13. Gaia

Hondarrak

13.1 Hondarren teorema

Definizioa. Izan bitez $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$, $z_0 \in \mathbb{C}$ f funtzioaren puntu singular isolatua eta $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ f-ren Laurenten seriea $0 < |z-z_0| < r$ eraztunean, $r > 0$ izanik. Orduan a_{-1} koefizientea f funtzioaren z_0 puntuko hondarra dela esaten da, $a_{-1} = \operatorname*{Res}_{z=z_0} f(z)$ edo $a_{-1} = \operatorname{Res}(f, z_0)$ idatzi ohi delarik.

Laurenten seriearen koefizienteen formularen arabera,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} f(z)\,dz, \quad 0 < \rho < r \text{ izanik.}$$

Teorema 13.1 (Hondarren teorema). *Izan bitez $\Omega \subset \mathbb{C}$, f funtzio analitikoa Ω-n, puntu kopuru finitu batean izan ezik eta γ f-ren puntu singularretatik pasatzen ez den bide itxia Ω-n, orientazio positiboarekin. γ-ren barrualdean geratzen diren puntu singular isolatuak $z_1, \ldots, z_m$ baldin badira, orduan*

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z).}$$

Froga. Kontsidera ditzagun $|z-z_i| = r_i$ zirkunferentziak, non r_i erradioak aukeratzen diren zirkunferentzia guztien barruko aldeak binaka disjuntuak izan daitezen eta denak γ-ren barrualdean gera daitezen. Cauchyren teorema integrala aplikatuz,

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{m} \int_{|z-z_i|=r_i} f(z)\,dz$$

eta hondarraren definizioaren ondorioz, teoremaren berdintza dugu □

Oharra. f-k infinitu puntu singular baldin baditu, denak isolatuak, teorema ere aplika daiteke, kurbaren barrualdean puntu singularren kopuru finitu bat geratzen baita soilik.

Definizioa. Izan bitez $R > 0$ eta f $|z| > R$ multzoan analitikoa (∞ f-ren puntu singular isolatua da, beraz). Izan bedi $\sum_{n=-\infty}^{\infty} a_n z^n$ f-ren Laurenten seriea $|z| > R$ multzoan. Orduan f-ren ∞-ko hondarra $-a_{-1}$ zenbakia da, $\operatorname{Res}_{z=\infty} f(z) = -a_{-1}$.

Kasu honetan, honako formulak emanda dago hondarra,

$$\operatorname{Res}_{z=\infty} f(z) = -\frac{1}{2\pi i}\int_{|z|=\rho} f(z)\,dz, \quad \rho > R \text{ izanik.}$$

Oharrak. Izan bitez $R > 0$ eta f $|z| > R$ multzoan analitikoa.

(i) $|z| = \rho$ zirkunferentziak ∞-tik begiratuta, noranzko negatiboa dauka, hortaz zeinu negatiboa.

(ii) f-k $z_0 \in \mathbb{C}$ puntuan singularitate gaindigarria baldin badauka, $\operatorname{Res}_{z=z_0} f(z) = 0$. Hala ere, ∞ singularitate gaindigarria izan daiteke eta $\operatorname{Res}_{z=\infty} f(z) \neq 0$.

13.2 Hondarrak kalkulatzeko metodoak

Definizioa erabili.

Ikusi dugunaren arabera, f-ren Laurenten seriea $0 < |z - z_0| < r$ eraztunean, $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ bada,

$$\operatorname{Res}_{z=z_0} f(z) = a_{-1}.$$

Adibidea. $f(z) = \dfrac{1}{\sin z - z}$ funtzioaren $z_0 = 0$ puntuko hondarra kalkulatuko dugu.

$z_0 = 0$ f-ren 3. mailako poloa da, beraz

$$\frac{1}{\sin z - z} = \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + \dots$$

Orduan,

$$\begin{aligned} 1 &= \Big(\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + \dots\Big)(\sin z - z) \\ &= \Big(\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + \dots\Big)\Big(-\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\Big) \end{aligned}$$

z-ren berreturen koefizienteak berdinduz,

$$\begin{aligned} &(z^0) \quad 1 = -\frac{a_{-3}}{3!} \implies a_{-3} = -3! = -6 \\ &(z^1) \quad 0 = -\frac{a_{-2}}{3!} \implies a_{-2} = 0 \\ &(z^2) \quad 0 = \frac{a_{-3}}{5!} - \frac{a_{-1}}{3!} \implies a_{-1} = \frac{3!a_{-3}}{5!} = -\frac{6}{20} = -\frac{3}{10} = \operatorname{Res}_{z=0}\frac{1}{\sin z - z} \end{aligned}$$

m mailako poloen hondarra

z_0 f-ren m mailako poloa baldin bada, orduan

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$$

Biderkatuz $(z-z_0)^m$-rekin,

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \ldots$$

$(z-z_0)^m f(z)$ funtzioak singularitate gaindigarria du z_0 puntuan eta a_{-1} bere Taylor-en seriearen $m-1$-garren koefizientea da, beraz,

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)^m f(z)\big).}$$

Adibidea. $f(z) = \dfrac{1}{\sin z - z}$ funtzioaren $z_0 = 0$ puntuko hondarra kalkulatuko dugu.
↳ 3. ordenako poloa!
Ikusitakoaren arabera,

$$\operatorname*{Res}_{z=0} \frac{1}{\sin z - z} = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\Big(\frac{z^3}{\sin z - z}\Big).$$

$$\begin{aligned}
g(z) =& \frac{z^3}{\sin z - z} \\
g'(z) =& \frac{3z^2}{\sin z - z} - \frac{z^3(\cos z - 1)}{(\sin z - z)^2} \\
g''(z) =& \frac{6z}{\sin z - z} - \frac{3z^2(\cos z - 1)}{(\sin z - z)^2} - \frac{3z^2(\cos z - 1) - z^3 \sin z}{(\sin z - z)^2} + \frac{2z^3(\cos z - 1)^2}{(\sin z - z)^3}.
\end{aligned}$$

Beraz,

$$\begin{aligned}
\operatorname*{Res}_{z=0} \frac{1}{\sin z - z} =& \frac{1}{2} \lim_{z\to 0}\Big(\frac{6z}{\sin z - z} - \frac{6z^2(\cos z - 1)}{(\sin z - z)^2} + \frac{z^3 \sin z}{(\sin z - z)^2} + \frac{2z^3(\cos z - 1)^2}{(\sin z - z)^3}\Big) \\
=& \cdots = -\frac{3}{10}
\end{aligned}$$

Metodo hau luzea izan daiteke poloaren maila altua denean, baina polo sinpleen hondarra kalkulatzeko oso egokia da, kasu horretan formula honela geratzen baita

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z).$$

Adibidea. $f(z) = \dfrac{1}{z^2+1}$ funtzioaren $z_0 = i$ eta $z_0 = -i$ polo sinpleen hondarrak kalkulatuko ditugu.

$$\begin{aligned}
\operatorname*{Res}_{z=i} \frac{1}{z^2+1} &= \lim_{z\to i}(z-i)\frac{1}{z^2+1} = \lim_{z=i} \frac{1}{z+i} = \frac{1}{2i} = -\frac{i}{2}, \\
\operatorname*{Res}_{z=-i} \frac{1}{z^2+1} &= \lim_{z\to -i}(z+i)\frac{1}{z^2+1} = \lim_{z=i} \frac{1}{z-i} = \frac{1}{-2i} = \frac{i}{2}.
\end{aligned}$$

$f(z) = \dfrac{g(z)}{h(z)}$ moduko funtzio batzuen hondarrak

Izan bedi $f(z) = \dfrac{g(z)}{h(z)}$, non $g(z_0) \neq 0$, $h(z_0) = 0$ eta $h'(z_0) \neq 0$ diren. z_0 f-ren polo simplea da zeren eta $\left(\dfrac{h(z)}{g(z)}\right)' = \dfrac{h'(z)g(z) - h(z)g'(z)}{(g(z))^2}$ eta ondorioz $\dfrac{1}{f}$ anulatzen da z_0 puntuan baina bere deribatua ez. Aurreko atalean ikusitakoaren arabera,

$$\begin{aligned}
\operatorname*{Res}_{z=z_0} f(z) &= \lim_{z\to z_0} (z - z_0)\frac{g(z)}{h(z)} \\
&= \lim_{z\to z_0} \frac{(z - z_0)\big(g(z_0) + g'(z_0)(z - z_0) + \dots\big)}{h'(z_0)(z - z_0) + \dfrac{h''(z_0)}{2}(z - z_0)^2 + \dots} \\
&= \lim_{z\to z_0} \frac{g(z_0) + g'(z_0)(z - z_0) + \dots}{h'(z_0) + \dfrac{h''(z_0)}{2}(z - z_0) + \dots} \\
&= \frac{g(z_0)}{h'(z_0)}.
\end{aligned}$$

Adibidea. $f(z) = \dfrac{z}{z^2+1}$ funtzioaren $z_0 = i$ puntuko hondarra kalkulatuko dugu. f funtzioa $g(z) = z$ eta $h(z) = z^2 + 1$ funtzioen arteko zatidura da. g ez da $z_0 = i$ puntuan anulatzen, h bai baina bere deribatua ez. Orduan,

$$\operatorname*{Res}_{z=i} \frac{z}{z^2+1} = \left.\frac{z}{2z}\right|_{z=i} = \frac{1}{2}.$$

∞-ko hondarra

Izan bedi $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $|z| > R$ denean. Orduan

$$f\Big(\frac{1}{z}\Big) = \sum_{n=-\infty}^{\infty} a_n \frac{1}{z^n} = \dots + \frac{a_2}{z^2} + \frac{a_1}{z} + a_0 + a_{-1}z + a_{-2}z^2 + \dots, \quad 0 < |z| < \frac{1}{R} \text{ denean,}$$

beraz,

$$\frac{1}{z^2} f\Big(\frac{1}{z}\Big) = \dots + \frac{a_2}{z^4} + \frac{a_1}{z^3} + a_0\frac{1}{z^2} + a_{-1}\frac{1}{z} + a_{-2} + \dots,, \quad 0 < |z| < \frac{1}{R} \text{ denean.}$$

Ondorioz,

$$\boxed{\operatorname*{Res}_{z=\infty} f(z) = -a_{-1} = -\operatorname*{Res}_{z=0} \frac{1}{z^2} f\Big(\frac{1}{z}\Big).}$$

Adibidea. Kalkulatuko dugu $f(z) = \dfrac{1}{z^4+1}$ funtzioaren hondarra ∞-n.

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} \frac{1}{z^2} f\Big(\frac{1}{z}\Big) = -\operatorname*{Res}_{z=0} \frac{1}{z^2}\frac{1}{\dfrac{1}{z^4}+1} = \operatorname*{Res}_{z=0} \frac{z^2}{z^4+1} = 0.$$

Proposizioa 13.2. *Izan bedi f $\mathbb{C}$ osoan analitikoa $z_1, \ldots, z_m$ puntuetan izan ezik. Orduan,*

$$\boxed{\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.}$$

Proposizio hau erabili daiteke, beraz, $\operatorname*{Res}_{z=\infty} f(z)$ kalkulatzeko.

Adibidea. $f(z) = \dfrac{1}{z^4+1}$ funtzioaren ∞-ko hondarra kalkulatuko dugu.

f-ren puntu singularrak $z_k = e^{\frac{\pi+2k\pi}{4}i}$ puntuak dira, $k = 0, 1, 2, 3$ izanik, hots, $z_0 = e^{\frac{\pi i}{4}}$, $z_1 = e^{\frac{3\pi i}{4}}$, $z_2 = e^{-\frac{\pi i}{4}}$, $z_3 = e^{-\frac{3\pi i}{4}}$, eta denak polo sinpleak dira.

$$\operatorname*{Res}_{z=z_k} f(z) = \frac{1}{4z_k^3} = \frac{z_k}{4z_k^4} = -\frac{z_k}{4}.$$

Orduan,

$$\begin{aligned}\operatorname*{Res}_{z=\infty} f(z) = -\sum_{k=0}^{3} \operatorname*{Res}_{z=z_k} f(z) &= \frac{z_0+z_1+z_2+z_3}{4} = \frac{e^{\frac{\pi i}{4}} + e^{\frac{3\pi i}{4}} + e^{-\frac{\pi i}{4}} + e^{-\frac{3\pi i}{4}}}{4} \\ &= \frac{2\cos\frac{\pi}{4} + 2\cos\frac{3\pi}{4}}{4} = 0.\end{aligned}$$

Adibidea. $f(z) = \dfrac{e^{iz}}{z^4}$ funtzioaren puntu singularrak $z_0 = 0$ eta ∞ dira. $z_0 = 0$ 4. mailako poloa da.

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{3!}\frac{d^3}{dz^3}\Big(z^4\frac{e^{iz}}{z^4}\Big) = \frac{1}{6} i^3\, e^{iz}\big|_{z=0} = -\frac{i}{6}.$$

∞-ko hondarra kalkulatzeko bi aukera ditugu.

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} \frac{1}{z^2} f\Big(\frac{1}{z}\Big) = -\operatorname*{Res}_{z=0} z^2 e^{i/z}.$$

$z_0 = 0$ $z^2e^{i/z}$ funtzioaren puntu singular esentziala da, beraz Laurenten seriezko garapena egin behar dugu,

$$\begin{aligned}z^2e^{i/z} &= z^2 \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{i}{z}\Big)^n \\ &= z^2\Big(1 + \frac{i}{z} - \frac{1}{2!z^2} - \frac{i}{3!z^3} + \frac{1}{4!z^4} + \ldots\Big) \\ &= \Big(z^2 + iz - 1 - \frac{i}{6z} + \frac{1}{24z^2} + \ldots\Big)\end{aligned}$$

hau da,

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} z^2e^{i/z} = \frac{i}{6}.$$

Beste aukera da aurreko proposizioa kontuan hartzea,

$$\operatorname*{Res}_{z=\infty} = -\operatorname*{Res}_{z=0} f(z) = \frac{i}{6}.$$

13.3 Funtzio trigonometrikoen integral erreal mugatuak

$\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$ moduko integralak kalkula daitezke $F\Big(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})\Big)\frac{1}{iz}$ aldagai konplexuko funtzioa $|z| = 1$ zirkunferentzian integratuz.

$z = e^{i\theta}$, $\theta \in [0, 2\pi]$, zirkunferentziaren parametrizazioa hartuz,

$$\frac{1}{2}\Big(z + \frac{1}{z}\Big) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos\theta,$$
$$\frac{1}{2i}\Big(z - \frac{1}{z}\Big) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin\theta,$$
$$\frac{dz}{iz} = \frac{ie^{i\theta}}{ie^{i\theta}} d\theta = d\theta.$$

Adibidea. $\displaystyle\int_0^{2\pi} \frac{dt}{a+\cos t}$, $a > 1$ izanik.

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{a+\cos t} &= \int_{|z|=1} \frac{1}{a + \frac{1}{2}\Big(z+\frac{1}{z}\Big)}\frac{dz}{iz} = \int_{|z|=1} \frac{2z}{2az + z^2 + 1}\frac{dz}{iz}\\ &= -\,2i\int_{|z|=1}\frac{dz}{z^2+2az+1}.\end{aligned}$$

$f(z) = \dfrac{1}{z^2+2az+1}$ funtzioaren puntu singularrak $z_1 = -a + \sqrt{a^2-1}$ eta $z_2 = -a - \sqrt{a^2-1}$ dira, biak polo sinpleak.

$|-a-\sqrt{a^2-1}| = a+\sqrt{a^2-1} > a > 1$, beraz z_2 ez da zirkunferentziaren barrualdean geratzen. Aldiz, $|-a+\sqrt{a^2-1}| = \Big|\dfrac{a^2-a^2+1}{a+\sqrt{a^2-1}}\Big| = \dfrac{1}{a+\sqrt{a^2-1}} < 1$ eta ondorioz, z_1 zirkunferentzia unitarioaren barrualdean dago. Beraz,

$$\begin{aligned}\int_0^{2\pi}\frac{dt}{a+\cos t} &= -\,2i\int_{|z|=1}\frac{dz}{z^2+2az+1} = -2i\cdot 2\pi i \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{1}{z^2+2az+1}\\ &=4\pi\frac{1}{(2z+2a)|_{z=-a+\sqrt{a^2-1}}} = \frac{4\pi}{2(-a+\sqrt{a^2-1}+a)} = \frac{2\pi}{\sqrt{a^2-1}}.\end{aligned}$$

13.4 Aldagai errealeko integral inpropioak eta balio nagusiak

Lehenengo eta behin, gogora dezagun zer den aldagai errealeko funtzio baten integral inpropio konbergentea eta zer den integral inpropio baten balio nagusia.

Definizioa. Izan bedi $f\colon \mathbb{R} \to \mathbb{R}$ bornatua.

(i) $\displaystyle\lim_{R_1\to+\infty}\int_{-R_1}^{0} f(x)\,dx$ eta $\displaystyle\lim_{R_2\to+\infty}\int_{0}^{R_2} f(x)\,dx$ finituak badira $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ integral inpropioa konbergentea dela diogu eta

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to+\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to+\infty}\int_{0}^{R_2} f(x)\,dx.$$

(ii) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ integral inpropioaren balio nagusia honela definitzen da

$$p.v. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to+\infty}\left(\int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx\right) = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$

Oharrak. Izan bedi $f\colon \mathbb{R}\to\mathbb{R}$ bornatua.

(i) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ konbergentea bada, orduan

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = p.v. \int_{-\infty}^{\infty} f(x)\,dx.}$$

Baina, integral inpropioa dibergentea izan daiteke eta bere balio nagusia finitua.

(ii) f bikoitia baldin bada, orduan

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx \ \text{ konbergentea } \iff p.v. \int_{-\infty}^{\infty} f(x)\,dx \ \text{ konbergentea.}}$$

Definizioa. Izan bitez $a, b \in \mathbb{R}$ eta $f\colon [a,b]\to\mathbb{R}$ eta demagun existitzen dela $c\in(a,b)$ non $\lim_{x\to c}|f(x)| = +\infty$.

Gauza bera jauzi infinituekin

(i) $\displaystyle\lim_{\epsilon_1\to0^+}\int_{a}^{c-\epsilon_1} f(x)\,dx$ eta $\displaystyle\lim_{\epsilon_2\to0^+}\int_{c+\epsilon_2}^{b} f(x)\,dx$ finituak baldin badira, $\displaystyle\int_{a}^{b} f(x)\,dx$ integral inpropioa konbergentea dela esaten da eta

$$\int_{a}^{b} f(x)\,dx = \lim_{\epsilon_1\to0^+}\int_{a}^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to0^+}\int_{c+\epsilon_2}^{b} f(x)\,dx.$$

(ii) $\displaystyle\int_{a}^{b} f(x)\,dx$ integral inpropioaren balio nagusia honela definitzen da

$$p.v. \int_{a}^{b} f(x)\,dx = \lim_{\epsilon\to0^+}\left(\int_{a}^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right).$$

Adibidea. $\displaystyle\int_{-1}^{1} \frac{dx}{x}$ integral inpropioa da $\displaystyle\lim_{x\to 0} \frac{1}{|x|} = +\infty$ delako.

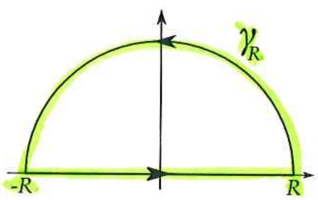
$$\lim_{\epsilon\to 0} \int_{\epsilon}^{1} \frac{dx}{x} = \lim_{\epsilon\to 0} (\log 1 - \log \epsilon) = +\infty,$$

beraz, integral inpropioa dibergentea da. Aldiz,

$$p.v. \int_{-1}^{1} \frac{dx}{x} = \lim_{\epsilon\to 0} \Big(\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} \Big) = \lim_{\epsilon\to 0} (\log \epsilon - \log|-1| + \log 1 - \log \epsilon) = 0$$

Funtzio arrazionalen integral inpropioak

Izan bitez P eta Q polinomioak, $\deg Q \geq \deg P + 2$ izanik eta $Q(x) \neq 0$, $\forall x \in \mathbb{R}$. $F(z) = \dfrac{P(z)}{Q(z)}$ definituz, F-k ez dauka puntu singular errealik. $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$ kalkulatzeko integratuko dugu F aldagai konplexuko funtzioa γ_R bidean, γ_R $\Omega = \{z \in \mathbb{C} \colon |z| < R, 0 < \arg z < \pi\}$ multzoaren muga izanik,



eta gero limitea hartuko dugu R-k ∞-rantz jotzen duenean.

Lema 13.3. *Izan bitez $R_0 > 0$ eta f analitikoa $|z| > R_0$, $\operatorname{Im} z > 0$ multzoan. Baldin eta $\displaystyle\lim_{R\to\infty} \max_{|z|=R} |zf(z)| = 0$ bada, orduan*

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0,$$

C_R $|z| = R$, $\operatorname{Im} z > 0$ zirkunferentzierdia izanik.

Froga. Frogatuko dugu $\displaystyle\lim_{R\to\infty} \Big| \int_{C_R} f(z)\, dz \Big| = 0$ dela.

$$\begin{aligned} \Big| \int_{C_R} f(z)\, dz \Big| =& \Big| \int_{C_R} zf(z) \frac{dz}{z} \Big| \leq \int_{C_R} |zf(z)| \Big| \frac{dz}{z} \Big| \\ \leq& \max_{|z|=R} |zf(z)| \int_0^{\pi} \Big| \frac{Rie^{it}}{Re^{it}} \Big|\, dt \\ =& \pi \max_{|z|=R} |zf(z)| \to 0, \quad R \to \infty \text{ denean.} \end{aligned}$$

□

Korolarioa 13.4. *Izan bitez P, Q polinomioak, $\deg Q \geq \deg P + 2$ izanik. Orduan*

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{P(z)}{Q(z)}\,dz\right| = 0.$$

Froga. Izan bedi $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Orduan

$$\lim_{z\to\infty}\left|\frac{P(z)}{a_n z^n}\right| = 1$$

denez, existitzen da $R_1 > 0$ non $|z| > R_1$ denean,

$$\frac{1}{2}|a_n||z|^n \leq |P(z)| \leq 2|a_n||z|^n$$

Modu berean, $Q(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0$ bada, existitzen da $R_2 > 0$ non $|z| > R_1$ denean,

$$\frac{1}{2}|b_m||z|^n \leq |Q(z)| \leq 2|b_m||z|^m$$

Orduan, $R = \max\{R_1, R_2\}$ bada,

$$\max_{|z|=R}\left|z\frac{P(z)}{Q(z)}\right| \leq \max_{|z|=R}|z|\frac{2|a_n||z|^n}{\frac{1}{2}|b_m||z|^m} = \frac{4|a_n|}{|b_m|}|z|^{n-m+1} \to 0, \quad R\to\infty \text{ denean}$$

eta aurreko teoremaren arabera,

$$\lim_{R\to\infty}\int_{C_R}\frac{P(z)}{Q(z)}\,dz = 0.$$

□

Adibidea. $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$.

Kalkulatuko dugu $\displaystyle\int_{\gamma_R}\frac{dz}{1+z^2}$ integrala γ_R $|z| < R$, $\operatorname{Im} z > 0$ eremuaren muga izanik, eta $R > 1$. $F(z) = \dfrac{1}{z^2+1}$ funtzioaren puntu singularrak $z = i$ eta $z = -i$ dira, baina $z = -i$ kurbaren kanpoaldean dago, beraz, hondarren teoremaren arabera,

$$\int_{\gamma_R}\frac{dz}{z^2+1} = 2\pi i \operatorname*{Res}_{z=i}\frac{1}{z^2+1} = 2\pi i\frac{1}{2i} = \pi.$$

Bestalde,

$$\int_{\gamma_R}\frac{dz}{z^2+1} = \int_{L_R}\frac{dz}{z^2+1} + \int_{C_R}\frac{dz}{z^2+1},$$

non L_R ardatz errealaren gainean dagoen zuzenkia den eta C_R jatorrian zentratutako eta R erradiodun goiko zirkunferentzierdia.

$$\left|z\frac{1}{z^2+1}\right| \leq \frac{|z|}{|z|^2-1} = \frac{R}{R^2-1}, \quad |z| = R \text{ denean},$$

beraz, $\lim\limits_{R\to\infty}\max\limits_{|z|=R}|zF(z)|=0$ eta lemaren arabera,

$$\lim_{R\to\infty}\int_{C_R}\frac{dz}{z^2+1}=0.$$

Bestalde, zuzenkia $\gamma(t)=t$, $t\in[-R,R]$ parametrizazioaren bidez deskriba daiteke, integrala honela geratzen delarik,

$$\int_{L_R}\frac{dz}{z^2+1}=\int_{-R}^{R}\frac{dt}{t^2+1}.$$

Beraz, limiteak hartuz,

$$\pi=\lim_{R\to\infty}\int_{\gamma_R}\frac{dz}{z^2+1}=\lim_{R\to\infty}\int_{L_R}\frac{dz}{z^2+1}+\lim_{R\to\infty}\int_{C_R}\frac{dz}{z^2+1}=\int_{-\infty}^{\infty}\frac{dx}{x^2+1}.$$

Adibidea. $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^4+1}=\frac{\pi}{\sqrt{2}}.$

Funtzio arrazionalen eta trigonometrikoen arteko biderkadurak

Izan bitez P eta Q polinomioak, $\deg Q\geq\deg P+1$ eta $Q(x)\neq 0$, $\forall x\in\mathbb{R}$, eta $a>0$. $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cos(ax)\,dx$ edo $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin(ax)\,dx$ kalkulatzeko $F(z)=\dfrac{P(z)}{Q(z)}e^{iaz}$ funtzioa γ_R bidean integratuko dugu, γ_R, aurreko atalean bezala, $|z|<R$, $\operatorname{Im}z>0$ eremuaren muga izanik.

Lema 13.5 (Jordanen lema). *Izan bitez $R_0>0$ eta f analitikoa $|z|>R_0$, $\operatorname{Im}z>0$ eremuan. $\lim\limits_{R\to\infty}\max\limits_{|z|=R}|f(z)|=0$ baldin bada, orduan $\forall\,\lambda>0$*

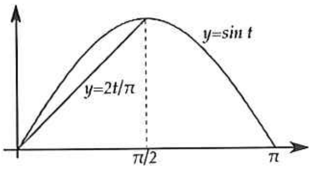
$$\lim_{R\to\infty}\int_{C_R}e^{i\lambda z}f(z)\,dz=0,$$

C_R jatorrian zentratutako eta R erradiodun goiko zirkunferentzierdia izanik.

Froga. Frogatuko dugu $\lim\limits_{R\to\infty}\left|\displaystyle\int_{C_R}e^{i\lambda z}f(z)\,dz\right|=0$ dela.

$$\begin{aligned}\left|\int_{C_R}e^{i\lambda z}f(z)\,dz\right|&\leq\int_{C_R}|e^{i\lambda z}||f(z)||dz|\leq\max_{|z|=R}|f(z)|\int_0^{\pi}|e^{i\lambda Re^{it}}||Rie^{it}|\,dt\\&=R\max_{|z|=R}|f(z)|\int_0^{\pi}e^{-\lambda R\sin t}\,dt=2R\max_{|z|=R}|f(z)|\int_0^{\pi/2}e^{-\lambda R\sin t}\,dt\end{aligned}$$

non, azken berdintzan $\sin t$ funtzioaren simetria erabili den.



Gainera, $t \in [0, \pi/2]$ denean, $\sin t \geq \frac{2t}{\pi}$, beraz, $\lambda > 0$ denez,

$$e^{-\lambda R \sin t} \leq e^{-\frac{2\lambda R}{\pi}t}, \quad 0 \leq t \leq \frac{\pi}{2} \text{ denean.}$$

Ondorioz,

$$\begin{aligned}\left|\int_{C_R} e^{i\lambda z} f(z)\,dz\right| \leq & 2R \max_{|z|=R} |f(z)| \int_0^{\pi/2} e^{-\frac{2\lambda R}{\pi}t}\,dt \\ = & 2R \max_{|z|=R} |f(z)| \frac{\pi}{2\lambda R}(1 - e^{-\lambda R}) \to 0, \quad R \to \infty \text{ denean.} \quad \square\end{aligned}$$

Adibidea. $\displaystyle\int_0^\infty \frac{x \sin ax}{x^2 + b^2}\,dx$, $a > 0$, $b > 0$.

$F(z) = \dfrac{ze^{iaz}}{z^2 + b^2}$ aldagai konplexuko funtzioa γ_R bidean integratuko dugu, γ_R $|z| < R$, $\operatorname{Im} z > 0$ eremuaren muga izanik.

F-ren puntu singularrak $z = bi$ eta $z = -bi$ dira, biak polo simpleak. $R > b$ bada, kurbaren barrualdean geratzen den bakarra $z = bi$ da, beraz, hondarren teoremaren arabera,

$$\int_{\gamma_R} \frac{ze^{iaz}}{z^2 + b^2}\,dz = 2\pi i \operatorname*{Res}_{z=bi} \frac{ze^{iaz}}{z^2 + b^2} = 2\pi i \frac{bie^{iaib}}{2bi} = e^{-ab}\pi i.$$

Bestalde, L_R $-R$-tik R-ra doan zuzenkia, eta C_R jatorrian zentratutako eta R erradiodun goiko zirkunferentzierdia baldin badira,

$$\int_{\gamma_R} \frac{ze^{iaz}}{z^2 + b^2}\,dz = \int_{L_R} \frac{ze^{iaz}}{z^2 + b^2}\,dz + \int_{C_R} \frac{ze^{iaz}}{z^2 + b^2}\,dz.$$

Ikus dezagun $f(z) = \dfrac{z}{z^2 + b^2}$ funtzioak Jordanen lemaren baldintza betetzen duela.

$$|f(z)| = \left|\frac{z}{z^2 + b^2}\right| \leq \frac{|z|}{|z^2 + b^2|} \leq \frac{|z|}{|z|^2 - b^2} = \frac{R}{R^2 - b^2}, \quad |z| = R \text{ denean,}$$

beraz $\lim\limits_{R\to\infty} \max\limits_{|z|=R} |f(z)| = 0$ eta

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{iaz}}{z^2 + b^2}\,dz = 0.$$

Gainera, zuzenkia parametrizatuz,

$$\int_{L_R} \frac{ze^{iaz}}{z^2+b^2}\,dz = \int_{-R}^{R} \frac{xe^{iax}}{x^2+b^2}\,dx = \int_{-R}^{R} \frac{x}{x^2+b^2}(\cos ax + i\sin ax)\,dx.$$

Beraz, limiteak hartuz,

$$\begin{aligned} e^{-ab}\pi i = \lim_{R\to\infty}\int_{\gamma_R} \frac{ze^{iaz}}{z^2+b^2}\,dz &= \lim_{R\to\infty}\int_{L_R} \frac{ze^{iaz}}{z^2+b^2}\,dz + \lim_{R\to\infty}\int_{C_R} \frac{ze^{iaz}}{z^2+b^2}\,dz \\ &= \int_{-\infty}^{\infty} \frac{x}{x^2+b^2}(\cos ax + i\sin ax)\,dx. \end{aligned}$$

$\frac{x\sin ax}{x^2+b^2}$ funtzio bikoitia denez, $\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx$. Zati irudikariak hartuz goiko berdintzan,

$$\int_0^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = \frac{e^{-ab}\pi}{2}.$$

Definizioa. Izan bedi $f\colon \mathbb{R} \to \mathbb{R}$. f-ren Fourierren transformatua honako funtzio hau da:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx = \int_{-\infty}^{\infty} \cos(\omega x) f(x)\,dx + i\int_{-\infty}^{\infty} sin(\omega x) f(x)\,dx.$$

Izan bedi f plano konplexu osoan analitikoa den funtzioa, agian puntu kopuru finitu batean izan ezik eta suposa dezagun

$$\lim_{R\to\infty} \max_{|z|=R} |f(z)| = 0$$

dela. Kalkulatuko dugu f-ren zuzen errealerako murrizketaren Fourierren transformatua.

$\omega > 0$ bada, kontsidera dezagun goian definitutako γ_R kurba, hau da, $|z| = R$ erradioko goiko zirkunferentzierdia eta $-R$ eta R puntuak batzen dituen zuzenkia. Izan bitez $\zeta_1, \ldots \zeta_m$ f-ren puntu singularrak, $\operatorname{Im}\zeta_j > 0$ izanik $j = 1, \ldots, m$ denean. Orduan, $R > \max\{|\zeta_1|, \ldots, |\zeta_m|\}$ hartuz, hondarren teoremaren arabera,

$$\int_{\gamma_R} e^{i\omega z} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=\zeta_j} e^{i\omega z} f(z).$$

Bestalde,

$$\int_{\gamma_R} e^{i\omega z} f(z)\,dz = \int_{-R}^{R} e^{i\omega x} f(x)\,dx + \int_{C_R} e^{i\omega z} f(z)\,dz,$$

eta Jordanen lemaren arabera,

$$\lim_{R\to\infty}\int_{C_R} e^{i\omega z} f(z)\,dz = 0$$

denez,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{i\omega x} f(x)\, dx = \sqrt{2\pi} i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} e^{i\omega z} f(z).$$

$\omega < 0$ bada, γ_R kurbaren gainean integratu beharrean, $|z| = R$ zirkunferentziaren beheko erdia hartu behar da $-R$ eta R puntuak batzen dituen zuzenkiarekin batera eta antzeko modu batean honako hau lortzen da

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx = -\sqrt{2\pi} i \sum_{j=1}^{p} \operatorname*{Res}_{z=\eta_j} e^{i\omega z} f(z),$$

$\eta_1, \ldots, \eta_p$ f-ren puntu singularrak izanik, $\mathrm{Im}(\eta_j) < 0$ delarik, $j = 1, \ldots, p$ denean. Kontuan izan orain zuzenkia eskuinetatik ezkerretara hartzen dela, eta horregatik zeinu negatiboa.

Adibidea. Kalkula dezagun $f(x) = \dfrac{x}{x^4+4}$ funtzioaren Fourierren transformatua.

$f(z) = \dfrac{z}{z^4+4}$ funtzioak lau puntu singular ditu: $z_1 = \sqrt{2}e^{\pi i/4} = 1+i$, $z_2 = \sqrt{2}e^{3\pi i/4} = -1+i$, $z_3 = \sqrt{2}e^{-\pi i/4} = 1-i$ eta $z_4 = \sqrt{2}e^{-3\pi i/4} = -1-i$.

$\omega > 0$ bada, parte irudikari positiboa duten puntu singularrak z_1 eta z_2 direnez,

$$\begin{aligned}
\hat{f}(\omega) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \frac{x}{x^4+4}\, dx \\
=& \sqrt{2\pi} i \Big(\operatorname*{Res}_{z=z_1} e^{i\omega z} \frac{z}{z^4+1} + \operatorname*{Res}_{z=z_2} e^{i\omega z} \frac{z}{z^4+1} \Big) \\
=& \sqrt{2\pi} i \Big(\frac{z e^{i\omega z}}{4z^3}_{|z=1+i} + \frac{z e^{i\omega z}}{4z^3}_{|z=-1+i} \Big) \\
=& \frac{\sqrt{2\pi} i}{4} \Big(\frac{e^{i(1+i)\omega}}{(1+i)^2} + \frac{e^{i(-1+i)\omega}}{(-1+i)^2} \Big) = \cdots = i \frac{\sqrt{2\pi} e^{-\omega} \sin\omega}{4}.
\end{aligned}$$

Antzeko modu batean, $\omega < 0$ bada, parte irudikari negatiboa duten puntu singularrak z_3 eta z_4 direnez,

$$\begin{aligned}
\hat{f}(\omega) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \frac{x}{x^4+4}\, dx \\
=& -\sqrt{2\pi} i \Big(\operatorname*{Res}_{z=z_3} e^{i\omega z} \frac{z}{z^4+1} + \operatorname*{Res}_{z=z_4} e^{i\omega z} \frac{z}{z^4+1} \Big) \\
=& -\sqrt{2\pi} i \Big(\frac{z e^{i\omega z}}{4z^3}_{|z=1-i} + \frac{z e^{i\omega z}}{4z^3}_{|z=-1-i} \Big) \\
=& -\frac{\sqrt{2\pi} i}{4} \Big(\frac{e^{i(1-i)\omega}}{(1-i)^2} + \frac{e^{i(-1-i)\omega}}{(-1-i)^2} \Big) = \cdots = i \frac{\sqrt{2\pi} e^{\omega} \sin\omega}{4}.
\end{aligned}$$

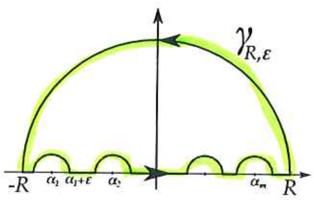
Hau da,

$$\hat{f}(\omega) = i\sqrt{\pi 2}\frac{e^{-|\omega|}\sin\omega}{2}, \quad \forall \omega \in \mathbb{R}.$$

Funtzio arrazionalen eta trigonometrikoen arteko biderkadura batzuen balio nagusiak

Izan bitez P eta Q polinomioak, $\deg Q \geq \deg P + 1$ eta Q-ren zero errealak $\cos ax$ edo $\sin ax$ funtzioaren zeroak izanik.

Kasu honetan $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos ax \, dx$ eta $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin ax \, dx$ orokorrean ez dira konbergenteak eta balio nagusiak kalkulatu daitezke bakarrik. Horretarako, $F(z) = \dfrac{P(z)}{Q(z)} e^{iaz}$ funtzioa integratzen da irudian agertzen den $\gamma_{R,\epsilon}$ bidean, $\alpha_1, \ldots, \alpha_n$ Q-ren zero errealak izanik.



Lema 13.6. *Izan bedi z_0 f-ren polo sinplea, eta izan bedi C_ϵ hurrengo moduan parametrizatu daitekeen z_0 puntuan zentratutako eta ϵ erradioko zirkunferentziaren arkua, $C_\epsilon(t) = z_0 + \epsilon e^{it}$, $t_0 \leq t \leq t_1$, $0 \leq t_0 < t_1 \leq 2\pi$ izanik. Orduan*

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) \, dz = i(t_1 - t_0) \operatorname*{Res}_{z=z_0} f(z).$$

Froga. $A = \operatorname*{Res}_{z=z_0} f(z)$ idazten badugu, $f(z) = \dfrac{A}{z - z_0} + h(z)$ dugu z_0-ren ingurune batean, h analitikoa izanik. Alde batetik,

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} h(z) \, dz = 0$$

da, h bornatua delako $|z - z_0| \leq \delta$ erako multzo batean; beste alde batetik, enuntziatuan ematen den kurbaren parametrizazioa erabiliz,

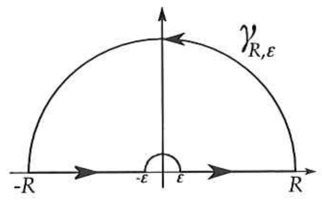
$$\int_{\gamma_\epsilon} \frac{A}{z - z_0} \, dz = \int_{t_0}^{t_1} \frac{A}{\epsilon e^{it}} i\epsilon e^{it} \, dt = (t_1 - t_0) iA. \qquad \square$$

Adibidez, C_ϵ zirkunferentzierdi bat bada, noranzko positiboan hartuta,

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) \, dz = i\pi \operatorname*{Res}_{z=z_0} f(z).$$

Adibidea. $\displaystyle\int_0^{\infty} \frac{\sin x}{x} \, dx.$

$f(z) = \dfrac{e^{iz}}{z}$ funtzioa $\gamma_{R,\epsilon}$ bidean integratuko dugu, $\gamma_{R,\epsilon}$ hurrengoa izanik:



f analitikoa da $\gamma_{R,\epsilon}$ bidearen ingurune batean, beraz, Cauchy-Goursaten teoremaren arabera, ϵ eta R guztietarako, $0 < \epsilon < R$ izanik,

$$\int_{\gamma_{R,\epsilon}} \frac{e^{iz}}{z}\, dz = 0.$$

Bestalde, zuzenkiak parametrizatuz,

$$\int_{\gamma_{R,\epsilon}} \frac{e^{iz}}{z}\, dz = \int_{\epsilon}^{R} \frac{e^{ix}}{x}\, dx + \int_{C_R} \frac{e^{iz}}{z}\, dz + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\, dx - \int_{C_{0,\epsilon}} \frac{e^{iz}}{z}\, dz,$$

non $C_{0,\epsilon}$ eta C_R jatorrian zentratutako goiko zirkunferentzierdiak diren, ϵ eta R erradiodunak, hurrenez hurren.

$\lim\limits_{R\to\infty} \max\limits_{|z|=R} \left|\frac{1}{z}\right| = \lim\limits_{R\to\infty} \frac{1}{R} = 0$ denez, Jordanen lemaren arabera

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, dz = 0.$$

Gainera, aurreko lemaren arabera, $z_0 = 0$ f-ren polo sinplea denez,

$$\lim_{\epsilon\to 0} \int_{C_{0,\epsilon}} \frac{e^{iz}}{z}\, dz = i\pi \operatorname*{Res}_{z=0} \frac{e^{iz}}{z} = i\pi e^{i0} = i\pi.$$

Bestalde, $\displaystyle\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\, dx$ integralean, $x = -t$ aldaketa eginez,

$$\begin{aligned}\int_{\epsilon}^{R} \frac{e^{ix}}{x}\, dx + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\, dx &= \int_{\epsilon}^{R} \frac{e^{ix}}{x}\, dx + \int_{R}^{\epsilon} \frac{e^{-it}}{-t}\,(-dt) = \int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\, dx \\ &= 2i \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx.\end{aligned}$$

Orduan, limiteak hartuz, $R \to \infty$ eta $\epsilon \to 0$ direnean,

$$0 = \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{\gamma_{R,\epsilon}} \frac{e^{iz}}{z}\, dz = 2i \int_0^{\infty} \frac{\sin x}{x}\, dx + 0 - i\pi.$$

Beraz, $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$

Adibidea. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$

Integral hau kalkulatzeko, lehenengo eta behin, integrakizuna berridatzi behar dugu:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{1-\cos 2x}{x^2}\,dx.$$

Orain, $f(z) = \dfrac{1-e^{2iz}}{z^2}$ funtzioa integratuko dugu aurreko adibideko $\gamma_{R,\epsilon}$ bidean. Cauchy-Goursaten teoremaren arabera,

$$\int_{\gamma_{R,\epsilon}} \frac{1-e^{2iz}}{z^2}\,dz = 0.$$

Bestalde, C_R eta $C_{0,\epsilon}$ aurreko adibidean bezala definituz,

$$\begin{aligned}&\int_{\gamma_{R,\epsilon}} \frac{1-e^{2iz}}{z^2}\,dz\\ &\quad= \int_\epsilon^R \frac{1-e^{2ix}}{x^2}\,dx + \int_{C_R} \frac{1-e^{2iz}}{z^2}\,dz + \int_{-R}^{-\epsilon} \frac{1-e^{2ix}}{x^2}\,dx - \int_{C_{0,\epsilon}} \frac{1-e^{2iz}}{z^2}\,dz.\end{aligned}$$

$\displaystyle\lim_{R\to\infty}\max_{|z|=R}\left|z\,\frac{1-e^{2iz}}{z^2}\right| = \lim_{R\to\infty}\max_{t\in[0,\pi]}\frac{1}{R}(1-e^{-2R\sin t}) \leq \lim_{R\to\infty}\frac{2}{R} = 0$, beraz

$$\lim_{R\to\infty}\int_{C_R} \frac{1-e^{2iz}}{z^2}\,dz = 0.$$

Gainera, $z_0 = 0$ f-ren polo sinplea denez

$$\begin{aligned}\int_{C_{0,\epsilon}} \frac{1-e^{2iz}}{z^2}\,dz &= \pi i \operatorname*{Res}_{z=0}\frac{1-e^{2iz}}{z^2} = \pi i \lim_{z\to 0} z\frac{1-e^{2iz}}{z^2}\\ &= \pi i \lim_{z\to 0}\frac{1-e^{2iz}}{z} = \pi i(-2i) = 2\pi.\end{aligned}$$

Orduan, limiteak hartuz,

$$0 = \lim_{\substack{R\to\infty\\ \epsilon\to 0}}\int_{\gamma_{R,\epsilon}} \frac{1-e^{2iz}}{z^2}\,dz = \int_0^\infty \frac{1-e^{2ix}}{x^2}\,dx + 0 + \int_{-\infty}^0 \frac{1-e^{2ix}}{x^2}\,dx - 2\pi.$$

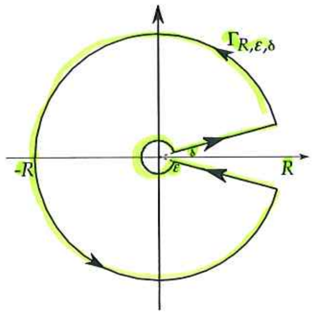
Zati errealak hartuz,

$$\begin{aligned}0 = \int_{-\infty}^\infty \frac{1-\cos 2x}{x^2}\,dx - 2\pi \quad &\Longrightarrow \quad 2\int_0^\infty \frac{1-\cos 2x}{x^2}\,dx = 2\pi\\ &\Longrightarrow \quad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{1-\cos 2x}{x^2}\,dx = \frac{\pi}{2}.\end{aligned}$$

Funtzio arrazionalen eta berretura ez-osoen arteko biderkadurak

Izan bitez P, Q polinomioak, Q erro erreal ez-negatiborik ez dituena, $a \in \mathbb{R} - \mathbb{Z}$, eta demagun $\lim_{x\to 0} \frac{P(x)}{Q(x)} x^{1+a} = \lim_{x\to\infty} \frac{P(x)}{Q(x)} x^{1+a} = 0$.

$\int_0^\infty \frac{P(x)}{Q(x)} x^a \, dx$ integrala kalkulatzeko $F(z) = \frac{P(z)}{Q(z)} z^a$ aldagai konplexuko funtzioa integratuko dugu $\Gamma_{R,\epsilon,\delta}$ bidean. $z^a = e^{a \log z}$ moduan definitzen da, non $\log z = \log|z| + i\theta(z)$ den, $\theta(z) \in \operatorname{Arg} z \cap [0, 2\pi)$ izanik. $\Gamma_{R,\epsilon,\delta}$ bidea hurrengo irudian agertzen dena da:

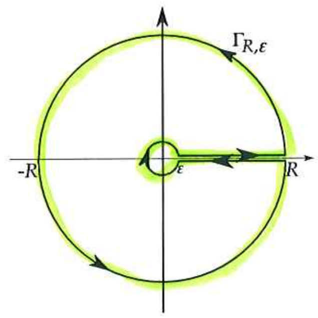


$\Gamma_{R,\epsilon,\delta}$ parametrizatzeko lau zati kontsideratzen dira: bi zirkunferentzien arkuak eta bi zuzenkiak. Zuzenkien kasuan, honako hau da parametrizazio bat:

$$\begin{aligned} L_1(t) =& te^{i\delta}, \quad t \in [\epsilon, R], \\ L_2(t) =& te^{(2\pi-\delta)i}, \quad t \in [\epsilon, R]. \end{aligned}$$

Gero, limiteak hartu behar dira $\mathbb{R} \to \infty$, $\epsilon \to 0$ eta $\delta \to 0$ direnean.

Normalean, suposatzen da $\delta = 0$ dela eta L_1 zuzenkian argumentua 0 eta L_2 zuzenkian argumentua 2π direla kontsideratzen da. Beraz, $\Gamma_{R,\epsilon} = \gamma_1 + \gamma_R - \gamma_2 - \gamma_\epsilon$ bidearen gainean integratzen dugu, non γ_R eta γ_ϵ bideak jatorrian zentratutako eta R eta ϵ erradioetako zirkunferentziak diren, hurrenez hurren eta $\gamma_1(t) = te^{0i}$, $t \in [\epsilon, R]$, $\gamma_2(t) = te^{2\pi i}$, $t \in [\epsilon, R]$ zuzenkiak.



Adibidea. $\displaystyle\int_0^\infty \frac{x^a}{x+b}\,dx$, $b>0$ eta $-1<a<0$ izanik.

$F(z)=\dfrac{z^a}{z+b}$ funtzioa integratuko dugu $\Gamma_{R,\epsilon}$ bidean, $z^a=e^{a\log z}$ izanik, non $\log z=\log|z|+i\theta(z)$, $\theta(z)\in \operatorname{Arg} z\cap(0,2\pi)$.

Hondarren teoremaren arabera, $\epsilon<b<R$ badira,

$$\begin{aligned}\int_{\Gamma_{R,\epsilon}}\frac{z^a}{z+b}\,dz =&2\pi i \operatorname*{Res}_{z=-b}\frac{z^a}{z+b}=2\pi i\lim_{z\to -b}(z+b)\frac{z^a}{z+b}\\ =&2\pi i(-b)^a=2\pi i e^{a\log(-b)}=2\pi i e^{a(\log b+\pi i)}\\ =&2\pi i b^a e^{a\pi i}.\end{aligned}$$

Bestalde, bidearen parametrizazioa erabiliz,

$$\begin{aligned}\int_{\Gamma_{R,\epsilon}}\frac{z^a}{z+b}\,dz =&\int_\epsilon^R \frac{e^{a(\log x+0i)}}{x+b}e^{0i}dx+\int_{\gamma_R}\frac{z^a}{z+b}\,dz\\ &-\int_\epsilon^R\frac{e^{a(\log x+2\pi i)}}{xe^{2\pi i}+b}e^{2\pi i}dx-\int_{\gamma_\epsilon}\frac{z^a}{z+b}\,dz.\end{aligned}$$

Gainera, $a<0$ denez,

$$\left|\int_{\gamma_R}\frac{z^a}{z+b}\,dz\right|\le\int_{\gamma_R}\left|\frac{z^a}{z+b}\right||dz|\le\int_{\gamma_R}\frac{e^{a\log|z|}}{|z|-b}|dz|=\frac{R^a}{R-b}2\pi R\to 0,\quad R\to\infty,$$

Antzera, $a>-1$ denez,

$$\left|\int_{\gamma_\epsilon}\frac{z^a}{z+b}\,dz\right|\le\int_{\gamma_\epsilon}\left|\frac{z^a}{z+b}\right||dz|\le\int_{\gamma_\epsilon}\frac{e^{a\log|z|}}{b-|z|}|dz|=\frac{\epsilon^a}{b-\epsilon}2\pi\epsilon\to 0,\quad \epsilon\to 0 \text{ denean.}$$

Beraz, limiteak hartuz,

$$2\pi i b^a e^{\pi a i}=\lim_{\substack{R\to\infty\\ \epsilon\to 0}}\int_{\Gamma_{R,\epsilon}}\frac{z^a}{z+b}\,dz=\int_0^\infty\frac{x^a}{x+b}\,dx-e^{2\pi a i}\int_0^\infty\frac{x^a}{x+b}\,dx.$$

Hau da,

$$\int_0^\infty\frac{x^a}{x+b}\,dx=2\pi i b^a\frac{e^{\pi a i}}{1-e^{2\pi a i}}=\frac{2\pi i b^a}{e^{-\pi a i}-e^{\pi a i}}=\frac{\pi b^a}{\sin(-\pi a)}.$$

Funtzio arrazionalen eta logaritmoen arteko biderkadurak

Izan bitez P eta Q polinomioak, $\deg Q\ge \deg P+2$ eta $Q(x)\ne 0$, $\forall x\ge 0$. $\displaystyle\int_0^\infty\frac{P(x)}{Q(x)}\ln x\,dx$ moduko integralak kalkulatzeko, aurreko kasuan bezala, saia gaitezke

$\int_{\Gamma_{R,\epsilon}} \frac{P(z)}{Q(z)} \log z \, dz$ integral konplexua erabiltzen non $\log z$ $\mathbb{C} - [0, \infty)$ multzoan holomorfoa den logaritmoaren adar bat den, $\log z = \ln |z| + i\theta(z)$, $\theta(z) \in \operatorname{Arg} z \cap (0, 2\pi)$. Haatik, bidearen parametrizazioa erabiliz,

$$\begin{aligned}\int_{\Gamma_{R,\epsilon}} \frac{P(z)}{Q(z)} \log z \, dz &= \int_\epsilon^R \frac{P(x)}{Q(x)} \ln x \, dx + \int_{\gamma_R} \frac{P(z)}{Q(z)} \log z \, dz \\ &\quad - \int_\epsilon^R \frac{P(x)}{Q(x)} (\ln x + 2\pi i) \, dx - \int_{\gamma_\epsilon} \frac{P(z)}{Q(z)} \log z \, dz\end{aligned}$$

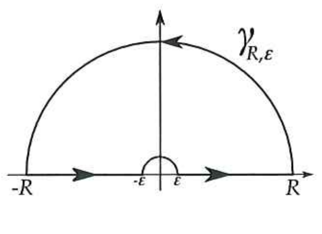
eta limiteak hartzean, kalkulatu nahi dugun integrala desagertzen da kontrako zeinuekin agertzen delako.

Konponbidea da $F(z) = \frac{P(z)}{Q(z)} \log^2 z$ funtzioa integratzea $\Gamma_{R,\epsilon}$ kurban, zeren eta, parametrizazioa erabiltzean,

$$\begin{aligned}\int_{\Gamma_{R,\epsilon}} \frac{P(z)}{Q(z)} \log^2 z \, dz &= \int_\epsilon^R \frac{P(x)}{Q(x)} \ln^2 x \, dx + \int_{\gamma_R} \frac{P(z)}{Q(z)} \log^2 z \, dz \\ &\quad - \int_\epsilon^R \frac{P(x)}{Q(x)} (\ln x + 2\pi i)^2 \, dx - \int_{\gamma_\epsilon} \frac{P(z)}{Q(z)} \log^2 z \, dz \\ &= \int_\epsilon^R \frac{P(x)}{Q(x)} \ln^2 x \, dx + \int_{\gamma_R} \frac{P(z)}{Q(z)} \log^2 z \, dz \\ &\quad - \int_\epsilon^R \frac{P(x)}{Q(x)} (\ln^2 x - 4\pi^2 + 4\pi i \ln x) \, dx - \int_{\gamma_\epsilon} \frac{P(z)}{Q(z)} \log^2 z \, dz.\end{aligned}$$

Orain, $\ln^2 x$ desagertu egiten da, baina $\ln x$ oraindik dago integral batean.

$\frac{P(x)}{Q(x)}$ bikoitia bada, badago beste aukera bat. $F(z) = \frac{P(z)}{Q(z)} \log z$ integra daiteke $\gamma_{R,\epsilon}$ kurba berrian,



Kasu honetan,

$$
\begin{aligned}
\int_{\gamma_{R,\epsilon}} \frac{P(z)}{Q(z)} \log z \, dz = & \int_{\epsilon}^{R} \frac{P(x)}{Q(x)} \ln x \, dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} \log z \, dz \\
& - \int_{\epsilon}^{R} \frac{P(-x)}{Q(-x)} (\ln x + \pi i) \, e^{\pi i} dx - \int_{\Gamma_\epsilon} \frac{P(z)}{Q(z)} \log z \, dz \\
= & 2 \int_{\epsilon}^{R} \frac{P(x)}{Q(x)} \ln x \, dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} \log z \, dz \\
& + \int_{\epsilon}^{R} \frac{P(x)}{Q(x)} \pi i \, dx - \int_{\Gamma_\epsilon} \frac{P(z)}{Q(z)} \log z \, dz.
\end{aligned}
$$

Adibidea. $\displaystyle\int_0^\infty \frac{\ln x}{x^2+b^2} \, dx$, $b > 0$ izanik.

Kalkulatuko dugu integral hau ikusi ditugun bi metodoen bidez.

Integra dezakegu $F(z) = \dfrac{\log^2 z}{z^2+b^2}$ funtzioa $\gamma_{R,\epsilon}$ bidean. F-k polo sinpleak ditu bi eta $-bi$ puntuetan. Hondarren teoremaren arabera,

$$
\begin{aligned}
\int_{\gamma_{R,\epsilon}} \frac{\log^2 z}{z^2+b^2} \, dz = & 2\pi i \Big(\operatorname*{Res}_{z=bi} \frac{\log^2 z}{z^2+b^2} + \operatorname*{Res}_{z=-bi} \frac{\log^2 z}{z^2+b^2} \Big) \\
= & 2\pi i \Big(\frac{\log^2 z}{2z} \Big|_{z=bi} + \frac{\log^2 z}{2z} \Big|_{z=-bi} \Big) \\
= & 2\pi i \Big(\frac{(\ln b + \frac{\pi}{2} i)^2}{2bi} + \frac{(\ln b + \frac{3\pi}{2} i)^2}{-2bi} \Big) \\
= & 2\pi i \frac{(\ln^2 b - \frac{\pi^2}{4} + \pi i \ln b) - (\ln^2 b - \frac{9\pi^2}{4} + 3\pi i \ln b)}{2bi} \\
= & \frac{\pi}{b} (2\pi^2 - 2\pi i \ln b).
\end{aligned}
$$

Bestalde,

$$
\begin{aligned}
\int_{\gamma_{R,\epsilon}} \frac{\log^2 z}{z^2+b^2} \, dz = & \int_{\epsilon}^{R} \frac{\ln^2 x}{x^2+b^2} \, dx + \int_{\gamma_R} \frac{\log^2 z}{z^2+b^2} \, dz \\
& - \int_{\epsilon}^{R} \frac{(\ln x + 2\pi i)^2}{x^2+b^2} \, dx - \int_{\gamma_\epsilon} \frac{\log^2 z}{z^2+b^2} \, dz \\
= & \int_{\gamma_R} \frac{\log^2 z}{z^2+b^2} \, dz + \int_{\epsilon}^{R} \frac{4\pi^2}{x^2+b^2} \, dx \\
& - 4\pi i \int_{\epsilon}^{R} \frac{\ln x}{x^2+b^2} \, dx - \int_{\gamma_\epsilon} \frac{\log^2 z}{z^2+b^2} \, dz.
\end{aligned}
$$

Gainera,

$$\Big|\int_{\gamma_R}\frac{\log^2 z}{z^2+b^2}dz\Big|\le\int_{\gamma_R}\Big|\frac{\log^2 z}{z^2+b^2}\Big||dz|\le\int_{\gamma_R}\frac{|\ln|z|+i\theta(z)|^2}{|z|^2-b^2}|dz|\le\frac{(\ln R+2\pi)^2}{R^2-b^2}2\pi R,$$
$$\Big|\int_{\gamma_\epsilon}\frac{\log^2 z}{z^2+b^2}dz\Big|\le\int_{\gamma_\epsilon}\Big|\frac{\log^2 z}{z^2+b^2}\Big||dz|\le\int_{\gamma_R}\frac{|\ln|z|+i\theta(z)|^2}{b^2-|z|^2}|dz|\le\frac{(\ln\epsilon+2\pi)^2}{b^2-\epsilon^2}2\pi\epsilon.$$

Beraz,

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{\log^2 z}{z^2+b^2}\,dz=\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{\log^2 z}{z^2+b^2}\,dz=0$$

eta limiteak hartuz goiko berdintzan

$$\frac{\pi}{b}(2\pi^2-4\pi i\log b)=\lim_{\substack{R\to\infty\\ \epsilon\to 0}}\int_{\gamma_{R,\epsilon}}\frac{\log^2 z}{z^2+b^2}\,dz=\int_0^\infty\frac{4\pi^2}{x^2+b^2}\,dx-2\pi i\int_0^\infty\frac{\ln x}{x^2+b^2}\,dx.$$

Azkenik, zati irudikariak hartuz,

$$-4\pi\int_0^\infty\frac{\ln x}{x^2+b^2}\,dx=-\frac{2\pi^2\ln b}{b}\quad\Longrightarrow\quad\int_0^\infty\frac{\ln x}{x^2+b^2}\,dx=\frac{\pi\ln b}{2b}.$$

$\dfrac{1}{x^2+b^2}$ bikoitia denez, beste aukera bat da $\dfrac{\log z}{z^2+b^2}$ funtzioa integratzea beste bide baten gainean, irudiko $\Gamma_{R,\epsilon}$ bidean hain zuzen ere. Funtzio honek polo sinpleak ditu ere bi eta $-bi$ puntuetan baina orain kurbaren barrualdean bakarrik bi polo sinplea geratzen da. Beraz, hondarren teoremaren arabera,

$$\int_{\Gamma_{R,\epsilon}}\frac{\log z}{z^2+b^2}\,dz=2\pi i\operatorname*{Res}_{z=bi}\frac{\log z}{z^2+b^2}=2\pi i\frac{\log(bi)}{2bi}=\frac{\pi}{b}(\ln b+\frac{\pi}{2}i).$$

Bestalde,

$$\begin{aligned}\int_{\Gamma_{R,\epsilon}}\frac{\log z}{z^2+b^2}\,dz=&\int_\epsilon^R\frac{\log x}{x^2+b^2}\,dx+\int_{\Gamma_R}\frac{\log z}{z^2+b^2}\,dz\\&-\int_\epsilon^R\frac{\ln x+\pi i}{x^2e^{2\pi i}+b^2}\,e^{\pi i}\,dx-\int_{\Gamma_\epsilon}\frac{\log z}{z^2+b^2}\,dz\\=&2\int_\epsilon^R\frac{\ln x}{x^2+b^2}\,dx+\int_{\gamma_R}\frac{\log z}{z^2+b^2}\,dz\\&+\int_\epsilon^R\frac{\pi i}{x^2+b^2}\,dx-\int_{\gamma_\epsilon}\frac{\log z}{z^2+b^2}\,dz.\end{aligned}$$

Lehen bezala, frogatzen da $\lim\limits_{R\to\infty}\displaystyle\int_{\Gamma_R}\frac{\log z}{z^2+b^2}\,dz=\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\frac{\log z}{z^2+b^2}\,dz=0$ direla, eta ondorioz, limiteak hartuz,

$$\frac{\pi}{b}(\ln b+\frac{\pi}{2}i)=\lim_{\substack{R\to\infty\\ \epsilon\to 0}}\int_{\Gamma_{R,\epsilon}}\frac{\log z}{z^2+b^2}\,dz=2\int_0^\infty\frac{\log x}{x^2+b^2}\,dx+\pi i\int_0^\infty\frac{dx}{x^2+b^2}.$$

Zati errealak hartuz,

$$2\int_0^\infty \frac{\log x}{x^2+b^2}\,dx = \frac{\pi}{b}\log b \quad \Longrightarrow \quad \int_0^\infty \frac{\ln x}{x^2+b^2}\,dx = \frac{\pi}{2b}\ln b.$$

Laplaceren transformatua eta alderantzizko transformatua

Definizioa. Izan bedi $f\colon (0,\infty) \to \mathbb{R}$ aldagai errealeko funtzioa. Bere Laplaceren transformatua honela definiten da:

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

Proposizioa 13.7. *Izan bedi F aldagai konplexuko funtzio analitikoa plano konplexu osoan agian puntu kopuru finitu batean izan ezik. Izan bedi L_R $\gamma - iR$ eta $\gamma + iR$ puntuak batzen dituen zuzenki bertikala, γ zenbaki erreala behar den bezain handia izanik F-ren puntu singular guztiak L_R-ren ezkerraldean gera daitezen. Defini dezagun*

$$\begin{aligned} f(t) =& \frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{zt}F(z)\,dz \\ =& \frac{1}{2\pi i} p.v. \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}F(z)\,dz, \qquad t>0. \end{aligned}$$

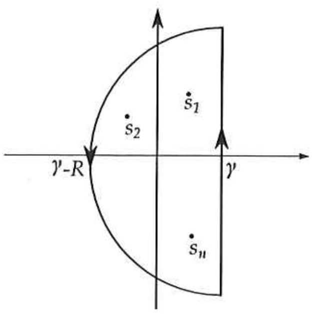
f funtzioa F-ren Laplaceren alderantzizko transformatua da, hau da,

$$F(z) = \int_0^\infty e^{-zx} f(x)\,dx$$

bada, orduan

$$f(t) = \frac{1}{2\pi i} p.v. \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}F(z)\,dz, \qquad t>0.$$

Izan bitez $s_1,\dots,s_n$ F-ren puntu singularrak eta izan bitez $\gamma > \max\{\operatorname{Re} s_j : j = 1,\dots,n\}$ eta $R_0 > \max\{|s_j| : i = 1,\dots,n\}$. Izan bedi Γ_R irudian agertzen den bidea, $R > \gamma + R_0$ izanik, F-ren puntu singular guztiak Γ_R-ren barruan gera daitezen.



Hondarren teoremaren arabera,

$$\int_{\Gamma_R} e^{zt}F(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z=s_j}(e^{zt}F(z)).$$

Existitzen bada $M_R \geq 0$ non $|F(z)| \leq M_R$ den $z \in C_R$ guztietarako, C_R $|z-\gamma| = R$ zirkunferentziaren ezkerreko erdia izanik, orduan froga daiteke

$$\lim_{R\to\infty} \int_{C_R} e^{zt}F(z)\,dz = 0$$

dela eta ondorioz,

$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{L_R} e^{zt}F(z)\,dz = \sum_{j=1}^{n} \operatorname*{Res}_{z=s_j}(e^{zt}F(z)).$$

Adibidea. $F(z) = \dfrac{z}{(z^2+a^2)^2}$ baldin bada, $a > 0$ izanik, kalkula dezagun bere Laplaceren alderantzizko transformatua,

$$f(t) = \frac{1}{2\pi i} p.v. \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{ze^{zt}}{(z^2+a^2)^2}\,dz, \quad t > 0.$$

(*) Goian ikusitakoaren arabera, $\dfrac{ze^{zt}}{(z^2+a^2)^2}$ funtzioaren hondarrak kalkulatu behar ditugu bere puntu singularretan, $z = ai$ eta $z = -ai$ hain zuzen ere, biak polo bikoitzak direlarik.

$$\begin{aligned}
\operatorname*{Res}_{z=ai}\Big(\frac{ze^{zt}}{(z^2+a^2)^2}\Big) &= \frac{d}{dz}\Big((z-ai)^2\frac{ze^{zt}}{(z^2+a^2)^2}\Big)_{|z=ai} \\
&= \frac{d}{dz}\Big(\frac{ze^{zt}}{(z+ai)^2}\Big)_{|z=ai} \\
&= \frac{(tze^{zt}+e^{zt})(z+ai)^2 - 2(z+ai)ze^{zt}}{(z+ai)^4}_{|z=ai} \\
&= \ldots = \frac{te^{ati}}{4ai} \\
\operatorname*{Res}_{z=-ai}\Big(\frac{ze^{zt}}{(z^2+a^2)^2}\Big) &= \frac{d}{dz}\Big((z+ai)^2\frac{ze^{zt}}{(z^2+a^2)^2}\Big)_{|z=-ai} \\
&= \frac{d}{dz}\Big(\frac{ze^{zt}}{(z-ai)^2}\Big)_{|z=-ai} \\
&= \frac{(tze^{zt}+e^{zt})(z-ai)^2 - 2(z-ai)ze^{zt}}{(z-ai)^4}_{|z=-ai} \\
&= \ldots = -\frac{te^{-ati}}{4ai}
\end{aligned}$$

(*) Hori baino lehen, egiaztatu,

$|z| = R$, $\quad |F(z)| = \dfrac{R}{|z^2+a^2|^2} \leq \dfrac{R}{(R^2-a^2)^2} \xrightarrow{R\to\infty} 0$

bestela alferrik gabiltza lanik egiten!

Beraz, $|F(z)| = \dfrac{|z|}{|z^2+a^2|^2} \leq \dfrac{|z|}{(|z|^2-a^2)^2} = \dfrac{R}{(R^2-a^2)^2}$, $|z| = R$ denean, eta $\lim\limits_{R\to\infty} \dfrac{R}{(R^2-a^2)^2} = 0$ denez, aurrekoaren arabera

$$f(t) = \operatorname*{Res}_{z=ai}\Big(\frac{ze^{zt}}{(z^2+a^2)^2}\Big) + \operatorname*{Res}_{z=-ai}\Big(\frac{ze^{zt}}{(z^2+a^2)^2}\Big) = \frac{te^{ati}}{4ai} - \frac{te^{-ati}}{4ai} = \frac{t\sin(at)}{2a}, \quad t > 0.$$

ANALISI BEKTORIALA ETA KONPLEXUA

13. Gaia: HONDARRAK

Ariketak

1. Kalkula itzazu:

   (i) $\operatorname*{Res}_{z=\infty} \frac{\sin z}{z^2}$ *Em.:* -1.

   (ii) $\operatorname*{Res}_{z=1} \frac{e^z}{(z-1)^2}$ *Em.:* e.

   (iii) $\operatorname*{Res}_{z=\infty} z^2 \sin \frac{\pi}{z}$ *Em.:* $\frac{\pi^3}{6}$.

   (iv) $\operatorname*{Res}_{z=1} z e^{\frac{1}{z-1}}$ *Em.:* $\frac{3}{2}$.

2. Kalkula itzazu hurrengo funtzioen hondarrak puntu singular isolatu finitu guztietan:

   (i) $\frac{1}{z^3+z}$ *Em.:* $\operatorname*{Res}_{z=0} \frac{1}{z^3+z} = 1$; $\operatorname*{Res}_{z=\pm i} \frac{1}{z^3+z} = \frac{1}{2}$.

   (ii) $\frac{z^2}{1+z^4}$ *Em.:* $\operatorname*{Res}_{z=e^{\pm\frac{\pi}{4}i}} \frac{z^2}{1+z^4} = \frac{1\mp i}{4\sqrt{2}}$; $\operatorname*{Res}_{z=e^{\pm\frac{3\pi}{4}i}} \frac{z^2}{1+z^4} = -\frac{1\pm i}{4\sqrt{2}}$.

   (iii) $\frac{1}{(z^2+1)^3}$ *Em.:* $\operatorname*{Res}_{z=\pm i} \frac{1}{(z^2+1)^3} = \mp\frac{3i}{16}$.

   (iv) $\frac{\sin \pi z}{(z-1)^3}$ *Em.:* $\operatorname*{Res}_{z=1} \frac{\sin \pi z}{(z-1)^3} = 0$.

   (v) $\frac{1}{\sin z^2}$ LOL *Em.:* $\operatorname*{Res}_{z=0} \frac{1}{\sin z^2} = 0$, $\operatorname*{Res}_{z=\pm(i)\sqrt{k\pi}} \frac{1}{\sin z^2} = \mp\frac{(-1)^k(i)}{2\sqrt{k\pi}}$, $k \in \mathbf{N}$;

   (vi) $\frac{1}{e^z+1}$ *Em.:* $\operatorname*{Res}_{z=(2k+1)\pi i} \frac{1}{e^z+1} = -1$, $\forall\, k \in \mathbf{Z}$.

   (vii) $\frac{1}{1-e^{z^2}}$ *Em.:* $\operatorname*{Res}_{z=0} \frac{1}{1-e^{z^2}} = 0$; $\operatorname*{Res}_{z=(\pm1\pm i)\sqrt{k\pi}} \frac{1}{1-e^{z^2}} = -\frac{1}{2(\pm1\pm i)\sqrt{k\pi}}$, $k \in \mathbf{N}$.

   LOL

   (viii) $\frac{z^{n-1}}{z^n+a^n}$ *Em.:* $\operatorname*{Res}_{z=ae^{\frac{(2k+1)\pi}{2}i}} \frac{z^{n-1}}{z^n+a^n} = \frac{1}{n}$, $k = 0, \ldots, n-1$.

   (ix) $\frac{e^{imz}}{(z^2+a^2)^2}$ *Em.:* $\operatorname*{Res}_{z=\pm ai} \frac{e^{imz}}{(z^2+a^2)^2} = -\frac{e^{\pm ma}(am\pm1)}{4a^3}i$.

3. Kalkula itzazu hurrengo funtzioen hondarrak puntu singular isolatu guztietan, ∞ barne:

   (i) $\frac{1+z^8}{z^6(z+2)}$ *Em.:* $\operatorname*{Res}_{z=0} \frac{1+z^8}{z^6(z+2)} = -\frac{1}{64}$, $\operatorname*{Res}_{z=-2} \frac{1+z^8}{z^6(z+2)} = \frac{257}{64}$, $\operatorname*{Res}_{z=\infty} \frac{1+z^8}{z^6(z+2)} = -4$.

   (ii) $\sin z \sin \frac{1}{z}$ *Em.:* $\operatorname*{Res}_{z=0} \sin z \sin \frac{1}{z} = 0$, $\operatorname*{Res}_{z=\infty} \sin z \sin \frac{1}{z} = 0$.

   (iii) $\frac{\sin z}{(z^2+1)^2}$ *Em.:* $\operatorname*{Res}_{z=\pm i} \frac{\sin z}{(z^2+1)^2} = -\frac{1}{4e}$, $\operatorname*{Res}_{z=\infty} \frac{\sin z}{(z^2+1)^2} = \frac{1}{2e}$.

4. Kalkulatu integral hauek:

(i) $\displaystyle\int_{|z|=2} \frac{dz}{(z+1)^2(z^2+2)}$ *Em.:* $\cancel{-\frac{32\pi i}{9}}$ 0.

(ii) $\displaystyle\int_{|z+1|=4} \frac{z}{e^z+3}\, dz$ *Em.:* $-i\frac{4\pi}{3}\ln 3$

(iii) $\displaystyle\int_{|z|=2} \frac{e^z}{z^3(1+z)}\, dz$ *Em.:* $\frac{e-2}{e}\pi i$

(iv) $\displaystyle\int_{|z|=1} \frac{z^2}{\sin^3 z \cos z}\, dz$ *Em.:* $2\pi i$

(v) $\displaystyle\int_{|z|=1/3} (z+1)e^{1/z}\, dz$ *Em.:* $3\pi i$

5. Kalkula itzazu

(i) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\sin^2\theta}$ *Em.:* $\frac{\pi}{2\sqrt{2}}$.

(ii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}, \quad |a| \neq 1$ *Em.:* $\frac{2\pi}{|a^2-1|}$.

6. Kalkula itzazu

(i) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}, \quad a>0,\ b>0,\ a\neq b;$ *Em.:* $\pi\frac{2b^3+a^3-3ab^2}{2ab^3(b^2-a^2)^2}$.

(ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos^2 x\, dx}{(x^2+a^2)(x^2+b^2)}, \quad a,b>0,\ a\neq b$ *Em.:* $\pi\frac{b-a+be^{-2a}-ae^{-2b}}{2ab(b^2-a^2)}$.

(iii) $\displaystyle\int_0^{+\infty} \frac{\sin \pi x}{x(1-x^2)}\, dx$ *Em.:* π.

(iv) $\displaystyle\int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2}\, dx, \quad a>0,\ b>0,\ a\neq b$ *Em.:* $\frac{\pi(b-a)}{2}$.

(v) p.v. $\displaystyle\int_0^{+\infty} \frac{\cos x}{a^2-x^2}\, dx, \quad a>0$ *Em.:* $\frac{\pi \sin a}{2a}$.

7. Kalkula ezazu $\frac{1}{(x^2+b^2)^2}$ funtzioaren Fourierren transformatua, $b>0$ izanik.

Em.: $\hat{f}(\omega) = \frac{1+\omega b}{4b^3 e^{\omega b}}\pi$.

8. Kalkula itzazu

(i) $\displaystyle\int_0^{+\infty} \frac{x^{a-1}}{x^2+2x+2}\, dx, \quad 0<a<2$ *Em.:* $\dfrac{\cos\frac{\pi a}{4} - \sin\frac{\pi a}{4}}{\sin \pi a} 2^{\frac{a-2}{2}}\pi$.

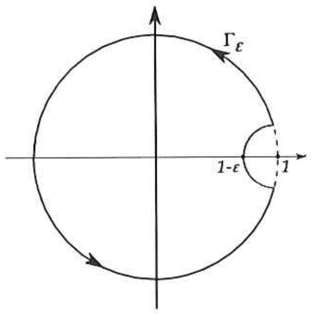
(ii) $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}}{(x+1)(x+2)}\, dx$ *Em.:* $(\sqrt{2}-1)\pi$.

(iii) $\displaystyle\int_0^{+\infty} \frac{\log x}{x^2-1}\, dx$ *Em.:* $\frac{\pi^2}{4}$.

(iv) $\displaystyle\int_0^{+\infty} \frac{\log^2 x}{x^2+1}\, dx$ *Em.:* $\frac{\pi^3}{8}$.

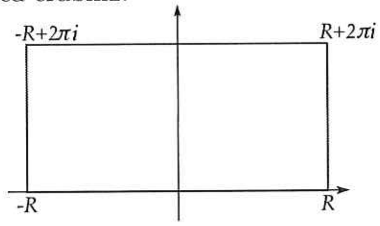
9. Integra ezazu $f(z) = \dfrac{\log(1-z)}{z}$ funtzioa emandako γ_ϵ bidean, egin $\epsilon \to 0$ eta zati irudikaria hartuz, froga ezazu hurrengoa

$$\int_0^{2\pi} \log\left(\sin\frac{\theta}{2}\right) d\theta = -2\pi \log 2.$$



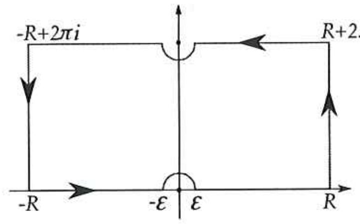
10. Kalkula itzazu hurrengo integralak ematen diren funtzioak eta bideak erabiliz

(i) $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{ax}}{(e^x+1)(e^x+2)}\,dx$, $0 < a < 2$ izanik, $f(z) = \dfrac{e^{az}}{(e^z+1)(e^z+2)}$ funtzioa eta γ_R bidea erabiliz.



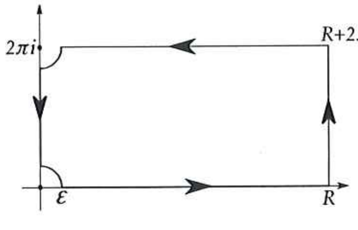
Em.: $\dfrac{\pi(1-2^{a-1})}{\sin(\pi a)}$.

(ii) $\displaystyle\int_0^{+\infty} \frac{\sin ax}{\sinh x}\,dx$, $a \neq 0$ izanik, $f(z) = \dfrac{e^{iaz}}{e^z - e^{-z}}$ funtzioa eta $\gamma_{R,\epsilon}$ bidea erabiliz.



Em.: $\dfrac{\pi}{2}\dfrac{\cosh(\pi a) - 1}{\sinh(\pi a)}$.

(iii) $\displaystyle\int_0^{+\infty} \frac{\sin ax}{e^x - 1}\,dx$, $a \neq 0$ izanik, $f(z) = \dfrac{e^{iaz}}{e^z - 1}$ funtzioa eta $\Gamma_{R,\epsilon}$ bidea erabiliz.



Em.: $\dfrac{\pi}{2}\operatorname{cotanh}(\pi a) - \dfrac{1}{2a}$.

11. Kalkula ezazu $f(s) = \dfrac{1}{(s-1)^2}$ funtzioaren Laplaceren alderantzizko transformatua.

Em.: xe^x.

1. ARIKETA

i) $$\operatorname*{Res}_{z=\infty} \frac{\sin z}{z^2} = -\operatorname*{Res}_{z=0} \frac{\sin z}{z^2} = -\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right] =$$

z=0 puntu singular bakarra

2. mailako poloa da.

$$= -\lim_{z=0}(\sin z)' = -\lim_{z=0}\cos z = -1$$

ii) $$\operatorname*{Res}_{z=1} \frac{e^z}{(z-1)^2} = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right] = \lim_{z\to 1}\frac{d}{dz}(e^z) = \lim_{z\to 1} e^z =$$

z=1 2. mailako poloa da.

$$= e$$

iii) $$\operatorname*{Res}_{z=\infty} z^2\cdot\sin\frac{\pi}{z} = -\operatorname*{Res}_{z=0}\frac{1}{z^2} f\left(\frac{1}{z}\right) = -\operatorname*{Res}_{z=0}\frac{\sin(\pi z)}{z^4} =$$

$$= \frac{-1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}\left[z^4\cdot f(z)\right] = \frac{-1}{6}\lim_{z\to 0}\frac{d^3}{dz^3}(\sin \pi z) =$$

$$= \frac{-1}{6}\lim_{z\to 0}\pi^3(-\cos \pi z) = \frac{\pi^3}{6}$$

iv) $$\operatorname*{Res}_{z=1} z\cdot e^{\frac{1}{z-1}} = -\operatorname*{Res}_{z=\infty} z\cdot e^{\frac{1}{z-1}} = \operatorname*{Res}_{z=0}\frac{1}{z^2}\cdot\frac{1}{z}\, e^{\frac{1}{\frac{1}{z}-1}} =$$

$$= \operatorname*{Res}_{z=0}\frac{1}{z^3} e^{\frac{z}{1-z}} = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2} z^3\frac{1}{z^3} e^{\frac{z}{1-z}} = (*)$$

~~non a_{-1} Laurent seriearen zati nagusiaren lehenengo gaia den:~~

$$\frac{1}{z^3} e^{\frac{z}{1-z}} = \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{1-z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{z^{n-3}}{(1-z)^n} =$$

$$= \sum_{m=-3}^{\infty}\frac{1}{(m+3)!}\cdot\frac{z^m}{(1-z)^{m+3}} \longrightarrow a_{-1} =$$

$$(*) = \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2} e^{\frac{z}{1-z}} = \frac{1}{2} \lim_{z\to 0} g''(z)$$

Deribatuak aparte hartuko ditugu:

$$g'(z) = e^{\frac{z}{1-z}} \cdot \frac{1-z+z}{(1-z)^2} = e^{\frac{z}{1-z}} \cdot \frac{1}{(1-z)^2}$$

$$g''(z) = e^{\frac{z}{1-z}} \frac{1}{(1-z)^4} + e^{\frac{z}{1-z}} \cdot \frac{2}{(1-z)^3}$$

Beraz, $\underset{z=1}{\text{Res}}\, z e^{\frac{1}{z-1}} = \frac{1}{2} \lim_{z\to 0} g''(z) = \frac{1}{2}\left[e^0 \cdot 1 + e^0 \cdot 2\right] = \frac{3}{2}$

2. ARIKETA

i) $f(z) = \dfrac{1}{z^3+z}$

Puntu singular isolatuak: $z^3+z=0 \longleftrightarrow z(z^2+1)=0$

$\longrightarrow z=0,\ z=\pm i \longrightarrow$ Hirurak polo sinpleak dira.

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} z\cdot f(z) = \lim_{z\to 0} \frac{1}{z^2+1} = 1$$

$$\underset{z=\pm i}{\text{Res}}\, f(z) = \lim_{z\to \pm i} (z\mp i) f(z) = \lim_{z\to \pm i} \frac{1}{z\cdot(z\pm i)} = \frac{1}{\pm i\cdot(\pm i \pm i)} =$$

$$= -1/2$$

ii) $f(z) = \dfrac{z^2}{1+z^4}$

Puntu singular isolatuak: $z^4+1=0 \longrightarrow z=(-1)^{1/4}$

$z = e^{i\frac{\pi+2\pi k}{4}}$, $k=0,1,2,3$.

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) = \frac{z^2}{(1+z^4)'}\bigg|_{z=e^{i\frac{\pi}{4}}} = \frac{z^2}{4z^3}\bigg|_{z=e^{i\frac{\pi}{4}}} = \frac{1}{4}\cdot\frac{e^{i\frac{\pi}{2}}}{e^{i\frac{3\pi}{4}}} =$$

$$= \frac{1}{4}\cdot e^{i\cdot\frac{-\pi}{4}} = \frac{1-i}{4\sqrt{2}}$$

$$\operatorname*{Res}_{z=e^{i\frac{3\pi}{4}}} f(z) = \frac{z^2}{4z^3}\bigg|_{z=e^{i\frac{3\pi}{4}}} = \frac{1}{4}\cdot\frac{e^{i\frac{3\pi}{2}}}{e^{i\frac{9\pi}{4}}} = \frac{1}{4}e^{i\cdot\frac{-3\pi}{4}} =$$

$$= -\frac{1+i}{4\sqrt{2}}$$

$$\operatorname*{Res}_{z=e^{i\cdot\frac{5\pi}{4}}} f(z) = \frac{1}{4}\cdot\frac{e^{i\frac{5\pi}{2}}}{e^{i\cdot\frac{15\pi}{4}}} = \frac{1}{4}\cdot e^{-i\frac{5\pi}{4}} = \frac{-1+i}{4\sqrt{2}}$$

$$\operatorname*{Res}_{z=e^{i\cdot\frac{7\pi}{4}}} f(z) = \frac{1}{4}\cdot\frac{e^{i\cdot\frac{7\pi}{2}}}{e^{i\cdot\frac{21\pi}{4}}} = \frac{1}{4}e^{-i\cdot\frac{7\pi}{4}} = \frac{1+i}{4\sqrt{2}}$$

iii) $f(z) = \dfrac{1}{(z^2+1)^3}$

Puntu singularrak: $z^2+1=0 \longrightarrow z=\pm i$, 3 ordenako poloak.

$$\operatorname*{Res}_{z=\pm i} f(z) = \frac{1}{2}\lim_{z\to\pm i}\frac{d^2}{dz^2}(z\mp i)^3 f(z) = \frac{1}{2}\lim_{z\to\pm i}\frac{d^2}{dz^2}\left(\frac{1}{(z\pm i)^3}\right) =$$

$$= \frac{1}{2}\lim_{z\to\pm i}\frac{d}{dz}\left(\frac{-3}{(z\pm i)^4}\right) = \frac{1}{2}\lim_{z\to\pm i}\frac{12}{(z\pm i)^5} = \frac{12}{(\pm 2i)^5}\cdot\frac{1}{2} =$$

$$= \pm\frac{2^2\cdot 3}{2^6\cdot i^5} = \pm\frac{3}{2^4 i} = \mp\frac{3i}{16}$$

iv) $f(z) = \dfrac{\sin(\pi z)}{(z-1)^3}$

Singularitatea: $z=1$, 3 ordenako poloa.

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2} (z-1)^3 f(z) =$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2} (\sin \pi z) = \frac{1}{2} \lim_{z\to 1} \frac{d}{dz} (\pi \cos \pi z) =$$

$$= \frac{1}{2} \lim_{z\to 1} (-\pi^2 \sin \pi z) = 0$$

v) $f(z) = \dfrac{1}{\sin z^2}$

Singularitateak: $z^2 = \pi k,\ k \in \mathbb{N} \longrightarrow z = \pm\sqrt{\pi k}$, $K \in \mathbb{N}$

Bakoitzaren izaera aztertzeko:

$$\lim_{z\to 0} f(z) = \infty \notin \mathbb{C}\ ;\ \lim_{z\to 0} z\cdot f(z) = \lim_{z\to 0} \frac{z}{\sin z^2} \overset{L'H}{=}$$

$$= \lim_{z\to 0} \frac{1}{\cos z^2 \cdot 2z} = \infty \notin \mathbb{C}$$

$$\lim_{z\to 0} z^2 \cdot f(z) = \lim_{z\to 0} \frac{z^2}{\sin z^2} \overset{L'H}{=} \lim_{z\to 0} \frac{2z}{\cos z^2 \cdot 2z} = \lim_{z\to 0} \frac{1}{\cos z^2} = 1$$

$z=0$ 2 ordenako poloa da.

$$\lim_{z\to \pm\sqrt{\pi k}} f(z) = \infty \notin \mathbb{C}\ ;\ \lim_{z\to \pm\sqrt{\pi k}} z\cdot f(z) = \lim_{z\to \pm\sqrt{\pi k}} \frac{z}{\sin z^2} \overset{L'H}{=}$$

$$= \lim_{z\to \pm\sqrt{\pi k}} \frac{1}{\cos z^2 \cdot 2z} = \frac{\pm 1}{2\pi k} \in \mathbb{C}$$

$z = \pm\sqrt{\pi k}$ polo sinpleak dira, $K \in \mathbb{N} - \{0\}$.

Hondarrak, hortaz,

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{d}{dz} z^2 f(z) = \lim_{z\to 0} \frac{d}{dz} \frac{z^2}{\sin z^2} =$$

$$= \lim_{z\to 0} \frac{2z\cdot \sin z^2 + 2z^3 \cos z^2}{(\sin z^2)^2} = \lim_{z\to 0} \frac{2z + 2z^3 \cdot \frac{1}{\tan z^2}}{\sin z^2} \overset{L'H}{=}$$

$$= \lim_{z\to 0} \frac{2 + 2\left(\frac{3z^2}{\tan z^2} + z^2\cdot(-2z)\cdot\left(1+\frac{1}{\tan^2 z^2}\right)\right)}{2\cos z^2 \cdot z}$$

$= \lim_{z\to 0}$ ja xd lol.

vi) $f(z) = \dfrac{1}{e^z+1}$

Singularitateak: $e^z = -1 \longleftrightarrow z = (2k+1)\pi i, \; k\in\mathbb{Z}$

$$\operatorname*{Res}_{z=(2k+1)\pi i} f(z) = \frac{1}{(e^z+1)'}\Bigg|_{z=(2k+1)\pi i} = \frac{1}{e^z}\Bigg|_{z=(2k+1)\pi i} =$$

$$= \frac{1}{-1} = -1$$

vii) $f(z) = \dfrac{1}{1-e^{z^2}}$

Singularitateak: $e^{z^2} = 1 \longrightarrow z^2 = 2\pi k i, \; k\in\mathbb{Z}$

$$\longrightarrow z = \sqrt{2\pi k i} = \sqrt{\pi k}\cdot\sqrt{2e^{i\frac{\pi}{2}}} = \sqrt{2\pi k}\, e^{i\cdot\frac{\pi+2\pi n}{2}}, \; n = 0,1$$

$$= \sqrt{\pi k \cdot 2}\cdot e^{i\frac{\pi}{2}} = \sqrt{\pi k}\,(1+i)$$

edo

$$= \sqrt{2\pi k}\; e^{i\frac{3\pi}{2}} = -\sqrt{\pi k}\,(1+i)$$

$z_0 = \pm\sqrt{\pi k}\,(1+i), \; k\in\mathbb{N}$

Singularitateen izaera:

$$\lim_{z\to z_0} f(z) = \infty \notin \mathbb{C}, \quad \lim_{z\to z_0} z\cdot f(z) \overset{\lim_{z\to z_0}}{=} \frac{z}{1-e^{z^2}}$$

$z_0 = 0$ bada,

$$\lim_{z\to 0} \frac{z}{1-e^{z^2}} \overset{L'H}{=} \lim_{z\to 0} \frac{1}{-e^{z^2}\cdot 2z} = \infty \notin \mathbb{C}$$

$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} \frac{z^2}{1-e^{z^2}} = \lim_{z\to 0} \frac{2z}{-e^{z^2}\cdot 2z} = \lim_{z\to 0} \frac{-1}{e^{z^2}} = -1 \in \mathbb{C}$$

$z_0 = 0$ 2 ordenako poloa da.

$z_0 = \pm\sqrt{\pi k}\,(1+i)$, $k \in \mathbb{N}-\{0\}$ bada,

$$\lim_{z\to z_0} \frac{z}{1-e^{z^2}} = \infty \notin \mathbb{C}$$

$$\lim_{z\to z_0} z^2 f(z) = \lim_{z\to z_0} \frac{z^2}{1-e^{z^2}} \overset{L'H}{=} \lim_{z\to z_0} \frac{2z}{-e^{z^2}\cdot 2z} = \lim_{z\to z_0} \frac{-1}{e^{z^2}} = -1 \in \mathbb{C}$$

2 ordenako poloak dira denak.

$$\cancel{\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{d}{dz} z^2\cdot f(z) = \lim_{z\to z_0} \frac{d}{dz} \frac{z^2}{1-e^{z^2}} =}$$

$$\cancel{= \lim_{z\to z_0} \frac{2z(1-e^{z^2}) + 2z^3 e^{z^2}}{(1-e^{z^2})^2}}$$

$$\operatorname*{Res}_{z=z_0} f(z) = \left.\frac{1}{(1-e^{z^2})'}\right|_{z=z_0} = \left.\frac{-1}{2z e^{z^2}}\right|_{z=z_0} \overset{z_0\neq 0}{=} \frac{-1}{\pm 2(1+i)\sqrt{\pi k}}$$

$z=0$ bada, me come los huevos...

viii) $f(z) = \dfrac{z^{n-1}}{z^n + a^n}$

Puntu singularrak: $z_k^n + a^n = 0 \longrightarrow z_k^n = -a^n \longrightarrow$

$\longrightarrow z_k = a \cdot (-1)^{1/n} = a (e^{-i\pi})^{1/n} = a \cdot e^{\frac{\pi + 2\pi k}{n} i} =$

$= a \cdot e^{\frac{(2k+1)\pi}{n} i}, \quad k = 0, 1, \dots, n-1.$

$$\lim_{z \to z_k} f(z) = \lim_{z \to z_k} \frac{z_k^{n-1}}{0} = \infty \notin \mathbb{C}$$

$$\lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} \frac{(z - z_k) z^{n-1}}{z^n + a^n} \overset{L'H}{=} \lim_{z \to z_k} \frac{z^{n-1} + (n-1)(z - z_k) z^{n-2}}{n z^{n-1}} =$$

$= \dfrac{1}{n} \in \mathbb{C} \longrightarrow$ Polo simpleak.

$$\operatorname*{Res}_{z = z_k} f(z) = \lim_{z \to z_k} (z - z_k) f(z) = \frac{1}{n}, \quad k = 0, 1, \dots, n-1.$$

ix) $f(z) = \dfrac{e^{imz}}{(z^2 + a^2)^2}$

Puntu singularrak: $z^2 + a^2 = 0 \longrightarrow z = \pm ai.$

Bigarren mailako poloak dira:

$$\operatorname*{Res}_{z = \pm ai} f(z) = \lim_{z \to \pm ai} \frac{d}{dz} \left[(z \mp ai)^2 f(z) \right] =$$

$$= \lim_{z \to \pm ai} \frac{d}{dz} \left(\frac{e^{imz}}{(z \pm ai)^2} \right) = \lim_{z \to \pm ai} \frac{im e^{imz} (z \pm ai)^2 - 2 e^{imz} (z \pm ai)}{(z \pm ai)^3} =$$

$$= \lim_{z \to \pm ai} \frac{im (z \pm ai) e^{imz} - 2 e^{imz}}{(z \pm ai)^2} =$$

$$= \frac{im \cdot (\pm 2ai) - 2}{(\pm 2ai)^2} \cdot e^{\pm imai} = \frac{\mp 2am - 2}{-4a^2} e^{\mp ma} =$$

$$= e^{\mp ma} \cdot \frac{-(1 \pm ma)}{-2a^2} = \frac{1 \pm ma}{2a^2} e^{\mp ma}$$

$\boxed{\text{3. ARIKETA}}$

i) $$f(z) = \frac{1+z^8}{z^6(z+2)}$$

Puntu singularrak: $z=0 \longrightarrow$ 6 ordenako poloa

$z=-2 \longrightarrow$ Polo sinplea.

∞.

$$\operatorname*{Res}_{z=-2} f(z) = \lim_{z\to -2} (z+2) f(z) = \lim_{z \to -2} \frac{1+z^8}{z^6} = \frac{1+2^8}{2^6} =$$

$$= \frac{257}{64}$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = -\operatorname*{Res}_{z=0} \frac{1}{z^2} \cdot z^6 \cdot \frac{1+(1/z)^8}{\frac{1}{z}+2} =$$

$$= -\operatorname*{Res}_{z=0} z^4 \cdot \frac{\frac{1}{z^8}(z^8+1)}{\frac{1}{z}(1+2z)} = -\operatorname*{Res}_{z=0} \frac{1}{z^3} \cdot \frac{z^8+1}{1+2z} =$$ 3 ordenako poloa

$$= -\frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left[\frac{z^8+1}{1+2z}\right] = -\frac{1}{2} \lim_{z \to 0} \frac{d}{dz}\left(\frac{8z^7(1+2z) - 2(z^8+1)}{(1+2z)^2}\right) =$$

$$= -\frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left(\frac{14z^8 + 8z^7 - 2}{(1+2z)^2}\right) =$$

$$= -\frac{1}{2} \lim_{z \to 0} \frac{(112z^7 + 56z^6)(1+2z)^2 - 2(1+2z)\cdot 2(14z^8+8z^7-2)}{(1+2z)^4} =$$

$$= -\frac{1}{2} \cdot \frac{-2\cdot 1 \cdot 2 \cdot (-2)}{1} = -4$$

$$\operatorname*{Res}_{z=0} f(z) = -\operatorname*{Res}_{z=-2} f(z) - \operatorname*{Res}_{z=\infty} f(z) = -\frac{257}{64} + 4 = -\frac{1}{64}$$

ii) $\sin z \cdot \sin\frac{1}{z} = f(z)$

Punto singularrak: $z=0 \longrightarrow$ Esentziala.
$z=\infty \longrightarrow$ Esentziala.

$$f(z) = \sin z \cdot \sin\frac{1}{z} = \cancel{\frac{1}{2}\left[\cos\left(z-\frac{1}{z}\right) - \cos\left(z+\frac{1}{z}\right)\right]} =$$

$$\cancel{= \frac{1}{2}\left[\cos\left(\frac{z^2-1}{z}\right) - \cos\left(\frac{z^2+1}{z}\right)\right]}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{z}\right)^{2k+1} =$$

$$= \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \dots\right) \cdot \left(\frac{1}{z} - \frac{1}{6}\frac{1}{z^3} + \frac{1}{120}\frac{1}{z^5} - \dots\right) =$$

$$= \dots + a_{-4} \cdot \frac{1}{z^4} + a_{-2}\frac{1}{z^2} + a_0 + a_2 z^2 + a_4 z^4 + \dots$$

Beraz, Laurent seriearen a_{-1} gaia: $a_{-1} = 0 \longrightarrow$

$$\longrightarrow \operatorname*{Res}_{z=0} f(z) = 0$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} f(z) = 0.$$

iii) $f(z) = \dfrac{\sin z}{(z^2+1)^2}$

Puntu singularrak: $z = \pm i \longrightarrow$ 2 ordenako poloak.
$z = \infty$.

$$\operatorname*{Res}_{z=\mp i} f(z) = \lim_{z\to\mp i} \frac{d}{dz}\left[(z\mp i)^2 f(z)\right] = \lim_{z\to\pm i} \frac{d}{dz}\left[\frac{\sin z}{(z\pm i)^2}\right] =$$

$$= \lim_{z\to\pm i} \frac{\cos z\,(z\pm i)^2 - 2\sin z\,(z\pm i)}{(z\pm i)^4} =$$

$$= \frac{\cos(\pm i)\cdot(\pm 2i)^2 - 2\sin(\pm i)\cdot(\pm 2i)}{(\pm 2i)^4} = \frac{-4\cos i \mp 4i\sin(\pm i)}{16} =$$

$$= \frac{-4\cosh(1) - 4i\sin i}{16} = \frac{1}{4}\left(-\cosh 1 + \sinh 1\right) =$$

$$= \frac{1}{4}\left(-\frac{e+\frac{1}{e}}{2} + \frac{e-\frac{1}{e}}{2}\right) = \frac{1}{8}\left(-e-\frac{1}{e}+e-\frac{1}{e}\right) = -\frac{1}{4e}$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=i} f(z) - \operatorname*{Res}_{z=-i} f(z) = -\left(2\cdot\frac{-1}{4e}\right) = \frac{1}{2e}$$

4. ARIKETA

i) $$\int_{|z|=2} \frac{dz}{(z+1)^2(z^2+2)}$$

$$f(z) = \frac{1}{(z+1)^2(z^2+2)} = \frac{1}{(z+1)^2(z-\sqrt{2}i)(z+\sqrt{2}i)}$$

f-ren puntu singularrak $|z|<2$ eremuan: $z_1=-1$, $z_{2,3}=\pm\sqrt{2}i$ non $z=-1$ 2 ordenako poloa den eta beste biak polo sinpleak.

Beraz, hondarren teorema aplikatuz,

$$\int_{|z|=2} f(z)\,dz = 2\pi i \cdot \sum_{k=1}^{3} \operatorname*{Res}_{z=z_k} f(z) =$$

$$= 2\pi i \left(\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=\sqrt{2}i} f(z) + \operatorname*{Res}_{z=-\sqrt{2}i} f(z) \right) =$$

$$= 2\pi i \left(\lim_{z\to -1} \frac{d}{dz}\left((z+1)^2 f(z)\right) + \lim_{z\to\sqrt{2}i} (z-\sqrt{2}i) f(z) + \lim_{z\to -\sqrt{2}i} f(z)(z+\sqrt{2}i) \right) =$$

$$= 2\pi i \left(\lim_{z\to -1} \frac{d}{dz} \frac{1}{(z^2+2)} + \lim_{z\to\sqrt{2}i} \frac{1}{(z+1)^2(z+\sqrt{2}i)} + \lim_{z\to -\sqrt{2}i} \frac{1}{(z+1)^2(z-\sqrt{2}i)} \right) =$$

$$= 2\pi i \left(\lim_{z\to -1} \frac{-2z}{(z^2+2)^2} + \frac{1}{(\sqrt{2}i+1)^2 \cdot 2\sqrt{2}i} + \frac{1}{(-\sqrt{2}i+1)^2 \cdot (-2\sqrt{2}i)} \right) =$$

$$= 2\pi i \left(\frac{2}{3^2} + \frac{1}{2\sqrt{2}i(1+2\sqrt{2}i-2)} - \frac{1}{2\sqrt{2}i(1-2\sqrt{2}i-2)} \right) =$$

$$= 2\pi i \left(\frac{2}{9} + \frac{1-2\sqrt{2}i-2-1-2\sqrt{2}i+2}{2\sqrt{2}i(1+2\sqrt{2}i-2)(1-\sqrt{2}\cdot 2i-2)} \right) =$$

$$= 2\pi i \left(\frac{2}{9} + \frac{-4\sqrt{2}i}{2\sqrt{2}i(-1+2\sqrt{2}i)(-1-2\sqrt{2}i)} \right)$$

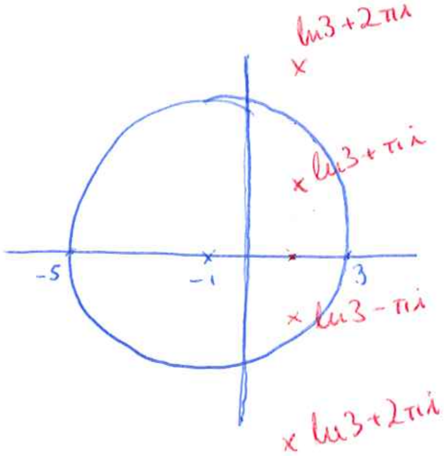
$$= 2\pi i \left(\frac{2}{9} + \frac{-4\sqrt{2}i}{2\sqrt{2}i(1+8)} \right) = 2\pi i \left(\frac{2}{9} - \frac{2}{9} \right) = 0$$

ii) $\displaystyle\int_{|z+1|=4} \frac{z}{e^z+3}\,dz$

$f(z) = \dfrac{z}{e^z+3}$. f-ren singularitateak:

$e^z + 3 = 0 \longrightarrow z = \log(-3) = \ln|3| + i(\arg 3 + 2\pi k) =$

$= \ln 3 + 2\pi i k + \pi i, \quad k \in \mathbb{Z}$



$|z+1|<4$ eremuan bakarrik $\ln 3 \pm \pi i$ singularitateak ditugu. Beraz, hondarren teorema aplikatuz:

$$\int_{|z+1|=4} f(z)\,dz = 2\pi i \sum_{k=0}^{1} \operatorname*{Res}_{z=z_k} f(z) =$$

$$= 2\pi i \left[\left. \frac{z}{(e^z+3)'} \right|_{z=\ln 3+\pi i} + \left. \frac{z}{(e^z+3)'} \right|_{z=\ln 3-\pi i} \right] =$$

$$= 2\pi i \left(\left. \frac{z}{e^z} \right|_{z=\ln 3+\pi i} + \left. \frac{z}{e^z} \right|_{z=\ln 3-\pi i} \right) =$$

$$= 2\pi i \left(\frac{\ln 3+\pi i}{e^{\ln 3+\pi i}} + \frac{\ln 3-\pi i}{e^{\ln 3-\pi i}} \right) =$$

$$= 2\pi i \left(\frac{\ln 3+\pi i}{3\cdot(-1)} + \frac{\ln 3-\pi i}{3}\cdot(-1) \right) = \frac{2\pi i}{3}\left(-\ln 3-\pi i-\ln 3+\pi i\right) =$$

$$= -\frac{4\pi i}{3}\cdot \ln 3$$

iii) $\displaystyle\int_{|z|=2} \frac{e^z}{z^3(1+z)}\,dz$

$f(z) = \dfrac{e^z}{z^3(1+z)}$. f-ren puntu singularrak $|z|<2$ eremuan, $z_1=0$, $z_2=-1$; non z_1 3 ordenako poloa den eta z_2 polo sinplea.

Hondarren teorema aplikatuz,

$$\int_{|z|=2} \frac{e^z}{z^3(z+1)}\,dz = 2\pi i \sum_{k=1}^{2} \operatorname*{Res}_{z=z_k} f(z) =$$

$$= 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z) \right) =$$

$$= 2\pi i \left(\frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left[z^3 f(z) \right] + \lim_{z\to -1} (z+1) f(z) \right) =$$

$$= 2\pi i \left(\frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{e^z}{1+z} \right) + \lim_{z\to -1} \frac{e^z}{z^3} \right) =$$

$$= 2\pi i \left(\frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left(\frac{e^z(1+z)-e^z}{(1+z)^2} \right) + \frac{e^{-1}}{(-1)^3} \right) =$$

$$= 2\pi i \left(\frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left(\frac{z e^z}{(1+z)^2} \right) - \frac{1}{e} \right) =$$

$$= \frac{-2\pi i}{e} + \pi i \cdot \lim_{z\to 0} \frac{(e^z + z e^z)(1+z)^2 - 2(1+z) z e^z}{(1+z)^2} =$$

$$= \frac{-2\pi i}{e} + \pi i \cdot \frac{1}{1} = \pi i \left(\frac{-2}{e} + 1 \right) = \frac{e-2}{e}\pi i$$

iv) $$\int_{|z|=1} \frac{z^2}{\sin^3 z \cdot \cos z}$$

$f(z) = \dfrac{z^2}{\sin^3 z \cdot \cos z}$. f-ren singularitate bakarra $|z|<1$ eremuan $z=0$ da. Beraz, hondarren teorema erabiliz, Polo sinplea

$$\int_{|z|=1} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) \overset{\text{Polo sinplea}}{=} 2\pi i \lim_{z\to 0} z f(z) =$$

$$= 2\pi i \lim_{z\to 0} \frac{z^3}{\sin^3 z \cos z} \sim 2\pi i \lim_{z\to 0} \frac{z^3}{z^3 \cdot \cos z} = 2\pi i \lim_{z\to 0} \frac{1}{\cos z} =$$

$$= 2\pi i$$

v) $$\int_{|z|=1/3} (z+1) e^{1/z}\, dz$$

$f(z) = (z+1)e^{1/z}$. f-ren singularitate bakarra $z=0$ da, $|z|<1/3$ eremuan dagoena. Hondarren teorema erabiliz:

$$\int_{|z|=1/3} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i \operatorname*{Res}_{z=\infty} f(z) =$$

$$= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f(1/z) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2}(1/z + 1)\cdot e^{z} =$$

$$= 2\pi i \cdot \operatorname*{Res}_{z=0} \frac{1+z}{z^3} e^{z} =$$

$z=0$ $\frac{1}{z^2} f(1/z)$ funtzioaren 3 ordenako poloa da.

$$= 2\pi i \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left[(1+z)e^{z}\right] = \pi i \lim_{z\to 0} \frac{d}{dz}\left[e^{z}(1+z) + e^{z}\right] =$$

$$= \pi i \lim_{z\to 0} \frac{d}{dz}\left[e^{z}(2+z)\right] = \pi i \lim_{z\to 0}\left[e^{z}(2+z) + e^{z}\right] =$$

$$= \pi i \lim_{z\to 0} e^{z}(z+3) = 3\pi i$$

5. ARIKETA

i) $\displaystyle\int_0^{\pi/2} \frac{d\varphi}{1+\sin^2\varphi} = \frac{1}{4}\int_{-\pi}^{\pi} \frac{d\varphi}{1+\sin^2\varphi} =$

Funtzio bikoitia, π periododuna

Aldagai-aldaketa:

$z = e^{i\varphi}$

$\sin\varphi = \dfrac{z-z^{-1}}{2i}$

$d\varphi = \dfrac{dz}{iz}$

$$= \frac{1}{4}\int_{|z|=1} \frac{1}{1+\left(\frac{z-1/z}{2i}\right)^2}\frac{dz}{iz} = \frac{1}{4}\int_{|z|=1}\frac{-4}{-4+(z-1/z)^2}\frac{dz}{iz} =$$

$$= i\int_{|z|=1}\frac{1}{z^2+1/z^2-6}\frac{dz}{z} = i\int_{|z|=1}\frac{z^2}{z^4-6z^2+1}\cdot\frac{dz}{z} =$$

$$= i\int_{|z|=1}\frac{z}{z^4-6z^2+1}\,dz = (*)$$

$f(z) = \dfrac{z}{z^4-6z^2+1}$. f-ren puntu singularrak:

$z^4-6z^2+1=0 \longrightarrow z^2 = \dfrac{6\pm\sqrt{36-4}}{2} = \dfrac{6\pm\sqrt{32}}{2} = 3\pm2\sqrt{2}$

baina baina gure eremu barneko bakarrak $\sqrt{3-2\sqrt{2}}$ eta $-\sqrt{3-2\sqrt{2}}\,i$

$$(*) = i\cdot 2\pi i\left[\operatorname*{Res}_{z=\sqrt{3-2\sqrt{2}}} f(z) + \operatorname*{Res}_{z=-\sqrt{3-2\sqrt{2}}\,i} f(z)\right] =$$

$$= -2\pi\cdot\left[\frac{z}{(z^4-6z^2+1)'}\bigg|_{z=\sqrt{3-2\sqrt{2}}=z_1} + \frac{z}{(z^4-6z^2+1)'}\bigg|_{z=\sqrt{3-2\sqrt{2}}\,i=z_2}\right] =$$

$$= -2\pi\left(\frac{z_1}{4z_1^3-12z_1} + \frac{z_2}{4z_2^3-12z_2}\right) =$$

$$= -2\pi\left(\frac{1}{4z_1^2-12} + \frac{1}{4z_2^2-12}\right) = -2\pi\frac{2}{4\cdot(3-2\sqrt{2})-12}$$

$$= -4\pi\cdot\frac{1}{-8\sqrt{2}} = \boxed{\frac{\pi}{2\sqrt{2}}}$$

ii) $\displaystyle\int_0^{2\pi} \frac{d\varphi}{1-2a\cos\varphi + a^2}, \quad |a| \neq 1.$

Aldagai-aldaketa eginez, $z = e^{i\varphi}$, $\cos\varphi = \dfrac{z + 1/z}{2}$, $d\varphi = \dfrac{dz}{iz}$

$$\int_{|z|=1} \frac{1}{1+a^2 - a(z+1/z)} \frac{dz}{iz} = -i \int_{|z|=1} \frac{z}{-az^2+(1+a^2)z - a} \cdot \frac{dz}{z} =$$

$$= i \int_{|z|=1} \frac{dz}{az^2 - (1+a^2)z + a} = (*)$$

$f(z) = \dfrac{dz}{az^2 - (1+a^2)z + a}$. f-ren puntu singularrak:

$$az^2 - (1+a^2)z + a = 0 \longrightarrow z = \frac{1+a^2 \pm \sqrt{(1+a^2)^2 - 4a^2}}{2a} =$$

$$= \frac{1+a^2 \pm \sqrt{a^4 - 2a^2 + 1}}{2a} = \frac{1+a^2 \pm (a^2-1)}{2a} \begin{cases} z_1 = a \\ z_2 = 1/a \end{cases}$$

$|a| \neq 1$ denez, bi singularitateetako bat baino ez da egongo $|z|<1$ eremuan. z_1 dela barnekoa suposatuko dugu, hots, $a<1$.

$$(*) = i \cdot 2\pi i \cdot \operatorname*{Res}_{z=a} f(z) = \quad \text{polo sinplea.}$$

$$= -2\pi \lim_{z\to a} (z-a) \cdot \frac{1}{(z-a)(z-1/a)} = \underbrace{-2\pi \cdot \frac{1}{a - 1/a}}_{a<1}$$

$a > 1$ bada,

$$(*) = i \cdot 2\pi i \cdot \operatorname*{Res}_{z=1/a} f(z) =$$

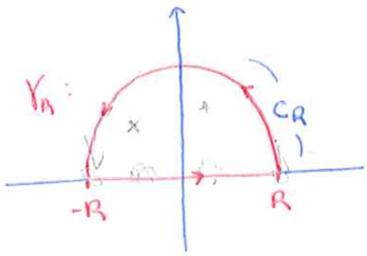
$$= -2\pi \cdot \lim_{z\to a} (z - 1/a) \cdot \frac{1}{(z-a)(z-1/a)} = \overbrace{-2\pi \frac{1}{1/a - a}}^{a>1}$$

$$\boxed{\frac{2\pi a}{|1-a^2|}}$$

6. ARIKETA

i) $$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}, \quad a>0,\ b>0,\ a\neq b.$$

$F(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)^2}$ funtzioa γ_R bidean integratuko dugu, non $R>a,b$ izanik.



F-ren singularitate bakarrak $z=\pm ai$ dira, non bakarrik $z_1=ai$ γ_R-k mugatutako eremu barnean dagoen, polo simplea izanik eta $z=\pm bi$ puntuak, bigarren ordenako poloak direnak eta $z_2=bi$ dagoelarik eremuan.

Hondarren teorema erabiliz,

$$\int_{\gamma_R} F(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=ai} F(z) + \operatorname*{Res}_{z=bi} F(z)\right) =$$

$$= 2\pi i\left[\lim_{z\to ai} (z-ai)F(z) + \lim_{z\to bi} \frac{d}{dz}(z-bi)^2F(z)\right] =$$

$$= 2\pi i\left[\lim_{z\to ai} \frac{1}{(z+ai)(z^2+b^2)^2} + \lim_{z\to bi} \frac{d}{dz}\frac{1}{(z^2+a^2)(z+bi)^2}\right] =$$

$$= 2\pi i\cdot\left[\frac{1}{2ai(b^2-a^2)^2} - \lim_{z\to bi}\frac{2z(z+bi)^2+2(z+bi)(z^2+a^2)}{(z^2+a^2)^2(z+bi)^4}\right] =$$

$$= 2\pi i\left[\frac{-i}{2a(b^2-a^2)^2} - \frac{-8b^3i+4bi(a^2-b^2)}{16b^4(a^2-b^2)^2}\right] =$$

$$= 2\pi i \left[\frac{-i}{2a(a^2-b^2)^2} - \frac{i(-2b^2+a^2-b^2)}{4b^3(a^2-b^2)^2} \right] =$$

$$= 2\pi i \cdot i \cdot \frac{-2b^3 - a(-b^2+a^2-b^2)}{4ab^3(a^2-b^2)^2} =$$

$$= -\pi \cdot \frac{-2b^3-a^3+3ab^2}{4ab^3(a^2-b^2)^2} = \pi \cdot \frac{2b^3+a^3-3ab^2}{4ab^3(a^2-b^2)^2}$$

Bestalde,

$$\int_{\gamma_R} F(z)\,dz = \int_{-R}^{R} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} + \int_{C_R} F(z)\,dz \qquad (*)$$

$|z|=R$ denean, $|zF(z)| = \dfrac{|z|}{|z^2+a^2|\cdot|z^2+b^2|^2} \leqslant \dfrac{R}{(R^2-a^2)(R^2-b^2)^2}$

dugu. Beraz, $\lim\limits_{R\to\infty} |zF(z)| = 0$ dugunez,

$$\lim_{R\to\infty} \int_{C_R} F(z)\,dz = 0 \quad \text{da.}$$

Beraz, $(*)$ adierazpenean $R \to \infty$ limitea hartuz:

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \pi \cdot \frac{2b^3+a^3-3ab^2}{4ab^3(a^2-b^2)^2}}$$

ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos^2 x\, dx}{(x^2+a^2)(x^2+b^2)}$, $a,b>0$, $a\neq b$.

$$\int_{-\infty}^{\infty} \frac{1}{2}\cdot\frac{1+\cos 2x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}}_{I_1} + \frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\frac{\cos 2x\, dx}{(x^2+a^2)(x^2+b^2)}}_{I_2}$$

I_1 integrala:

$F(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)}$ integratuko dugu γ_R bidean. Honek inguratzen duen eremuan $z=ai$ eta $z=bi$ singularitateak ditugunez.

$$\int_{\gamma_R} F(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=ai} F(z) + \operatorname*{Res}_{z=bi} F(z)\right) =$$

$$= 2\pi i\left[\lim_{z\to ai}\frac{1}{(z+ai)(z^2+b^2)} + \lim_{z\to bi}\frac{1}{(z+bi)(z^2+a^2)}\right] =$$

$$= 2\pi i\left[\frac{1}{-2ai(b^2-a^2)} + \frac{1}{-2bi(a^2-b^2)}\right] = 2\pi i\cdot\frac{b-a}{2abi(a^2-b^2)} =$$

$$= \frac{(b-a)\pi}{ab(a^2-b^2)}$$

Bestetik,

$$\int_{\gamma_R} F(z)\,dz = \int_{C_R} F(z)\,dz + \int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$|z|=R$ izanik, $|zF(z)| = \dfrac{|z|}{|z^2+a^2|\cdot|z^2+b^2|} \leq \dfrac{R}{(R^2-a^2)(R^2-b^2)}$

eta $|zF(z)| \xrightarrow{R\to\infty} 0$ denez, $\int_{C_R} F(z)\,dz = 0$ da.

Hortaz, $$\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{(b-a)\pi}{ab(a^2-b^2)}$$

I_2 integrala

$$F(z) = \frac{e^{2iz}}{(z^2+a^2)(z^2+b^2)}$$ γ_R bidean integratuko dugu ere.

Honek mugatutako eremuko singularitateak $z=ai$ eta $z=bi$ dira, polo sinpleak direnak.

$$\int_{\gamma_R} F(z) = 2\pi i \left(\operatorname*{Res}_{z=ai} F(z) + \operatorname*{Res}_{z=bi} F(z) \right) =$$

$$= 2\pi i \left[\lim_{z\to ai} \frac{e^{2iz}}{(z+ai)(z^2+b^2)} + \lim_{z\to bi} \frac{e^{2iz}}{(z+bi)(z^2+a^2)} \right] =$$

$$= 2\pi i \left[\frac{e^{-2a}}{-2ai(b^2-a^2)} + \frac{e^{-2b}}{-2bi(a^2-b^2)} \right] = 2\pi i \cdot \frac{be^{-2a} - ae^{-2b}}{2abi(a^2-b^2)} =$$

$$= \frac{(be^{-2a} - ae^{-2b})\pi}{ab(a^2-b^2)}$$

Bestetik,

$$\int_{\gamma_R} F(z)dz = \int_{C_R} F(z)dz + \int_{C_L} F(z)dz.$$

$|z|=R$ denean, $$|F(z)| = \frac{1}{|z^2+a^2|\cdot|z^2+b^2|} \leq \frac{1}{(R^2-a^2)(R^2-b^2)}$$

$|F(z)| \xrightarrow{R\to\infty} 0$ denez, $\int_{C_R} F(z)\,dz = 0$ dugu.

Beraz,

$$\int_{-\infty}^{+\infty} \frac{e^{2iz}}{(z^2+a^2)(z^2+b^2)} dz = \frac{(be^{-2a} - ae^{-2b})\pi}{ab(a^2-b^2)}$$

eta:

$$\int_{-\infty}^{+\infty} \frac{\cos(2x)}{(x^2+a^2)(x^2+b^2)} dx = \frac{(be^{-2a} - ae^{-2b})\pi}{ab(a^2-b^2)}$$

Dena batuz:

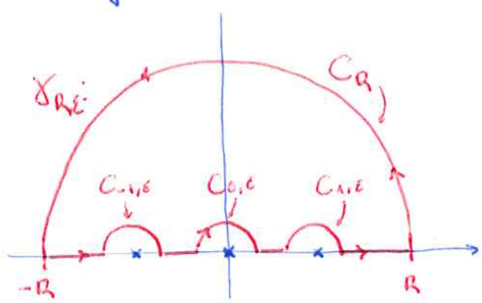
$$\frac{1}{2}I_1 + \frac{1}{2}I_2 = \frac{(b-a)\pi}{2ab(a^2-b^2)} + \frac{(be^{-2a}-ae^{-2b})\pi}{2ab(a^2-b^2)}$$

Edo,

$$\boxed{\int_{-\infty}^{+\infty} \frac{\cos^2 x}{(x^2+a^2)(x^2+b^2)}\,dx = \pi \cdot \frac{b-a+be^{-2a}-ae^{-2b}}{2ab(a^2-b^2)}}$$

iii) $\displaystyle\int_0^{+\infty} \frac{\sin(\pi x)}{x(1-x^2)}\,dx$

Integra dezagun $F(z)=\dfrac{e^{i\pi z}}{z(1-z^2)}$ funtzioa $\gamma_{R,\varepsilon}$ bidean zehar, $z=0$ eta $z=\pm 1$ puntuak gure funtzioaren puntu singular errealak izanik, hirurak polo simpleak.



F analitikoa denez $\gamma_{R,\varepsilon}$ bideak mugatutako eremuan, Cauchy teoremagatik, $\displaystyle\int_{\gamma_R} F(z)\,dz = 0.$

Bestetik,

$$\int_{\gamma_R} F(z)\,dz = \int_{-R}^{-1-\varepsilon} \frac{e^{i\pi x}}{x(1-x^2)}\,dx + \int_{-1+\varepsilon}^{-\varepsilon} \frac{e^{i\pi x}}{x(1-x^2)}\,dx + \int_{\varepsilon}^{1-\varepsilon} \frac{e^{i\pi x}}{x(1-x^2)}\,dx +$$

$$+ \int_{1+\varepsilon}^{R} \frac{e^{i\pi x}}{x(1-x^2)}\,dx + \int_{C_R} F(z)\,dz - \int_{C_{-1,\varepsilon}} F(z)\,dz - \int_{C_{0,\varepsilon}} F(z)\,dz - \int_{C_{1,\varepsilon}} F(z)\,dz. \quad (*)$$

$|z|=R$ denean, $\displaystyle |F(z)| = \frac{1}{|z||1-z^2|} \leq \frac{1}{R(R^2-1)} \xrightarrow{R\to\infty} 0$

Beraz, $R\to\infty$ limitean, Jordan lemagatik:

$$\int_{C_R} F(z)\,dz = 0 \qquad \text{dugu.}$$

Horrez gain, $\varepsilon \to 0$ limitean,

$$\lim_{\varepsilon\to 0} \int_{C_{-1,\varepsilon}} F(z)\,dz = i\cdot\pi\cdot \operatorname{Res}_{z=-1} \frac{e^{i\pi z}}{z(1-z^2)} = i\cdot\pi \lim_{z\to -1} \frac{e^{i\pi z}}{z(1-z)} =$$

$$= i\pi\cdot \frac{e^{-i\pi}}{-2} = i\frac{\pi}{2}$$

$$\lim_{\varepsilon\to 0} \int_{C_{0,\varepsilon}} F(z)\,dz = i\pi \operatorname{Res}_{z=0} \frac{e^{i\pi z}}{z(1-z^2)} = i\pi \lim_{z\to 0} \frac{e^{i\pi z}}{1-z^2} =$$

$$= i\pi\cdot \frac{e^{0}}{1} = i\pi$$

$$\lim_{\varepsilon\to 0} \int_{C_{1,\varepsilon}} F(z)\,dz = i\pi \operatorname{Res}_{z=1} \frac{e^{i\pi z}}{z(1-z^2)} = i\pi \lim_{z\to 1} \frac{-e^{i\pi z}}{z(1+z)} =$$

$$= i\pi\cdot \frac{-e^{i\pi}}{2} = i\cdot\frac{\pi}{2}$$

Horrela, ⊛ adierazpenean $R\to\infty$, $\varepsilon\to 0$ limiteak hartuz:

$$0 = \int_{-\infty}^{+\infty} \frac{e^{i\pi x}}{x(1-x^2)}\,dx + 0 - i\frac{\pi}{2} - i\pi - i\frac{\pi}{2} \longrightarrow \int_{-\infty}^{+\infty} \frac{e^{i\pi z}}{z(1-z^2)}\,dz = 2\pi i$$

eta parte irudikariak berdinduz,

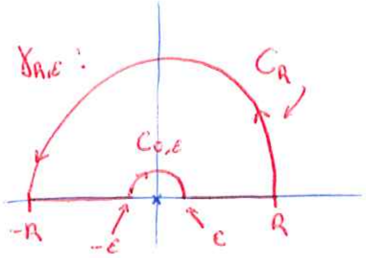
$$\int_{-\infty}^{+\infty} \frac{\sin(\pi x)}{x(1-x^2)}\,dx = 2\pi .$$

Azkenik, gure funtzioa bikoitia denez,

$$\boxed{\int_{0}^{+\infty} \frac{\sin(\pi x)}{x(1-x^2)}\,dx = \pi}$$

iv) $\displaystyle\int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2}\,dx, \quad a>0,\ b>0,\ a\neq b.$

Azter dezagun $F(z) = \dfrac{e^{iaz}}{z^2}$ funtzioa, $\gamma_{R,\varepsilon}$ bidean.



F analitikoa denez $\gamma_{R,\varepsilon}$ bideak mugaturiko eremuan, Cauchy teoremagatik,

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = 0.$$

Bestetik,

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = \int_{C_R} F(z)\,dz + \int_{-R}^{-\varepsilon} \frac{e^{iax}}{x^2}\,dx - \int_{C_{0,\varepsilon}} F(z)\,dz + \int_{\varepsilon}^{R} \frac{e^{iax}}{x^2}\,dx \quad (*)$$

$|z|=R$ denean, $|F(z)| = \dfrac{1}{|z^2|} = \dfrac{1}{R^2} \xrightarrow{R\to\infty} 0.$

Beraz, Jordan lemagatik, $R\to\infty$ limitean

$$\int_{C_R} F(z)\,dz = 0 \quad \text{dugu.}$$

Bestetik, $\varepsilon\to 0$ limitean, $z=0$ 2. ordenako poloa da.

$$\lim_{\varepsilon\to 0} \int_{C_{0,\varepsilon}} F(z)\,dz = \pi i \operatorname*{Res}_{z=0} F(z) = \pi i \cdot \lim_{z\to 0} \frac{d}{dz}\left[z^2 F(z)\right] =$$

$$= \pi i \cdot \lim_{z\to 0} \frac{d}{dz}\left(e^{iaz}\right) = \pi \cdot i \lim_{z\to 0} ia\, e^{iaz} = \pi i \cdot ia = -\pi a.$$

Beraz, $(*)$ adierazpenean $R\to\infty$, $\varepsilon\to 0$ limiteak hartuz,

$$0 = 0 + \pi a + \int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2}\,dx \longrightarrow \int_{-\infty}^{+\infty} \frac{\cos ax}{x^2}\,dx = -\pi a$$

Baina $\dfrac{\cos ax}{x^2}$ funtzio bikoitia denez, $\displaystyle\int_0^{+\infty} \frac{\cos ax}{x^2}\,dx = -\frac{\pi a}{2}$

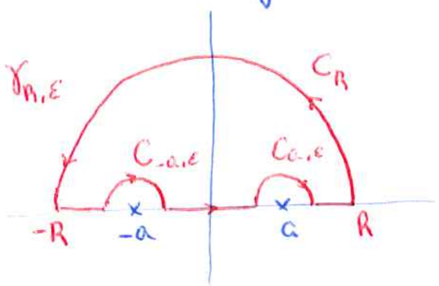
Beraz, $\displaystyle\int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \int_0^{+\infty} \frac{\cos ax}{x^2}\,dx - \int_0^{+\infty} \frac{\cos bx}{x^2}\,dx =$

$$= -\frac{\pi a}{2} - \left(-\frac{\pi b}{2}\right)$$

$$\boxed{\int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi(b-a)}{2}}$$

v) $\displaystyle \text{p.v.}\int_0^{+\infty} \frac{\cos x}{a^2 - x^2}\,dx \,, \quad a > 0 .$

Integrakizunaren puntu singularrak $x = \pm a$ errealak izanik, aztertu dezagun $F(z) = \dfrac{e^{iz}}{a^2 - z^2}$ funtzioa $\gamma_{R,\varepsilon}$ bidean:



F analitikoa denez $\gamma_{R,\varepsilon}$-k mugatzen duen eremuan, Cauchy teoremagatik,

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = 0 .$$

Bestetik,

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = \int_{-R}^{-a-\varepsilon} F(x)\,dx + \int_{-a+\varepsilon}^{a-\varepsilon} F(x)\,dx + \int_{a+\varepsilon}^{R} F(x)\,dx + \int_{C_R} F(z)\,dz -$$

$$- \int_{C_{-a,\varepsilon}} F(z)\,dz - \int_{C_{a,\varepsilon}} F(z)\,dz . \qquad (*)$$

$|z| = R$ -rako, $\displaystyle |F(z)| = \frac{1}{|a^2 - z^2|} \leq \frac{1}{R^2 - a^2} \xrightarrow{R\to\infty} 0$

Beraz, Jordan lemagatik,

$$\int_{C_R} F(z)\,dz = 0 .$$

Bestetik, $z=\pm a$ polo simpleak izanik,

$$\lim_{\varepsilon\to 0}\int_{C_{-a,\varepsilon}} F(z)\,dz = \pi i \operatorname{Res}_{z=-a} F(z) = \pi i \lim_{z\to -a}(z+a)F(z) =$$

$$= \pi i \cdot \lim_{z\to -a}\frac{e^{iz}}{a-z} = \pi i \cdot \frac{e^{-ia}}{2a}$$

$$\lim_{\varepsilon\to 0}\int_{C_{a,\varepsilon}} F(z)\,dz = \pi i \operatorname{Res}_{z=a} F(z) = \pi i \lim_{z\to a}(z-a)F(z) =$$

$$= \pi i \lim_{z\to a}\frac{-e^{iz}}{a+z} = \pi i \frac{-e^{ia}}{2a}$$

Horrela, $(*)$ adierazpenean $R\to\infty$ eta $\varepsilon\to 0$ limiteak hartuz,

$$0 = \int_{-\infty}^{+\infty}\frac{e^{ix}}{a^2-x^2}\,dx + 0 - \pi i\frac{e^{-ia}}{2a} - \pi i\frac{e^{ia}}{-2a}$$

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{a^2-x^2}\,dx = \pi i\frac{1}{2a}\left(-e^{ia}+e^{-ia}\right) = \pi i\cdot\frac{1}{\cancel{2}a}(\cancel{2}\cdot(-i)\cdot\sin a) =$$

$$= \pi\frac{\sin a}{a}$$

eta parte errealak berdinduz,

$$\int_{-\infty}^{+\infty}\frac{\cos x}{a^2-x^2}\,dx = \pi\cdot\frac{\sin a}{a}$$

Azkenik, gure funtzioa bikoitia denez,

$$\int_{0}^{+\infty}\frac{\cos x}{a^2-x^2}\,dx = \frac{\pi\cdot\sin a}{2a}\,,$$ konbergentea da:

$$\boxed{\text{p.v.}\int_{0}^{+\infty}\frac{\cos x}{a^2-x^2}\,dx = \frac{\pi\cdot\sin a}{2a}}$$

7. ARIKETA

$$f(x)=\frac{1}{(x^2+b^2)^2} \qquad \text{Fourier transformatua: } \hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}\,dx.$$

$|z|=R$ denean, $|f(z)|=\dfrac{1}{|z^2+b^2|^2}\leq\dfrac{1}{(R^2-b^2)^2}\xrightarrow{R\to\infty}0$

dugu. Berez, gure formulak erabil ditzakegu.

f-ren puntu singularrak $z=\pm bi$ dira, 2 ordenako poloak direnak.

$\omega\geq 0$ denean:

$$\hat{f}(\omega)=\sqrt{2\pi}\,i\operatorname*{Res}_{z=bi} f(z)e^{i\omega z}=\sqrt{2\pi}\,i\lim_{z\to bi}\frac{d}{dz}\left[(z-bi)^2f(z)e^{i\omega z}\right]=$$

$$=\sqrt{2\pi}\,i\lim_{z\to bi}\frac{d}{dz}\left[\frac{e^{i\omega z}}{(z+bi)^2}\right]=\sqrt{2\pi}\,i\lim_{z\to bi}\frac{i\omega e^{i\omega z}(z+bi)^2-2(z+bi)e^{i\omega z}}{(z+bi)^4}=$$

$$=\sqrt{2\pi}\,ie^{-b\omega}\cdot\frac{i\omega(2bi)^2-2\cdot 2bi}{(2bi)^{4}}=\sqrt{2\pi}\,ie^{-b\omega}\cdot\frac{-2b\omega-2}{-8b^3}=$$

$$=\sqrt{2\pi}\,i\,\frac{b\omega+1}{4b^3}\cdot e^{-b\omega},\qquad \omega\geq 0.$$

$\omega<0$ denean:

$$\hat{f}(\omega)=-\sqrt{2\pi}\,i\operatorname*{Res}_{z=-bi} f(z)e^{i\omega z}=-\sqrt{2\pi}\,i\lim_{z\to -bi}\frac{d}{dz}\left[(z+bi)^2f(z)e^{i\omega z}\right]=$$

$$=-\sqrt{2\pi}\,i\lim_{z\to -bi}\frac{d}{dz}\left[\frac{e^{i\omega z}}{(z-bi)^2}\right]=-\sqrt{2\pi}\,i\lim_{z\to -bi}\frac{i\omega e^{i\omega z}(z-bi)^2-2(z-bi)e^{i\omega z}}{(z-bi)^4}=$$

$$=-\sqrt{2\pi}\,ie^{b\omega}\cdot\frac{i\omega(-2bi)^2-2\cdot(-2bi)}{(-2bi)^{\cancel{4}3}}=-\sqrt{2\pi}\,ie^{b\omega}\cdot\frac{2b\omega-2}{8b^3}=$$

$$=-\sqrt{2\pi}\,i\,\frac{b\omega-1}{8b^3}\,e^{b\omega},\qquad \omega<0.$$

Beraz, gure Fourier transformatua,

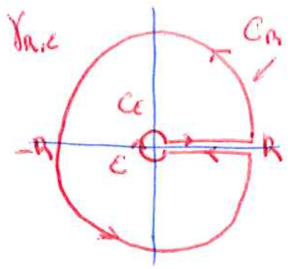
$$\boxed{\hat{f}(\omega) = i\sqrt{2\pi}\cdot\frac{1+|\omega|\cdot b}{4b^3}\, e^{-|\omega| b}}$$

8. ARIKETA

i) $\displaystyle\int_0^{+\infty} \frac{x^{a-1}}{x^2+2x+2}\,dx,\ 0<a<2.$ Izan bedi $F(z) = \dfrac{z^{a-1}}{z^2+2z+2},\ z\in\mathbb{C}$

Gure puntu singularrak $z = \pm i - 1$ dira.

Integra dezagun ondorengo $\gamma_{R,\varepsilon}$ bidean:



eta $z^{a-1} = e^{(a-1)\log z}$, non $\log z = \ln z + i\arg z$.

Hondarren teoremagatik:

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=i-1} F(z) + \operatorname*{Res}_{z=-i-1} F(z)\right) =$$

Polo sinpleak

$$= 2\pi i\left[\lim_{z\to i-1}(z-i+1)F(z) + \lim_{z\to -i-1}(z+i+1)F(z)\right] =$$

$$= 2\pi i\left(\lim_{z\to i-1}\frac{z^{a-1}}{z+i+1} + \lim_{z\to -i-1}\frac{z^{a-1}}{z-i+1}\right) =$$

$$= 2\pi i\left[\frac{(i-1)^{a-1}}{2i} + \frac{(-i-1)^{a-1}}{-2i}\right] = \pi\left[(i-1)^{a-1} - (-i-1)^{a-1}\right] =$$

$$= \pi e^{(a-1)\log(i-1)} - \pi e^{(a-1)\log(i-1)} =$$

$$= \pi e^{(a-1)(\ln\sqrt{2} + i\frac{3\pi}{4})} - \pi e^{(a-1)(\ln\sqrt{2}+i\frac{5\pi}{4})} =$$

$$= \pi e^{(a-1)\ln\sqrt{2}}\left[e^{(a-1)i\frac{3\pi}{4}} - e^{(a-1)i\frac{5\pi}{4}}\right]$$

Bestetik, (*) adierazpena:

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = \int_{\varepsilon}^{R} \frac{e^{(a-1)\cdot \ln x}}{x^2+2x+2}\,dx - \int_{\varepsilon}^{R} \frac{e^{(a-1)(\ln x + 2\pi i)}}{x^2+2x+2} + \int_{C_R} F(z)\,dz - \int_{C_\varepsilon} F(z)\,dz.$$

$$\left|\int_{C_R} F(z)\,dz\right| \leq \int_{C_R} |F(z)||dz| = \int_{C_R} \left| \frac{e^{(a-1)\log z}}{z^2+2z+2} \right| |dz| =$$

$$= \int_{C_R} \frac{e^{(a-1)\cdot \ln|z|}}{|z-i+1|\cdot|z+i+1|}|dz| \leq \frac{R^{a-1}}{(R-i+1)(R+i+1)} 2\pi R \xrightarrow{R\to\infty} 0 \quad (a<2 \text{ da})$$

$$\left|\int_{C_\varepsilon} F(z)\,dz\right| \leq \int_{C_\varepsilon} |F(z)||dz| = \int_{C_\varepsilon} \left| \frac{e^{(a-1)\log z}}{z^2+2z+2} \right| |dz| =$$

$$= \int_{C_\varepsilon} \frac{e^{(a-1)\ln|z|}}{|z-i+1|\cdot|z+i+1|}|dz| \leq \frac{\varepsilon^{a-1}}{(\varepsilon-i+1)(\varepsilon+i+1)} 2\pi\varepsilon \xrightarrow{\varepsilon\to 0} 0 . \quad (a>0 \text{ da})$$

Hortaz, (*) adierazpenean $R\to\infty$ eta $\varepsilon\to 0$ limiteak hartuz:

$$\pi e^{(a-1)\cdot\ln\sqrt{2}}\left[e^{(a-1)\cdot i\frac{3\pi}{2}} - e^{(a-1)i\frac{5\pi}{2}}\right] = \int_0^\infty \frac{e^{(a-1)\ln x}}{x^2+2x+2}\,dx \cdot \left(1 - e^{(a-1)2\pi i}\right)$$

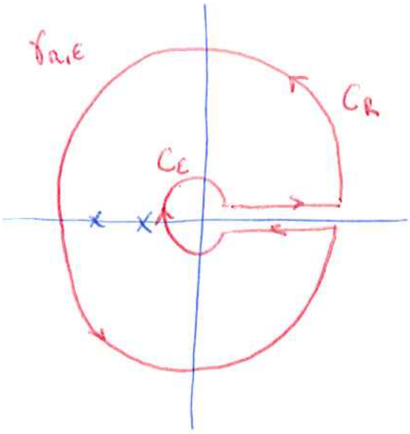
$$\boxed{\int_0^\infty \frac{x^a}{x^2+2x+2}\,dx = \pi e^{(a-1)\ln\sqrt{2}} \cdot \frac{e^{(a-1)i\cdot\frac{3\pi}{2}} - e^{(a-1)i\frac{5\pi}{2}}}{1-e^{(a-1)2\pi i}}}$$

ii) $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}}{(x+1)(x+2)}\,dx$

Izan bedi $F(z) = \dfrac{z^{1/2}}{(z+1)(z+2)}$, non $z^{1/2} = e^{\frac{1}{2}\log z}$.

Gure puntu singularrak $z=-1$ eta $z=-2$ dira.

Ondorengo $\gamma_{R,\varepsilon}$ bidean integratuko dugu F:



Hondarren teoremagatik:

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=-1} F(z) + \operatorname*{Res}_{z=-2} F(z) \right) =$$

$$= 2\pi i \left[\lim_{z\to -1} (z+1) F(z) + \lim_{z\to -2} (z+2) F(z) \right] =$$

$$= 2\pi i \left(\lim_{z\to -1} \frac{z^{1/2}}{z+2} + \lim_{z\to -2} \frac{z^{1/2}}{z+1} \right) = 2\pi i \left(\frac{(-1)^{1/2}}{1} + \frac{(-2)^{1/2}}{-1} \right) =$$

$$= 2\pi i \left(e^{\frac{1}{2}\log(-1)} - e^{\frac{1}{2}\log(-2)} \right) = 2\pi i \left(e^{\frac{1}{2}(\ln 1 + i\pi)} - e^{\frac{1}{2}(\ln 2 + i\pi)} \right) =$$

$$= 2\pi i \left(e^{i\frac{\pi}{2}} - e^{\frac{1}{2}\ln 2} \cdot e^{i\frac{\pi}{2}} \right) = 2\pi i \left(i - \sqrt{2}\cdot i \right) = 2\pi \left(-1 + \sqrt{2} \right)$$

Bestetik, (*) dugu bidearen parametrizazioagatik:

$$\int_{\gamma_{R,\varepsilon}} F(z)\,dz = \int_\varepsilon^R \frac{e^{\frac{1}{2}\ln x}}{(x+1)(x+2)}\,dx + \int_{C_R} F(z)\,dz - \int_\varepsilon^R \frac{e^{\frac{1}{2}(\ln x + 2\pi i)}}{(x+1)(x+2)}\,dx - \int_{C_\varepsilon} F(z)\,dz$$

$|z| = R$ -rako:

$$\left| \int_{C_R} F(z)\,dz \right| \leq \int_{C_R} |F(z)|\cdot|dz| = \int_{C_R} \left| \frac{e^{\frac{1}{2}\log z}}{(z+1)(z+2)} \right| |dz| = \int_{C_R} \frac{e^{\frac{1}{2}\ln|z|}}{|z+1|\cdot|z+2|} |dz| \leq$$

$$\leq \frac{R^{1/2}}{(R-1)(R-2)} 2\pi R \xrightarrow{R\to\infty} 0$$

$|z| = \varepsilon$ -rako:

$$\left| \int_{C_\varepsilon} F(z)\,dz \right| \leq \ldots \leq \frac{\varepsilon^{1/2}}{(\varepsilon-1)(\varepsilon-2)} 2\pi\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

Beraz, (*) adierazpenean $R\to\infty$ eta $\varepsilon\to 0$ limiteak hartzen baditugu:

$$2\pi(-1+\sqrt{2}) = (1-e^{i\pi}) \int_0^{+\infty} \frac{x^{1/2}}{(x+1)(x+2)}\, dx$$

Beraz,

$$\boxed{\int_0^{+\infty} \frac{\sqrt{x}}{(x+1)(x+2)}\, dx = \pi \cdot (\sqrt{2}-1)}$$

12. ARIKETA

Gure funtzioaren singularitateak: $s=1$ baino ez, 2 ordenako poloa dena.

Bestetik, $|f(s)| = \dfrac{1}{|s-1|^2} \leq \dfrac{1}{(R-1)^2} \xrightarrow{R\to+\infty} 0$ $|z|=R$ denean.

Beraz, f-ren alderantzizko transformatua:

$$\boxed{f(x)} = \operatorname*{Res}_{s=1} f(s)e^{sx} = \operatorname*{Res}_{s=1} \frac{e^{sx}}{(s-1)^2} = \lim_{s\to 1} \frac{d}{ds}\left[e^{sx}\right] =$$

$$= \lim_{s\to 1} xe^{sx} \boxed{= xe^x}$$

ANALISI BEKTORIALA ETA KONPLEXUA

Bigarren azterketa partziala. 2015eko maiatzaren 19a.

1. (a) Aurkitu $(e^z+1)^3+8=0$ ekuazioaren soluzio konplexu guztiak.

   (b) Izan bitez $a, b, z \in \mathbf{C}$, $|z|=1$ izanik. Frogatu honako berdintza hau:

$$|az+b| = |\overline{b}z + \overline{a}|.$$

2. Aurkitu $f = u + iv$ holomorfoa plano konplexu osoan baldin eta

$$u_x = 3(x^2-y^2)-4y, \qquad f(1+i)=0 \qquad \text{eta} \qquad f'(0)=0$$

   badira.

3. Izan bedi

$$f(z) = \frac{1+z-\cos z}{z+z^4-\sin z}.$$

   Aztertu f-ren $z_0=0$ puntuko singularitatea, zehaztuz ordena poloa baldin bada eta eman f-ren Laurenten seriearen parte singularra $0<|z|<r$ moduko eraztun batean, $r>0$ izanik.

4. Izan bitez $a>0$ eta $b>0$. Kalkulatu

$$\int_{-\infty}^{\infty} \frac{a\cos x + b\sin x}{x^2+a^2}\, dx.$$

5. Izan bitez $D \subset \mathbf{C}$ sinpleki konexua, $\gamma \subset D$ kurba itxi sinplea orientazio positiboarekin hartuta, $c, z_0 \in D-\gamma$ eta $f\colon D \to \mathbf{C}$ holomorfoa, $f(z) \neq 0$ $z \neq z_0$ guztietarako eta z_0 f-ren zero sinplea izanik. Kalkulatu

$$\int_\gamma \frac{1}{f(z)(z-c)}\, dz$$

   c eta z_0-ren kasu posible guztietarako.

ANALISI BEKTORIALA ETA KONPLEXUA

Fisika eta Ingeniaritza Elektronikoko graduetako 2. kurtsoa - 46. Taldea

Ohiko deialdiko azterketa. Bigarren lauhilabetea. 2015eko ekainaren 1a.

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1. (a) Aurkitu $\sin z = \cosh 4$ ekuazioaren soluzio konplexu guztiak.
   (b) Idatzi $(1+i)^n + (1-i)^n$ zenbakia forma binomikoan, $n \in \mathbf{N}$ izanik.

2. Aurkitu $f = u + iv$ funtzio holomorfoa plano konplexu osoan,

$$u(x,y) = xe^{-x}\cos y + ye^{-x}\sin y \qquad \text{eta} \qquad f(0) = 0$$

badira. Idatzi f z aldagaiaren menpe, $z = x + iy$ izanik.

3. Izan bedi

$$f(z) = \frac{z^2 - 1}{\cos(\pi z) + 1}.$$

   (a) Aurkitu f-ren puntu singularrak eta sailkatu, poloak baldin badaude, ordena zehaztuz.
   (b) Aurkitu $z_0 = 0$ puntuan zentratutako f-ren Taylorren seriearen lehen hiru batugaiak.
   (c) Zein da aurreko ataleko Taylorren seriearen konbergentzia-erradioa?

4. Kalkulatu $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2(1+x^2)}\,dx.$

5. (a) Izan bitez f holomorfoa $|z| < R_0$ zirkuluan, $R_0 > 0$ izanik eta $a \in \mathbf{C}$ non $|a| < R < R_0$. Frogatu berdintza hau:

$$f(a) = \frac{1}{2\pi i}\int_{|z|=R} \frac{R^2 - |a|^2}{(z-a)(R^2 - \overline{a}z)} f(z)\,dz.$$

   (b) Izan bedi Γ $\{z \in \mathbf{C}\colon |z| \leq 1, \operatorname{Im} z \geq 0\}$ multzoaren muga, erlojuaren orratzen kontrako orientazioarekin. Kalkulatu $\displaystyle\int_\Gamma |z|\overline{z}\,dz.$

ANALISI BEKTORIALA ETA KONPLEXUA

Bigarren azterketa partziala. 2016ko maiatzaren 20a.

1. (a) Izan bitez $0 < a < 1$, $z \in \mathbf{C}$ eta $w = \dfrac{z-a}{az-1}$. Frogatu $|w| < 1$ dela baldin eta soilik baldin $|z| < 1$ bada.
   (b) Aurkitu $\sin(z+i) = 1$ ekuazioaren soluzio konplexu guztiak.
   (c) Eman $(ai)^i$ adierazpenaren balio posible guztien parte erreala eta parte irudikaria, $a \in \mathbf{R} - \{0\}$ izanik.
   (d) Izan bedi γ erpinak $1+i$, $-1+i$, $-1-i$ eta $1-i$ puntuetan dituen laukia, erlojuaren orratzen kontrako orientazioarekin. Kalkulatu $\displaystyle\int_\gamma \overline{z}\, dz$.

2. Izan bedi $u(x,y) = 2xy + (e^y + e^{-y})\cos(ax)$. Aurkitu $a > 0$ parametroaren balioak u harmonikoa izan dadin eta, balio horietarako, aurkitu v funtzioa non $f = u+iv$ holomorfoa den eta $f(0) = 2+2i$.

3. Izan bedi $f\colon \mathbf{C} \to \mathbf{C}$ funtzioa holomorfoa plano konplexu osoan.
   (a) Izan bitez $z_1, z_2 \in \mathbf{C}$, $z_1 \neq z_2$ eta $R > \max\{|z_1|, |z_2|\}$. Frogatu honako formula hau:
   $$f(z_1) = f(z_2) + \frac{z_1 - z_2}{2\pi i}\int_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)}\, dz.$$
   (b) Demagun existitzen dela $M > 0$ non $|f(z)| \leq M$ den $z \in \mathbf{C}$ guztietarako. $z_1, z_2 \in \mathbf{C}$ eta $R > \max\{|z_1|, |z_2|\}$ guztietarako, frogatu honako desberdintza hau:
   $$|f(z_1) - f(z_2)| \leq \frac{|z_1 - z_2| M R}{(R - |z_1|)(R - |z_2|)}.$$
   (c) Frogatu **Liouvilleren teorema**: f funtzio osoa eta bornatua bada, orduan konstantea da.

4. Kalkulatu
$$\int_{|z|=1/3} \frac{e^{-1/z^2}}{z(z-1)^2}\, dz \quad \text{eta} \quad \int_{|z|=3} \frac{e^{-1/z^2}}{z(z-1)^2}\, dz.$$

5. Kalkulatu, aldagai konplexuko funtzio baten integral egoki bat erabiliz,
$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 4x + 5}\, dx.$$

ANALISI BEKTORIALA ETA KONPLEXUA

Fisika eta Ingeniaritza Elektronikoko graduetako 2. kurtsoa - 46. Taldea

Ohiko deialdiko azterketa. Bigarren lauhilabetea. 2015eko ekainaren 1a.

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1. (a) **(0.5 puntu)** Izan bitez $z, w \in \mathbf{C}$, $|z| = |w| = 1$. Kalkulatu $|z+w|^2 + |z-w|^2$.
   (b) **(1 puntu)** Aurkitu $i \sinh z + \cosh z = 2$ ekuazioaren soluzio konplexu guztiak.
   (c) **(0.5 puntu)** Aurkitu $2z^4 + 1 - \sqrt{3}i = 0$ ekuazioaren soluzio konplexu guztiak, forma binomikoan idatziz.

2. Izan bedi $f = u + iv$ funtzio holomorfoa jatorriaren ingurune batean, non

$$u(x,y) = e^{-y} \cos x + y - 1$$

den eta existitzen diren $m \in \mathbf{N}$ eta $l \in \mathbf{C} - \{0\}$,

$$\lim_{z \to 0} \frac{f(z)}{z^m} = l$$

delarik. Aurkitu f, m eta l.

3. Izan bedi $f(z) = \dfrac{1}{z^m \sin(\pi z^2)}$, $m \in \mathbf{N}$ izanik.
   (a) Aurkitu eta sailkatu f-ren puntu singularrak, m parametroaren balioen arabera.
   (b) Eman $0 < |z| < R$ moduko eraztun batean konbergentea den Laurenten seriearen parte nagusia, $R > 0$ izanik, $m = 1$ eta $m = 2$ balioetarako.

4. Izan bedi f funtzio holomorfoa $|z| < R$ diskoan, $R > 1$ izanik. Kalkulatu

$$\int_{|z|=1} \left(1 + \frac{2}{z} + \frac{1}{z^2}\right) f(z)\, dz \quad \text{eta} \quad \int_{|z|=1} \left(1 - \frac{2}{z} + \frac{1}{z^2}\right) f(z)\, dz.$$

f eta haren deribatuen balioen menpe.

$f(z) = z$ hartuz, ondorioztatu berdintza hau:

$$\int_0^{2\pi} \cos\theta \cos^2 \frac{\theta}{2}\, d\theta = \frac{\pi}{2}.$$

5. Izan bitez $a > 0$, $b > 0$. Kalkulatu

$$\int_0^{\infty} \frac{(x^2 - b^2)\sin(ax)}{(x^2 + b^2)x}\, dx.$$