$\int_\gamma z^{-1/2}\,dz$$, non $z^{1/2}$ definitzeko, $\sqrt{1}=-1$ dugun eta



$z^{1/2}=\sqrt{|z|}e^{i\cdot\frac{\arg z}{2}+ik\pi}$ dugu.

$\sqrt{1}=-1$ izateko,

$\sqrt{1}=e^{ik\pi}=-1$ izateko $k=1$ behar du.

Hortaz, $z^{1/2}=\sqrt{|z|}e^{i\cdot\frac{\arg z}{2}+i\pi}=-e^{i\cdot\frac{\arg z}{2}}\cdot\sqrt{|z|}$

$\gamma:[0,\pi]\longrightarrow\mathbb{C}$, $\gamma(\alpha)=e^{i\alpha}$, $\alpha\in[0,\pi]$ izanik:

$$\int_\gamma z^{-1/2}\,dz=\int_0^\pi\frac{1}{[\gamma(\alpha)]^{1/2}}\cdot\gamma'(\alpha)\,d\alpha=\int_0^\pi\frac{ie^{i\alpha}}{-e^{i\alpha/2}}\,d\alpha=-i\int_0^\pi e^{i\frac{\alpha}{2}}\,d\alpha=$$

$$=-i\cdot\left.\frac{e^{i\frac{\alpha}{2}}}{i/2}\right|_0^\pi=-2\left(e^{i\frac{\pi}{2}}-1\right)=\boxed{2-2i}$$

6. ARIKETA

Izan bedi $\gamma_R=\{z:|z|=R\}$.

Frogatu $$\left|\int_{\gamma_R}\frac{\log z}{z^2}\,dz\right|\leq\frac{2\pi(\ln R+\pi)}{R}.$$

Izan bedi gure bidearen ondorengo parametrizazioa:

$\gamma:(-\pi,\pi]\longrightarrow\mathbb{C}:\gamma(\alpha)=Re^{i\alpha}$

Horrela:

$$\left|\int_{\gamma_R}\frac{\log z}{z^2}\,dz\right|=\left|\int_\gamma\frac{\log(\gamma(\alpha))}{\gamma^2(\alpha)}\gamma'(\alpha)\,d\alpha\right|=$$

$$= \left| \int_{-\pi}^{\pi} \frac{\log(Re^{i\alpha})}{(Re^{i\alpha})^2} Ri e^{i\alpha} \, d\alpha \right| \leq \int_{-\pi}^{\pi} \left| \frac{\log(Re^{i\alpha})}{(Re^{i\alpha})^2} Rie^{i\alpha} \right| d\alpha =$$

$$= \int_{-\pi}^{\pi} \frac{|\log(Re^{i\alpha})|}{|R|^2 \cdot \underbrace{|e^{2i\alpha}|}_{1}} \cdot |R| \cdot \underbrace{|ie^{i\alpha}|}_{1} \cdot d\alpha = \frac{1}{R} \int_{-\pi}^{\pi} |\log(Re^{i\alpha})| \, d\alpha =$$

$$= \frac{1}{R} \int_{-\pi}^{\pi} |\ln R + i\alpha| \, d\alpha \leq \frac{1}{R} \int_{-\pi}^{\pi} (\ln R + |\alpha|) \, d\alpha \underset{\alpha \in (-\pi, \pi]}{\leq} \frac{1}{R} \int_{-\pi}^{\pi} (\ln R + \pi) \, d\alpha =$$

$$= \frac{\ln R + \pi}{R} \int_{-\pi}^{\pi} d\alpha = \frac{2\pi(\ln R + \pi)}{R}$$

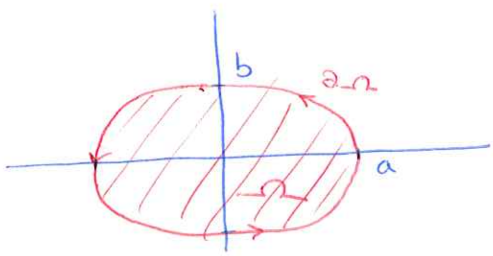
Hau da, $\boxed{\left| \int_{\gamma_R} \frac{\log z}{z^2} dz \right| \leq \frac{2\pi}{R}(\ln R + \pi)}$

7. ARIKETA

Egiaztatu z^{-1} funtzioak integral bera duela jatorrian zentratutako elipse guztietan.

Frogatu

$$\int_0^{2\pi} \frac{d\alpha}{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} = \frac{2\pi}{ab} \quad , \quad a, b > 0 \text{ izanik.}$$



$f(z) = \frac{1}{z}$ holomorfoa da $\Omega - \{z = 0\}$ eremuan.

$$\int_{\partial\Omega} \frac{dz}{z} = 2\pi i \cdot g(z) = 2\pi i \, ,$$ independentea dena elipsearen dimentsioekiko.

non $g(z) = 1$

Izan bedi $\partial\Omega$-ren γ parametrizazioa:

$\gamma : [0, 2\pi) \longrightarrow \mathbb{C} \ : \ \gamma(\varphi) = a\cos\varphi + b\sin\varphi\, i$

Beraz,

$$\int_{\partial\Omega} \frac{dz}{z} = \int_{\gamma} \frac{1}{\gamma(\varphi)} \cdot \gamma'(\varphi)\, d\varphi = \int_0^{2\pi} \frac{-a\sin\varphi + ib\cos\varphi}{a\cos\varphi + ib\sin\varphi}\, d\varphi =$$

$$= \int_0^{2\pi} \frac{(-a\sin\varphi + ib\cos\varphi)(a\cos\varphi - ib\sin\varphi)}{a^2\cos^2\varphi + b^2\sin^2\varphi}\, d\varphi = 2\pi i$$

Parte irudikaria hartuz:

$$\int_0^{2\pi} \frac{ab\sin^2\varphi + ab\cos^2\varphi}{a^2\cos^2\varphi + b^2\sin^2\varphi}\, d\varphi = 2\pi$$

Azkenik:

$$\boxed{\int_0^{2\pi} \frac{d\varphi}{a^2\cos^2\varphi + b^2\sin^2\varphi} = \frac{2\pi}{ab}}$$

8. ARIKETA

Frogatu $\displaystyle\int_{|z|=1} \frac{e^{kz}}{z}\, dz = 2\pi i$ dela.

Ondorioztatu: $\displaystyle\int_0^{2\pi} e^{k\cos\varphi} \cdot \cos(k\sin\varphi)\, d\varphi = 2\pi$

$\Omega = \{z : |z| = 1\}$ izanik, $f(z) = e^{kz}$ holomorfoa dela Ω-n dakigu. Cauchy-ren formula aplikatuz; $z_0 = 0 \in \Omega$ eginik...

$$\int_{|z|=1} \frac{e^{kz}}{z}\, dz = 2\pi i \cdot f(0) = 2\pi i\, e^{k\cdot 0} = 2\pi i.$$

Gure $\partial\Omega$ muga parametrizatuz:

$\gamma : [0, 2\pi) \longrightarrow \mathbb{C} \ : \ \gamma(\varphi) = e^{i\varphi}$

$$\int_{|z|=1} \frac{e^{kz}}{z}\,dz = \int_\gamma \frac{e^{k\gamma(\alpha)}}{\gamma(\alpha)}\gamma'(\alpha)\,d\alpha = \int_0^{2\pi} \frac{e^{k\cdot e^{i\alpha}}}{\cancel{e^{i\alpha}}}\, i\cancel{e^{i\alpha}}\,d\alpha =$$

$$= i\int_0^{2\pi} e^{k(\cos\alpha + i\sin\alpha)}\,d\alpha = i\int_0^{2\pi} e^{k\cos\alpha}\, e^{ik\sin\alpha}\,d\alpha =$$

$$= \cancel{i}\int_0^{2\pi} e^{k\cos\alpha}\left(\cos(k\sin\alpha) + i\sin(k\sin\alpha)\right)d\alpha = 2\pi\cancel{i}$$

Parte erreala berdinduz:

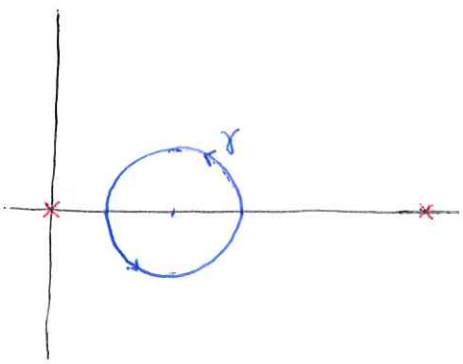
$$\boxed{\int_0^{2\pi} e^{k\cos\alpha}\cdot\cos(k\sin\alpha)\,d\alpha = 2\pi}$$

9. ARIKETA

Kalkulatu $\int_\gamma \frac{e^{z^2}}{z^2-6z}\,dz$ γ ezberdinetarako.

Etenguneak $\longrightarrow$ $z^2-6z=0 \longrightarrow \begin{cases} z=0 \\ z=6 \end{cases}$; $f(z) = \frac{e^{z^2}}{z^2-6z}$.

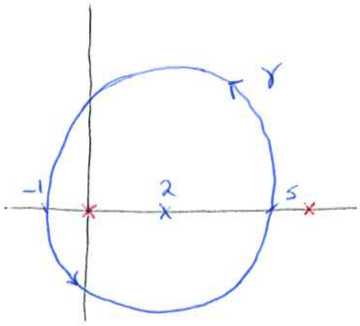
i) $|z-2|=1$



Etenguneak ez daude gure multzoan. Hortaz f holomorfoa da Ω-n eta Cauchy-ren teoremagatik:

$$\int_\gamma \frac{e^{z^2}}{z^2-6z}\,dz = 0$$

ii) $|z-2|=3$

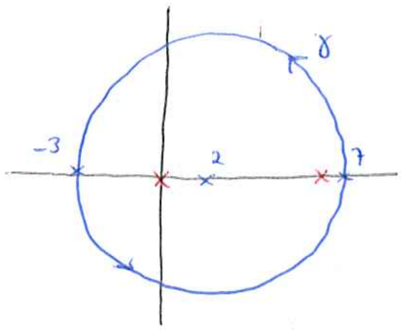


$z=0$ etengunea gure multzoan dago, ez da beharrezkoa frakzio sinpleetan deskonposatzea!

$$\int_\gamma \frac{e^{z^2}}{z^2-6z}\,dz = \int_{|z-2|=3} \frac{e^{z^2}}{z-6}\,\frac{dz}{z} = 2\pi i\, f(0)$$

non $f(z)=\dfrac{e^{z^2}}{z-6}$, $2\pi i f(0) = 2\pi i \cdot \dfrac{e^0}{0-6} = -\dfrac{\pi i}{3}$

iii) $|z-2|=5$



Frakzio sinpleetan deskonposatuz:

$$\frac{1}{z(z-6)} = \frac{A}{z} + \frac{B}{z-6} = \frac{A(z-6)+Bz}{z(z-6)}$$

$$A=-1/6 \ ; \quad B=1/6$$

Hortaz: $\displaystyle\int_\gamma \frac{e^{z^2}}{z^2-6z}\,dz = \int_{|z-2|=5} \frac{-1}{6}\cdot\frac{e^{z^2}}{z}\,dz + \int_{|z-2|=5} \frac{1}{6}\,\frac{e^{z^2}}{z-6}\,dz =$

$$= -\frac{1}{6}\,2\pi i\, f(0) + \frac{1}{6}\,2\pi i\, f(6) = -\frac{\pi i}{3}e^0 + \frac{\pi i}{3}e^{36} =$$

non $f(z)=e^{z^2}$ den $\qquad = \dfrac{\pi i}{3}\left(e^{36}-1\right)$

10. ARIKETA

i) $\displaystyle\int_{|z-1|=1/2} \frac{e^{1/z}}{z^2+z}\,dz$

Etenguneak: $z=0$

$z^2+z=0 \longrightarrow z(z+1)=0 \longrightarrow$

$\longrightarrow z=-1$

$f(z)=\dfrac{e^{1/z}}{z^2+z}$ holomorfoa da gure Ω multzoan.

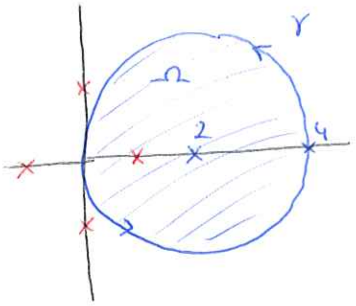
Beraz. Cauchy-ren teoremagatik:

$$\int_{|z-1|=1/2} \frac{e^{1/z}}{z(z+1)}\,dz = \boxed{0}$$

ii) $$\int_{|z-2|=2} \frac{\cosh(z)}{z^4-1}\,dz = \circledast$$

Etengouneak: $z^4-1=0 \longrightarrow z=\sqrt[4]{1} = e^{\frac{\pi}{2}ki}$, $k=0,1,2,3$

$z_0 = 1$; $z_1 = i$; $z_3 = -1$; $z_4 = -i$.



$z_0 = 1$ baino ez dago gure Ω multzoan:

$$\circledast = \int_{|z-2|=2} \frac{\cosh(z)}{z^3+z^2+z+1}\cdot\frac{dz}{z-1} = 2\pi i\, f(1) =$$

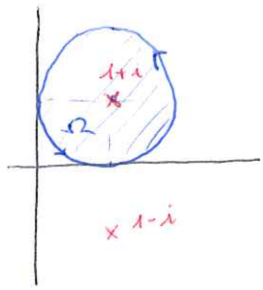
$$= 2\pi i\cdot\frac{\cosh(1)}{1+1+1+1} =$$

$f(z) = \dfrac{\cosh(z)}{z^3+z^2+z+1}$ izanik,

$$= \cancel{2}\pi i\cdot\frac{1}{4}\cdot\frac{e+e^{-1}}{\cancel{2}} =$$

$$\boxed{= \frac{\pi i}{4e}(e^2+1)}$$

iii) $$\int_{|z-1-i|=1} \frac{\sin[\pi(z-1)]}{z^2-2z+2}\,dz$$



Etengouneak: $z^2-2z+2=0 \longrightarrow z = 1\pm i$

Soilik $z_0 = 1+i$ dugu Ω multzoan

$$\int_{|z-1-i|=1} \frac{\sin[\pi(z-1)]}{z-1+i}\,\frac{dz}{z-1-i} = 2\pi i\cdot\frac{\sin[\pi(1+i-1)]}{1+i-1+i} = 2\pi i\cdot\frac{\sin(i\pi)}{2i} =$$

$$= \pi\sin(i\pi) = \boxed{i\pi\sinh(\pi)}$$

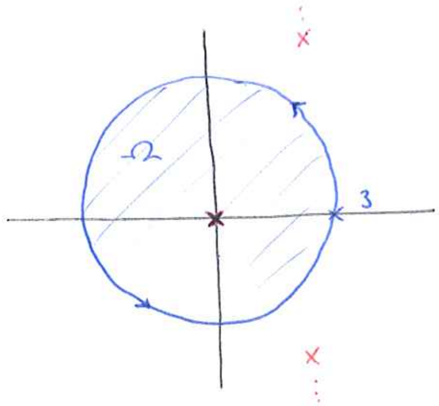
iv) $\displaystyle\int_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}\,dz$

Etengunеak:

$z=0$

$e^z+2=0 \longrightarrow z=\mathrm{Log}\,2 = \ln(2)+i\cdot 2k\pi\ ,\ k\in\mathbb{Z}$

Gure Ω multzoa:



Infinitu etengune ditu gure integrakizunak, baina gure multzoan bakarra, $z_0=0$.

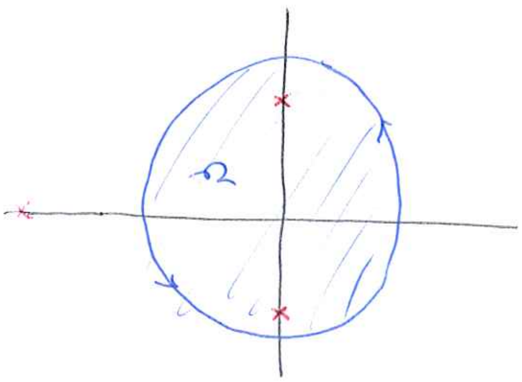
$f(z)=\dfrac{\cos(z+\pi i)}{e^z+2}$ izanik,

$$\int_{|z|=3} f(z)\cdot\frac{dz}{z} = 2\pi i\cdot f(0) = 2\pi i\cdot\frac{\cos(\pi i)}{e^0+2} = \boxed{\frac{2\pi}{3}\, i\cosh(\pi)}$$

v) $\displaystyle\int_{|z|=4} \frac{dz}{(z^2+9)(z+9)}$

Etengunеak: $z=-9$

$z^2=-9 \longrightarrow z=\pm 3i$



Gure multzoan $z=\pm 3i$ dira etengunеak.

$$\frac{1}{z^2+9} = \frac{A}{z+3i}+\frac{B}{z-3i} = \frac{A(z-3i)+B(z+3i)}{z^2+9}$$

$A+B=0 \longrightarrow A=-B=\frac{i}{6}$

$-3iA+3iB=1 \longrightarrow 6iB=1 \longrightarrow B=\frac{-i}{6}$

Hortaz, $f(z)=\dfrac{1}{z+9}$ bada,

$$\int_{|z|=4}\frac{dz}{(z^2+9)(z+9)} = \frac{i}{6}\int_{|z|=4} f(z)\,\frac{dz}{z+3i} - \frac{i}{6}\int_{|z|=4} f(z)\frac{dz}{z-3i} =$$

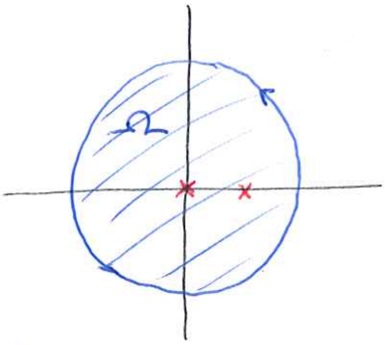
$$= \frac{i}{6} \cdot 2\pi i f(-3i) - \frac{i}{6} 2\pi i f(3i) =$$

$$= -\frac{\pi}{3} \cdot \frac{1}{-3i+9} + \frac{\pi}{3} \frac{1}{3i+9} = \frac{\pi}{3}\left(\frac{3i-9}{-9-81} - \frac{9+3i}{81+9} \right) =$$

$$= \frac{\pi}{3} \cdot 3 \left(\frac{i-3}{-90} - \frac{3+i}{90} \right) = \frac{\pi}{90}(-i+3-3-i) = \boxed{-\frac{\pi}{45} i}$$

vi) $$\int_{|z|=2} \frac{\sin z \cdot \sin(z-1)}{z^2 - z} dz$$

Etenguneak: $z=0$, $z=1$.



Frakzio sinpleetan deskonposatuz:

$$\frac{1}{z^2-z} = \frac{A}{z} + \frac{B}{z-1} \qquad \begin{matrix} A=-1 \\ B=1 \end{matrix}$$

$f(z) = \sin z \cdot \sin(z-1)$ izanik:

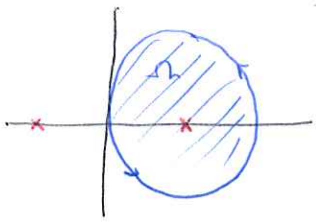
$$\int_{|z|=2} \frac{f(z)\,dz}{z^2-z} = -\int_{|z|=2} f(z) \frac{dz}{z} + \int_{|z|=2} f(z) \frac{dz}{z-1} = -2\pi i f(0) + 2\pi i f(1) =$$

$$\boxed{= 0}$$

11. ARIKETA

i) $$\int_{|z-1|=1} \frac{\sin(\pi z)}{(z^2-1)^2} dz = \int_{|z-1|=1} \frac{\sin(\pi z)}{(z-1)^2 (z+1)^2} dz.$$

Etenguneak: $z = \pm 1$



Bakarrik $z=1$ dago gure gunean,

hortaz, $f(z) = \dfrac{\sin(\pi z)}{(z+1)^2}$ bada,

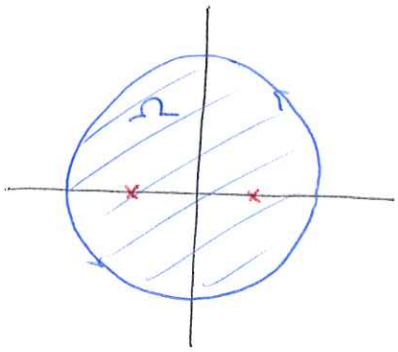
$$\int_{|z-1|=1} \frac{\sin(\pi z)}{(z+1)^2}\frac{dz}{(z-1)^2} = \frac{2\pi i}{1!} f'(1)$$

$$f'(z) = \frac{\pi\cos(\pi z)(z+1)^2 - 2\sin(\pi z)(z+1)}{(z+1)^4}$$

$$f'(1) = \frac{-4\pi}{2^4} = \frac{-\pi}{4}$$

$$\frac{2\pi i}{1!} f'(1) = 2\pi i \cdot \left(-\frac{\pi}{4}\right) = \boxed{-\frac{\pi^2 i}{2}}$$

ii) $\displaystyle\int_{|z|=2} \frac{\cosh z}{(z+1)^3(z-1)}\,dz$ Etengouneak: $z=\pm 1$ (biak gure eremuaren barne).



$$\frac{1}{(z+1)^3(z-1)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{(z+1)^3} + \frac{D}{z-1} = \frac{A(z-1)(z+1)^2 + B(z-1)(z+1) + C(z-1) + D(z+1)^3}{(z+1)^3(z-1)}$$

$D = 1/8$

$C = -1/2$

$A + D = 0 \rightarrow A = -1/8$

$A + B + 3D = 0 \longrightarrow B = -1/4$

$f(z) = \cosh z$ bada, orduan:

$$\int_{|z|=2} f(z)\frac{dz}{(z+1)^3(z-1)} = \frac{1}{8}\int_\gamma f(z)\frac{dz}{z-1} - \frac{1}{2}\int_\gamma f(z)\frac{dz}{(z+1)^3} -$$

$$-\frac{1}{4}\int_\gamma f(z)\frac{dz}{(z+1)^2} - \frac{1}{8}\int_\gamma f(z)\frac{dz}{z+1} =$$

$$= \frac{1}{8}2\pi i f(1) - \frac{1}{2}\frac{2\pi i}{2!}f''(-1) - \frac{1}{4}\frac{2\pi i}{1!}f'(-1) - \frac{1}{8}2\pi i f(-1) =$$

$$= \cancel{\frac{\pi}{4} i\cosh(1)} - \frac{\pi}{2} i\cosh(-1) - \frac{\pi}{2} i\sinh(-1) - \cancel{\frac{\pi}{4} i\cosh(-1)} =$$

$$= \frac{-\pi}{2} i\cosh(1) + \frac{\pi}{2} i\sinh(1) =$$

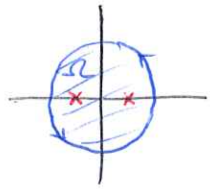
$$= -\frac{\pi}{2} i \cdot \frac{e + e^{-1}}{2} + \frac{\pi}{2} i \, \frac{e - e^{-1}}{2} = -\frac{\pi}{4} i \cdot \frac{e^2+1}{e} + \frac{\pi}{4} i \, \frac{e^2-1}{e} =$$

$$= \frac{\pi i}{4e} (e^2 - 1 - e^2 - 1) = \boxed{-\frac{\pi i}{2e}}$$

iii) $$\int_{|z|=2} \frac{z \cdot \sinh(z)}{(z^2-1)^2} \, dz = \int_{|z|=2} \frac{z \sinh(z)}{(z-1)^2 (z+1)^2} \, dz$$

Etenguneak: $z = \pm 1$

Biak gure eremuaren barne.



$$\frac{1}{(z-1)^2(z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1} + \frac{D}{(z+1)^2} =$$

$$= \frac{A(z-1)(z+1)^2 + B(z+1)^2 + C(z+1)(z-1)^2 + D(z-1)^2}{(z-1)^2 (z+1)^2}$$

$$B = 1/8 \; ; \quad D = 1/8$$

$$\left.\begin{array}{l} z^3 \longrightarrow A + C = 0 \\ z^0 \longrightarrow -A + C + B + D = 1 \longrightarrow C - A = 3/4 \end{array}\right\} \longrightarrow \begin{array}{l} A = -3/8 \\ C = 3/8 \end{array}$$

Horrela, $f(z) = z \cdot \sinh(z)$ bada

$f'(z) = \sinh(z) + z\cosh(z)$.

$$\int_\gamma f(z) \frac{dz}{(z^2-1)^2} = \frac{3}{8} \int_\gamma f(z) \frac{dz}{z+1} + \frac{1}{8} \int_\gamma f(z) \frac{dz}{(z+1)^2} - \frac{3}{8} \int_\gamma f(z) \frac{dz}{z-1} +$$

$$+ \frac{1}{8} \int_\gamma f(z) \frac{dz}{(z-1)^2} =$$

$$= \frac{3}{8} 2\pi i f(-1) + \frac{1}{8} \frac{2\pi i}{1!} f'(-1) - \frac{3}{8} 2\pi i f(1) + \frac{1}{8} \frac{2\pi i}{1!} f'(1) =$$

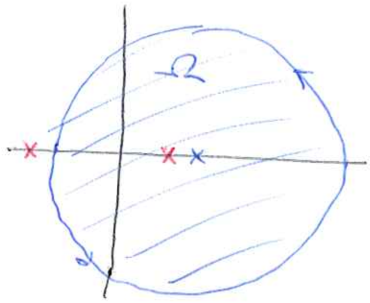
$$f(-z) = -z \sinh(-z) = z \sinh(z) = f(z)$$

$$f'(-z) = \sinh(-z) - z\cosh(-z) = -\sinh(z) - z\cosh(z) = -f'(z)$$

denez,

$$\int_{|z|=2} \frac{z \cdot \sinh(z)}{(z^2-1)^2}\, dz = \boxed{0}$$

iv) $$\int_{|z-3|=6} \frac{z\, dz}{(z-2)^3(z+4)}$$



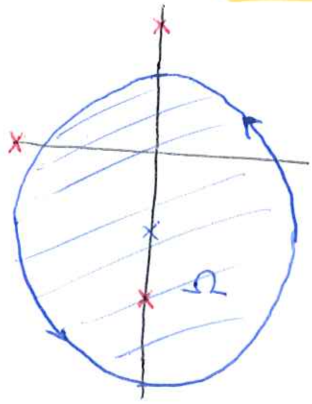
Etenguneak $z=2$ eta $z=-4$ dira, baina soilik $z=2$ dago gure eremuan.

$f(z) = \dfrac{z}{z+4}$ bada,

$$\int_{|z-3|=6} f(z) \frac{dz}{(z-2)^3} = \frac{2\pi i}{2!} f''(2) = \pi i \left(\frac{4}{(z+4)^2} \right)' \Bigg|_{z=2} =$$

$$= \pi i \frac{-8}{(z+4)^3} \Bigg|_{z=2} = -8\pi i \frac{1}{(2+4)^3} = \boxed{-\frac{\pi i}{27}}$$

v) $$\int_{|z+i|=2} \frac{e^{\frac{1}{z+2}}}{(z^2+4)^2}\, dz$$



Etenguneak:

$z = -2$

$z = \pm 2i$

Soilik $z=-2i$ dago gure eremuan.

$f(z) = \dfrac{e^{\frac{1}{z+2}}}{(z-2i)^2}$ izanik, eta

$$f'(z) = \frac{\frac{-1}{(z+2)^2} e^{\frac{1}{z+2}} (z-2i)^{\cancel{2}} - 2 e^{\frac{1}{z+2}} \cancel{(z-2i)}}{(z-2i)^{\cancel{4}3}}$$ izanik:

$$\int_\gamma f(z)\frac{dz}{(z+2i)^2} = \frac{2\pi i}{1!} f'(-2i) =$$

$$= -2\pi i \cdot e^{\frac{1}{z+2}} \cdot \frac{\frac{z-2i}{(z+2)^2}+2}{(z-2i)^3}\Bigg|_{z=-2i} = -2\pi i \cdot \frac{\frac{-4i}{(2(1-i))^2}+2}{(-4i)^3} \cdot e^{\frac{1}{2(1-i)}} =$$

$$= +2\pi i \frac{\frac{-4i}{4(1-i)^2}+2}{-4^3 \cdot i^3} e^{\frac{1}{2-2i}} = 2\pi \cancel{i} \cdot \frac{\frac{-i}{i^2-2i+1}+2}{-4^3 \cancel{i}} \cdot e^{\frac{1}{2(1-i)}} =$$

$$= 2\pi \cdot \frac{\frac{-i}{-2i}+2}{-4^3} e^{\frac{1}{2(1-i)}} = 2\pi \cdot \frac{5}{-2\cdot 4^3} e^{\frac{1}{2(1-i)}} = \frac{-5\pi}{64} e^{\frac{1}{2-2i}\cdot\frac{2+2i}{2+2i}} =$$

$$= -\frac{5\pi}{64} e^{\frac{2+2i}{4+4}} = -\frac{5\pi}{64} e^{\frac{1}{4}(1+i)} = \boxed{-\frac{5\pi}{64} \cdot e^{1/4} \cdot (\cos 1/4 + i \sin 1/4)}$$