TRATAMIENTO DE SEÑALES: Convocatoria ordinaria
Segundo Parcial

El tiempo estimado para resolver el examen es de hora y 30 minutos.
Las 3 cuestiones del problema 1 tienen el mismo valor.

PROBLEMA 1 (10 puntos, 30 minutos)

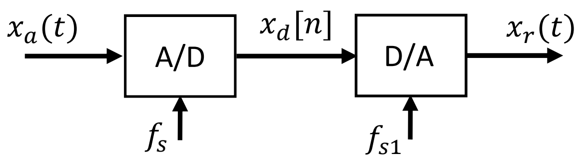
1. Indicar si las siguientes señales son periódicas. Obtener los coeficientes del desarrollo en serie de Fourier de aquellas que lo sean así como su periodo fundamental:

   a. $x(t) = 3\,cos\left(3\,t + \frac{\pi}{4}\right) + 15\,cos\left(7\,t + \frac{\pi}{2}\right)$

   b. $x(t) = 2\,cos\left(2\pi\,10\,t + \frac{\pi}{3}\right) + 7\,cos\left(2\pi\,15\,t + \frac{\pi}{6}\right)$

   c. $x(t) = 5\,cos\left(2\pi\,3\,t + \frac{\pi}{2}\right) + 4\,cos\left(6\,t + \frac{\pi}{3}\right)$

2. En el esquema de la figura se muestrea $x_a(t)$, sin filtro antialiasing, con $f_s = 2$kHz para obtener $x_d[n]$. Mediante un conversor digital analógico ideal trabajando con $f_{s1} = 6$kHz se obtiene $x_r(t)$ a partir de $x_d[n]$.



   Para $x_a(t) = A_1\,cos^2(2\pi f_1 t)$, con $f_1 = 800$Hz se pide:

   a. Calcular $x_d[n]$.

   b. Calcular $x_r(t)$.

3. La señal:

$$x(t) = A_1\,cos(2\pi\,300\,t + \theta_1) + A_2\,cos(2\pi\,750\,t + \theta_2)$$

   se muestrea a una frecuencia f_s sin utilizar filtros antialiasing. Seguidamente se realiza una conversión D/A ideal a una f_s' el doble de f_s obteniéndose la señal $x_r(t) = B_1\,cos(2\pi\,300\,t + \alpha_1) + B_2\,cos(2\pi\,600\,t + \alpha_2)$. Indicar razonadamente que f_s se ha utilizado de entre las siguientes posibilidades: f_s= 800 Hz, f_s= 900 Hz, f_s= 1000 Hz y f_s= 1600 Hz.

PROBLEMA 2 (10 puntos, 30 minutos)

Sea la señal $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$

a) Representar gráficamente *p(t)*, obtener sus coeficientes del desarrollo en serie de Fourier y representar gráficamente *P(ω).* **(2 puntos)**

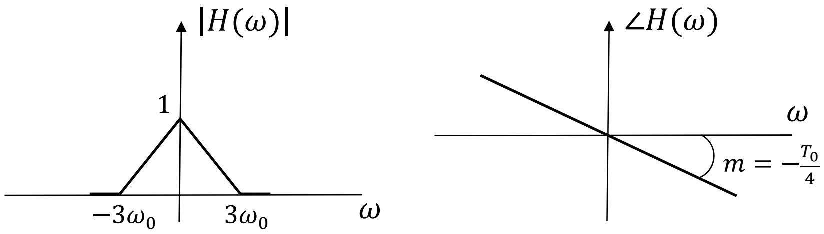
b) La señal *p(t)* es la entrada del sistema cuya respuesta frecuencial, *H(ω)*, en módulo y fase, se representa gráficamente en la figura siendo $\omega_o = \frac{2\pi}{T_o}$. Calcular la salida del sistema, *y(t)*, expresada como suma de funciones sinusoidales. **(2 puntos)**

Se forma la señal *x(t)* como: $x(t) = x_b(t) * p(t)$,

siendo: $x_b(t) = \Lambda\left(\frac{t}{T}\right) = \frac{1}{T} \cdot \prod\left(\frac{t}{T}\right) * \prod\left(\frac{t}{T}\right)$, con $T = \frac{T_o}{2}$

c) Calcular los coeficientes del desarrollo en serie de Fourier de *x(t)* (expresión analítica cerrada en función de *k* y T_o). **(4 puntos)**

d) Calcular la salida, *y(t)*, expresada como suma de funciones sinusoidales, del sistema, cuya respuesta frecuencial es *H(ω)*, cuando la entrada es *x(t)*. **(2 puntos)**



PROBLEMA 3 (10 puntos, 30 minutos)

a) Demostrar que un SLI de respuesta impulsional real para una entrada $x[n] = A_x \cos(\Omega_0 n + \theta_x)$ tiene como salida $y[n] = B_y \cos(\Omega_0 n + \theta_y)$. Encontrar las relaciones entre las amplitudes y fases en función de la respuesta frecuencial del sistema, $H(\Omega)$. **(5 puntos)**

Sea el sistema siguiente:

$$\mathrm{H}(\Omega) = \frac{1 - \mathrm{e}^{-\mathrm{j}2\Omega}}{1 - \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}\Omega} + \mathrm{e}^{-\mathrm{j}2\Omega}}$$

b) Calcular la ecuación en diferencias del sistema e indicar orden y tipo. ¿Es un sistema real? **(2 puntos)**

c) Calcular la respuesta $y[n]$ para las siguientes señales de entrada **(3 puntos)**

$x[n] = 3{,}2 + 2{,}1 \cos\left(\frac{\pi}{4} n\right)$

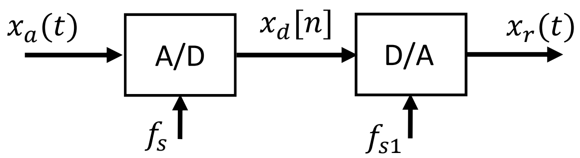
$x[n] = A(-1)^n$

SEINALEEN PROZESAKETA: OHIZKO DEIALDIA
2. partziala

Azterketa bukatzeko ordu eta 30 minutu dituzue. Azterketak 3 ariketa ditu. Ariketa bakoitzak 10 puntu balio ditu, eta 1. ariketako galdera guztiek pisu berdina dute.

1. ARIKETA (10 puntu, 30 minutu)

1. Arrazoitu honako seinaleak periodikoak diren. Kalkulatu Fourieren serieko koefizienteak eta periodoa periodikoak direnentzat:
   a. $x(t) = 3\,cos\left(3\,t + \frac{\pi}{4}\right) + 15\,cos\left(7\,t + \frac{\pi}{2}\right)$
   b. $x(t) = 2\,cos\left(2\pi\,10\,t + \frac{\pi}{3}\right) + 7\,cos\left(2\pi\,15\,t + \frac{\pi}{6}\right)$
   c. $x(t) = 5\,cos\left(2\pi\,3\,t + \frac{\pi}{2}\right) + 4\,cos\left(6\,t + \frac{\pi}{3}\right)$
2. Irudiko seinalea, $x_a(t)$, lagindu egiten da antialiasing iragazkirik gabe, $f_s = 2$kHz laginketa-maiztasuna erabiliz $x_d[n]$ lortzeko. D/A bihurgailu idealarekin, $f_{s1} = 6$kHz laginketa-maiztasunarekin $x_r(t)$ lortzen da.



   Hartu $x_a(t) = A_1\,cos^2(2\pi f_1 t)$, non $f_1 = 800$Hz den. Honakoak eskatzen dira:
   a. Kalkulatu $x_d[n]$.
   b. Kalkulatu $x_r(t)$.

3. Izan bedi honako seinalea,

$$x(t) = A_1\,cos(2\pi\,300\,t + \theta_1) + A_2\,cos(2\pi\,750\,t + \theta_2)$$

   f_s laginketa maiztasunarekin lagintzen dena, antialiasing iragazkirik gabe. Jarraian, D/A bihurketa ideala egiten da f_s' laginketa-maiztasuna erabiliz, f_s –ren bikoitza dena, honako seinalea lortzeko:

$$x_r(t) = B_1\,cos(2\pi\,300\,t + \alpha_1) + B_2\,cos(2\pi\,600\,t + \alpha_2).$$

   Arrazoitu zein f_s erabili den, honakoen artean: f_s= 800 Hz, f_s= 900 Hz, f_s= 1000 Hz edo f_s= 1600 Hz.

2. ARIKETA (10 puntu, 30 minutu)

Izan bedi honako seinalea $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$

a) Irudikatu *p(t)*, lortu bere Fourieren serieko koefizienteak, eta kalkulatu eta irudikatu *P(ω).* **(2 puntu)**

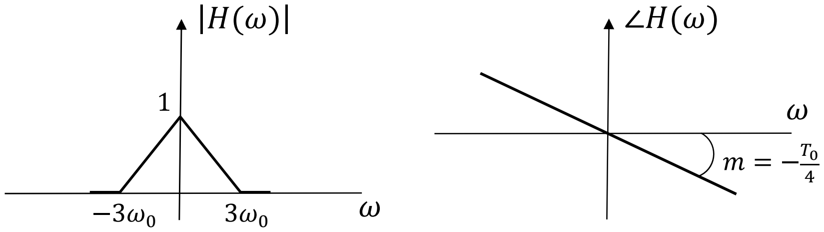
b) Aurreko seinalea, *p(t)*, sistema baten sarrera-seinalea da, *H(ω)* maiztasun erantzuna duena (modulua eta fasea irudian adierazita, non $\omega_o = \frac{2\pi}{T_o}$). Kalkulatu sistemaren irteera-seinalea, *y(t)*, eta sinusoidalen menpe adierazi. **(2 puntu)**

Honako *x(t)* seinalea osatzen da: $x(t) = x_b(t) * p(t)$,

non: $x_b(t) = \Lambda\left(\frac{t}{T}\right) = \frac{1}{T} \cdot \prod\left(\frac{t}{T}\right) * \prod\left(\frac{t}{T}\right)$ den, eta $T = \frac{T_o}{2}$

c) Kalkulatu *x(t)*–ren Fouerieren serieko koefizienteak (adierazpen itxi bat *k* eta T_o-ren menpe). **(4 puntu)**

d) Kalkulatu *y(t)* irteera-seinalea eta sinusoidalen batuketa bezala adierazi. Horretarako hartu *x(t)* sarrera-seinalea eta *H(ω)* maiztasun-erantzuna duen sistema. **(2 puntu)**



3. ARIKETA (10 puntu, 30 minutu)

a) Frogatu ezazu $h[n]$ erreala duen LTI sistemaren irteera-seinalea $y[n] = B_y \cos(\Omega_0 n + \theta_y)$ dela sarrera-seinalea $x[n] = A_x \cos(\Omega_0 n + \theta_x)$ denean. Lortu anplitudeen eta faseen arteko erlazioak sistemaren maiztasun-erantzunaren menpe, $H(\Omega)$-ren menpe alegia. **(5 puntu)**

Izan bedi honako sistema:

$$\mathrm{H}(\Omega) = \frac{1 - \mathrm{e}^{-\mathrm{j}2\Omega}}{1 - \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}\Omega} + \mathrm{e}^{-\mathrm{j}2\Omega}}$$

b) Kalkulatu sistemaren diferentzia-ekuazioa eta adierazi sistemaren ordena eta mota. Sistema erreala al da? **(2 puntu)**

c) Kalkulatu irteera-seinalea, $y[n]$, honako sarrera-seinaleentzat. **(3 puntu)**

$x[n] = 3{,}2 + 2{,}1 \cos\left(\frac{\pi}{4} n\right)$

$x[n] = A(-1)^n$

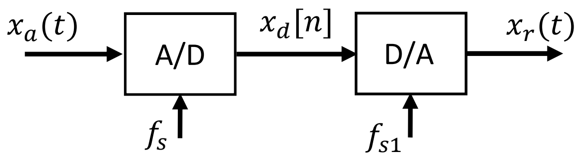
SIGNAL PROCESSING: Final exam
Second mid-term

The duration of the exam is one hour and 30 minutes.
All short questions in the first problem have the same value.

PROBLEM 1 (10 points, 30 minutes)

1. State if the following signals are periodic. If so, obtain their Fourier series coefficients and their fundamental period.
   a. $x(t) = 3\,cos\left(3\,t + \frac{\pi}{4}\right) + 15\,cos\left(7\,t + \frac{\pi}{2}\right)$
   b. $x(t) = 2\,cos\left(2\pi\,10\,t + \frac{\pi}{3}\right) + 7\,cos\left(2\pi\,15\,t + \frac{\pi}{6}\right)$
   c. $x(t) = 5\,cos\left(2\pi\,3\,t + \frac{\pi}{2}\right) + 4\,cos\left(6\,t + \frac{\pi}{3}\right)$

2. In the diagram of the figure the signal $x_a(t)$ is sampled without anti-aliasing filter and a sampling frequency of $f_s = 2$kHz, to obtain the signal $x_d[n]$. Using an ideal digital to analog converter with sampling frequency $f_{s1} = 6$kHz we recover the signal $x_r(t)$ from $x_d[n]$.



   Consider the signal $x_a(t) = A_1\,cos^2(2\pi f_1 t)$, with $f_1 = 800$Hz. Answer the following questions:
   a. Compute $x_d[n]$.
   b. Compute $x_r(t)$.

3. Consider the following signal:

$$x(t) = A_1\,cos(2\pi\,300\,t + \theta_1) + A_2\,cos(2\pi\,750\,t + \theta_2)$$

   sampled with frequency fs and without anti-aliasing filter. Then we do a digital to analog conversion with a frequency fs' twice the value of fs to obtain the signal $x_r(t) = B_1\,cos(2\pi\,300\,t + \alpha_1) + B_2\,cos(2\pi\,600\,t + \alpha_2)$. Indicate, in a justified manner, which fs was used, choose one of these four possibilities: fs= 800 Hz, fs= 900 Hz, fs= 1000 Hz and fs= 1600 Hz.

PROBLEM 2 (10 points, 30 minutes)

Consider the following signal: $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$

a) Sketch the signal *p(t)*, obtain its Fourier series coefficients and sketch its Fourier transform *P(ω).* **(2 points)**

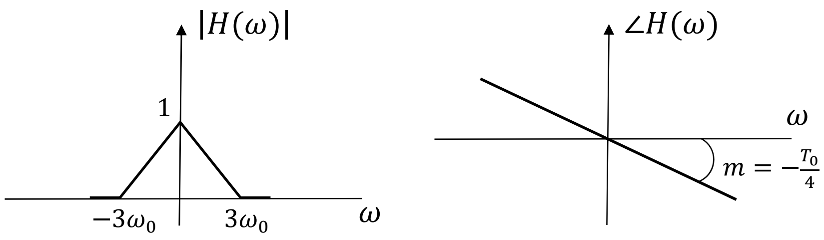
b) The signal *p(t)* is the input to a system whose frequency response, *H(ω)*, is shown in the figure in module and phase, where $\omega_o = \frac{2\pi}{T_o}$. Compute the output signal, *y(t)*, expressed as a sum of sinusoidal functions. **(2 points)**

We form the signal *x(t)* as: $x(t) = x_b(t) * p(t)$.

Where: $x_b(t) = \Lambda\left(\frac{t}{T}\right) = \frac{1}{T} \cdot \prod\left(\frac{t}{T}\right) * \prod\left(\frac{t}{T}\right)$, and $T = \frac{T_o}{2}$.

c) Compute the Fourier series coefficients of *x(t)* (closed form analytical expression in terms of k and T_o). **(4 points)**

d) Compute the output of the system, *y(t)*, expressed a sum of sinusoidal functions when the input is *x(t)* and the system's frequency response is the *H(ω)* shown in the figure. **(2 points)**



PROBLEM 3 (10 points, 30 minutes)

a) Consider a LTI system with a real impulse response. Show that for an input signal of the form $x[n] = A_x \cos(\Omega_0 n + \theta_x)$ the output is $y[n] = B_y \cos(\Omega_0 n + \theta_y)$. Obtain the relations between the amplitudes and phases of the input and output signals in terms of the system's frequency response: $H(\Omega)$. **(5 points)**

Consider the following system:

$$\mathrm{H}(\Omega) = \frac{1 - \mathrm{e}^{-\mathrm{j}2\Omega}}{1 - \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}\Omega} + \mathrm{e}^{-\mathrm{j}2\Omega}}$$

b) Obtain the system's difference equation and indicate the system type and its order. Is the system real? **(2 points)**

c) Compute the output $y[n]$ for the following input signals **(3 points)**

$x[n] = 3{,}2 + 2{,}1 \cos\left(\frac{\pi}{4}n\right)$

$x[n] = A(-1)^n$

CUESTION 1

a) $x(t) = 3\cos(3t + \pi/4) + 15\cos(7t + \pi/2)$ **Si es periodica**

$w_1 = 3$ $w_0 = M.C.D(w_1, w_2) = 1$

$w_2 = 7$ $w_0 = \frac{2\pi}{T_0} = 1$ **$T_0 = 2\pi$ s**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0t} = \frac{3}{2} e^{j\frac{\pi}{4}} e^{j3t} + \frac{3}{2} e^{-j\frac{\pi}{4}} e^{-j3t} + \frac{15}{2} e^{j\frac{\pi}{2}} e^{j7t} + \frac{15}{2} e^{-j\frac{\pi}{2}} e^{-j7t}$$

$a_3 = \frac{3}{2} e^{j\pi/4}$ $a_7 = \frac{15}{2} e^{j\pi/2}$ $a_k = 0 \quad \forall\, k \neq 3, -3, 7, -7$

$a_{-3} = \frac{3}{2} e^{-j\pi/4}$ $a_{-7} = \frac{15}{2} e^{-j\pi/2}$.

b) $x(t) = 2\cos(2\pi 10 t + \frac{\pi}{3}) + 7\cos(2\pi 15 t + \pi/6)$ **Si es periodica**

$w_1 = 2\pi \cdot 10 \Rightarrow f_1 = 10$ Hz $f_0 = M.C.D(10, 15) = 5$

$w_2 = 2\pi \cdot 15 \Rightarrow f_2 = 15$ Hz **$T_0 = \frac{1}{5} = 0'2$ s**

$w_0 = 2\pi \cdot 5$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0t} = e^{j\frac{\pi}{3}} e^{j2\pi \cdot 2 \cdot 5 t} + e^{-j\frac{\pi}{3}} e^{-j2\pi \cdot 2 \cdot 5 \cdot t} + \frac{7}{2} e^{j\frac{\pi}{6}} e^{j2\pi \cdot 3 \cdot 5 \cdot t} + \frac{7}{2} e^{-j\frac{\pi}{6}} e^{-j2\pi \cdot 3 \cdot 5 \cdot t}.$$

$a_2 = e^{j\pi/3}$ $a_3 = \frac{7}{2} e^{j\pi/6}$ $a_k = 0 \quad \forall\, k \neq 2, -2, 3, -3$

$a_{-2} = e^{-j\pi/3}$ $a_{-3} = \frac{7}{2} e^{-j\pi/6}$

c) $x(t) = 5 \cos(2\pi\, 3t + \frac{\pi}{2}) + 4 \cos(6t + \pi/3)$

$W_1 = 2\pi \cdot 3$

$W_2 = 6$

$W_0 = M.C.D(W_1, W_2) \Rightarrow$ No existe.

$\frac{W_1}{W_2}$ No es una relación de números enteros.

No es periódica.

CUESTION 2

$$x_d[n] = x_a(t)\Big|_{t=\frac{n}{f_s}} = A_1 \cos^2\left(2\pi \cdot 8 \cdot \frac{n}{20}\right)$$

$$x_d[n] = A_1 \cos^2\left(2\pi \cdot \frac{8}{20} n\right) = A_1 \cos^2\left(2\pi \frac{2}{5} n\right)$$

$$x_d[n] = A_1 \frac{1 + \cos\left(2\pi \frac{4}{5} n\right)}{2} = \frac{A_1}{2} + \frac{A_1}{2} \cos\left(2\pi \frac{4}{5} n\right)$$

$f_d = \frac{4}{5} > \frac{1}{2}$ Hay que corregir

$$f_d = \frac{4}{5} - 1 = \frac{-1}{5}$$

$$x_d[n] = \frac{A_1}{2} + \frac{A_1}{2} \cos\left(2\pi \left(-\frac{1}{5}\right) n\right) = \boxed{\frac{A_1}{2} + \frac{A_1}{2} \cos\left(2\pi \frac{1}{5} n\right)}$$

$x_d[n]$:
- $f_{d_1} = 0$
- $f_{d_2} = \frac{1}{5}$

dos componentes de frecuencias discretas.

En la conversión D/A ideal a $f_{s_1} = 6$ kHz.

las frecuencias discretas pasan a ser en analógico:

$$f_{a_1} = f_{d_1} \cdot f_{s_1} = 0 \cdot 6000 = 0 \text{ Hz}$$

$$f_{a_2} = f_{d_2} \cdot f_{s_1} = \frac{1}{5} \cdot 6000 = 1200 \text{ Hz.}$$

$$\boxed{x_r(t) = \frac{A_1}{2} + \frac{A_1}{2} \cos\left(2\pi \cdot 1200\, t\right)}$$

$$\boxed{x_r(t) = A_1 \cos^2\left(2\pi\, 600\, t\right)}$$

CUESTION 3

$$x(t) = A_1 \cos(2\pi 300 t + \theta_1) + A_2 \cos(2\pi 750 t + \theta_2)$$

Tiene componentes frecuenciales en $f_1 = 300$
$f_2 = 750$

En la conversión A/D se obtienen frecuencias discretas

$$fd_1 = \frac{f_1}{f_s}$$

No habrá que corregirlas si

$$f_s > 2 f_{máx}$$

Caso de $f_s = 1600 Hz$

$$fd_2 = \frac{f_2}{f_s}$$

En la conversión D/A salen las frecuencias

$$fa_1 = fd_1 \cdot f_s' = \frac{f_1}{f_s} \cdot 2 f_s = 2 f_1 = 600 \text{ Hz}$$

$$fa_2 = fd_2 \cdot f_s' = \frac{f_2}{f_s} \cdot 2 f_s = 2 f_2 = 1500 \text{ Hz}$$

Esta componente no está en la señal recuperada

$f_s = 1600$ Hz no puede ser

Caso de $f_s = 1000$ Hz

$fd_1 = \frac{300}{1000} = \frac{3}{10} < \frac{1}{2}$ No se corrige.

$fd_2 = \frac{750}{1000} = \frac{75}{100} > \frac{1}{2}$ Se corrige.

$fd_2' = \frac{75}{100} - 1 = \frac{-25}{100} = \frac{25}{100}$ por ser coseno.

$fa_1 = fd_1 \cdot f_s' = \frac{3}{10} \cdot 2000 = 600$ Hz ($2f_s$)

$fa_2 = fd_2' \cdot f_s' = \frac{25}{100} \cdot 2000 = 500$ Hz ← Esta componente no está en $x_r(t)$.

$f_s = 1000$ Hz no puede ser

$f_s = 900$ Hz

$$f_{d_1} = \frac{300}{900} = \frac{3}{9}$$

$$f_{d_2} = \frac{750}{900} = \frac{75}{90}$$ Hay que corregir.

$$f_{d_2}' = \frac{75}{90} - 1 = -\frac{15}{90} = \frac{15}{90}$$ (por ser coseno)

$$f_{a_1} = f_{d_1} \cdot f_s' = \frac{3}{9} \cdot 1800 = 600 \text{ Hz}$$

$$f_{a_2} = f_{d_2} \cdot f_s' = \frac{15}{90} \cdot 1800 = 300 \text{ Hz}$$

Son las dos componentes que están en $x_r(t)$

$f_s = 900$ Hz — Cumple la condición

Compruebo que pasa con $f_s = 800$ Hz

$$f_{d_1} = \frac{300}{800} = \frac{3}{8}$$

$$f_{d_2} = \frac{750}{800} = \frac{75}{80}$$ Hay que corregir. $f_{d_2}' = \frac{75}{80} - 1 = \frac{-5}{80} = \frac{-1}{16} ; \frac{1}{16}$ (por ser coseno)

$$f_{a_1}^* = f_{d_1} \cdot f_s' = \frac{3}{8} \cdot 1600 = 600 \text{ Hz}$$

$$f_{a_2} = f_{d_2}' \cdot f_s' = \frac{1}{16} \cdot 1600 = 100 \text{ Hz}$$ ← esta componente no existe en $x_r(t)$.

Conclusión: La única f_s posible es la de **900 Hz**

Otra forma:

$x(t) = A_1 \cos(2\pi 300 t) + A_2 \cos(2\pi 750 t)$

$x_r(t) = B_1 \cos(2\pi 300 t) + A_2 \cos(2\pi \cdot 600 t)$.

$f_s = 800, 900, 1000, 1600$ Hz

→ $f_1 = 300$ Hz para que no exista aliasing $f_s > 600$ lo cumplen todas. No existirá aliasing pero en la conversión D/A al ser a $f_s' = 2f_s$ se recupera una señal de 600 Hz

$fd_1 = \frac{f_1}{f_s}$ $\quad f_{1r} = fd_1 \, f_s' = \frac{f_1}{f_s} \cdot f_s'$

$f_{1r} = \frac{300}{f_s} \cdot 2 f_s = 600$ Hz

Entonces la componente de 600 en $x_r(t)$ es debida a la de 300.

Por tanto la componente de 300 en $x_r(t)$ tiene que venir provocada por la $f_2 = 750$ Hz por aliasing.

$f_s \leq 1500$ Hz $\quad$ La $f_s = 1600$ no cumple.

$f_s = 800, 900$ y 1000 producen aliasing en una de $f_2 = 750$ Hz

¿Cuál es la buena?

800 $\quad fd_2 = \frac{f_2}{f_s} = \frac{750}{800} = \frac{15}{80}$ $\quad f_2' = fd_2' \cdot f_s' = \frac{5}{80} \cdot 1600 = 100$ Hz ✗ NO VALE

$\frac{75}{80} - 1 = -\frac{5}{80} \underset{\cos}{\Rightarrow} \frac{5}{80}$

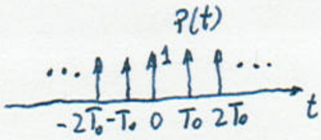
900 $\quad fd_2 = \frac{750}{900} = \frac{75}{90}$ $\quad \frac{75}{90} - 1 = -\frac{15}{90} \underset{\cos}{\Rightarrow} \frac{15}{90}$ $\quad f_2' = \frac{15}{90} \cdot 1800 = 300$ ✓ VALE

1000 $\quad fd_2 = \frac{750}{1000} = \frac{75}{100}$ $\quad \frac{75}{100} - 1 = -\frac{25}{100} \underset{\cos}{\Rightarrow} \frac{1}{4}$ $\quad f_2' = \frac{1}{4} \cdot 2000 = 500$ ✗ NO VALE

La f_s correcta es 900 Hz

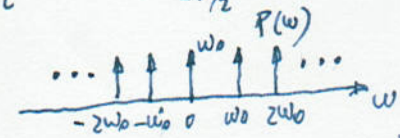
PROBLEMA 2

a) $p(t)=\sum_k \delta(t-kT_0)$



$a_k=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} p(t)e^{-jk\omega_0 t}\,dt=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\underbrace{\delta(t)e^{-jk\omega_0 t}}_{\delta(t)\cdot 1}\,dt=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)\,dt=\frac{1}{T_0}\ \forall k$

$P(\omega)=\sum_k 2\pi a_k\,\delta(\omega-k\omega_0)=\sum_k \omega_0\,\delta(\omega-k\omega_0)$

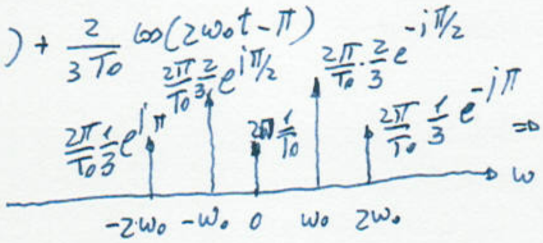


b) Una forma: $y(t)$ periódica de periodo T_0 con c.d.s.F. $b_k=a_k H(\omega)\big|_{\omega=k\omega_0}$ $\Rightarrow$
$$\begin{cases} b_0=a_0\cdot H(0)=a_0=\frac{1}{T_0} \\ b_{\pm1}=a_{\pm1}H(\pm\omega_0)=\frac{1}{T_0}\frac{2}{3}e^{\mp j\pi/2} \\ b_{\pm2}=a_{\pm2}H(\pm2\omega_0)=\frac{1}{T_0}\frac{1}{3}e^{\mp j\pi} \\ b_k=0 \quad \text{otro } k\end{cases}$$

$\Rightarrow y(t)=\sum_{k=-2}^{2} b_k e^{jk\omega_0 t}=\frac{1}{T_0}+\frac{2}{3T_0}e^{-j\pi/2}e^{j\omega_0 t}+\frac{2}{3T_0}e^{j\pi/2}e^{-j\omega_0 t}+\frac{1}{3T_0}e^{-j\pi}e^{j2\omega_0 t}+\frac{1}{3T_0}e^{j\pi}e^{-j2\omega_0 t}$

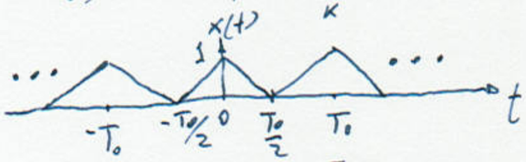
$\Rightarrow y(t)=\frac{1}{T_0}+\frac{4}{3T_0}\cos(\omega_0 t-\pi/2)+\frac{2}{3T_0}\cos(2\omega_0 t-\pi)$

También: $Y(\omega)=P(\omega)\cdot H(\omega)\Rightarrow$



$\Rightarrow y(t)=\frac{1}{T_0}+\frac{4}{3T_0}\cos(\omega_0 t-\pi/2)+\frac{2}{3T_0}\cos(2\omega_0 t-\pi)$

c) $x(t)=x_b(t)*p(t)=\sum_k x_b(t-kT_0)$ periódica de periodo T_0 con $x_b(t)=\Lambda\left(\frac{t}{T}\right)=\frac{1}{T}\underbrace{\Pi\left(\frac{t}{T}\right)*\Pi\left(\frac{t}{T}\right)}_{x_1(t)}$; $T=\frac{T_0}{2}$



$a_k=\frac{1}{T_0}X_b(\omega)\big|_{\omega=k\omega_0}$ $\quad X_b(\omega)=\frac{1}{T}X_1(\omega)\cdot X_1(\omega)=\frac{1}{T}X_1^2(\omega)$

$x_1(t)=\Pi\left(\frac{t}{T}\right)\xrightarrow{\mathcal{F}}X_1(\omega)=2\frac{\operatorname{sen}(\omega T/2)}{\omega}\Rightarrow X_b(\omega)=\frac{1}{T_0/2}\cdot\frac{4\operatorname{sen}^2(\omega T_0/4)}{\omega^2}$

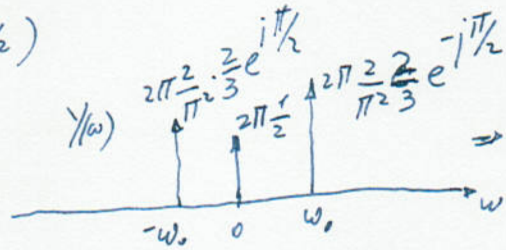
$a_k=\frac{1}{T_0}\cdot\frac{1}{T_0/2}\cdot\frac{4\operatorname{sen}^2\left(k\frac{2\pi}{T_0}\cdot\frac{T_0}{4}\right)}{k^2\frac{4\pi^2}{T_0^2}}\Rightarrow a_k=2\frac{\operatorname{sen}^2(k\pi/2)}{k^2\pi^2}\Rightarrow\begin{cases}a_0=\frac{1}{2}\\ a_{\pm1}=\frac{2}{\pi^2}\\ a_{\pm2}=0\end{cases}$

d) Una forma: $y(t)$ periódica de periodo T_0 con c.d.s.F. $b_k=a_k H(k\omega_0)=\begin{cases}b_0=a_0\,1=\frac{1}{2}\\ b_{\pm1}=a_{\pm1}H(\pm\omega_0)=\frac{2}{\pi^2}\cdot\frac{2}{3}e^{\mp j\pi/2}\\ b_k=0 \quad \text{otro } k\end{cases}$

$\Rightarrow y(t)=\sum_{k=-1}^{1} b_k e^{jk\omega_0 t}=\frac{1}{2}+\frac{4}{3\pi^2}e^{-j\pi/2}e^{j\omega_0 t}+\frac{4}{3\pi^2}e^{j\pi/2}e^{-j\omega_0 t}\Rightarrow$

$\Rightarrow y(t)=\frac{1}{2}+\frac{8}{3\pi^2}\cos(\omega_0 t-\pi/2)$

También: $Y(\omega)=X(\omega)\cdot H(\omega)\Rightarrow$



$\Rightarrow y(t)=\mathcal{F}^{-1}\{Y(\omega)\}=\frac{1}{2}+\frac{8}{3\pi^2}\cos(\omega_0 t-\pi/2)$

P2 - 3. ARIKETA

a) $h[n]$ real $\rightarrow$ $H(\Omega)$ simetría hermítica, $|H(\Omega)| = |H(-\Omega)|$

$\sphericalangle H(\Omega) = -\sphericalangle H(-\Omega)$

$x[n] = A_x \cdot \cos(\Omega_0 n + \theta_x)$

$x[n] = \frac{A_x}{2} e^{j\theta_x} e^{j\Omega_0 n} + \frac{A_x}{2} e^{-j\theta_x} e^{-j\Omega_0 n}$

Tras un LTI:

$y[n] = \frac{A_x}{2} e^{j\theta_x} e^{j\Omega_0 n} \cdot H(\Omega_0) + \frac{A_x}{2} e^{-j\theta_x} e^{-j\Omega_0 n} H(-\Omega_0)$

$y[n] = \frac{A_x}{2} e^{j\theta_x} e^{j\Omega_0 n} |H(\Omega_0)| e^{j\sphericalangle H(\Omega_0)} + \frac{A_x}{2} e^{-j\theta_x} e^{-j\Omega_0 n} |H(-\Omega_0)| e^{j\sphericalangle H(-\Omega_0)}$

$|H(-\Omega_0)| = |H(\Omega_0)|$, $\quad \sphericalangle H(-\Omega_0) = -\sphericalangle H(\Omega_0)$

$y[n] = \underbrace{A_x |H(\Omega_0)|}_{B_y} \cos(\Omega_0 n + \underbrace{\theta_x + \sphericalangle H(\Omega_0)}_{\theta_y})$

b) $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 - e^{-j2\Omega}}{1 - \frac{1}{\sqrt{2}} e^{-j\Omega} + e^{-j2\Omega}}$

$y[n] - 1/\sqrt{2}\, y[n-1] + y[n-2] = x[n] - x[n-2]$ IIR, orden 2

Real, p.q los coeficientes son reales.

c) • $x[n] = 3'2 + 2'1 \cos\left(\frac{\pi}{4} n\right)$

$y[n] = 3'2 \cdot H(\Omega=0) + 2'1 |H(\pi/4)| \cos\left(\frac{\pi}{4} n + \sphericalangle H(\pi/4)\right)$

$H(0) = 0$

$H(\pi/4) = 2 \angle \pi$

$y[n] = 4'2 \cos\left(\frac{\pi}{4} n + \frac{\pi}{2}\right)$

• $x[n] = A(-1)^n = A e^{j\pi n}$

$y[n] = A\, H(\Omega=\pi) \cdot e^{j\pi n} = \emptyset \quad \forall n$

$H(\pi) = \emptyset$

TRATAMIENTO DE SEÑALES: EXAMEN FINAL (Primer parcial)

La puntuación total del examen es de 30 puntos divididos en 3 problemas del mismo valor. Las cuestiones del primer problema tienen el mismo peso. La duración del examen es de una hora 30 minutos.

PROBLEMA 1 (10 puntos, 30 minutos)

1. Sea el siguiente sistema que añade una eco a la señal de entrada. Conocemos además su respuesta impulsional.

$$y[n] = x[n] + \alpha x[n-k] \qquad \text{y} \qquad h[n] = \{\underline{1}, 0,0,0,0, \tfrac{1}{5}\}$$

   Obtener la expresión analítica de la respuesta impulsional en función de α y k. Hallar los valores de α y k.

2. Se sabe que la salida de un sistema LTI es $y(t) = 2\wedge\left(\frac{t-1}{4}\right)$ y que la respuesta impulsional del sistema es $h(t) = \prod\left(\frac{t-2}{4}\right)$. Representar la señal de entrada al sistema $x(t)$.

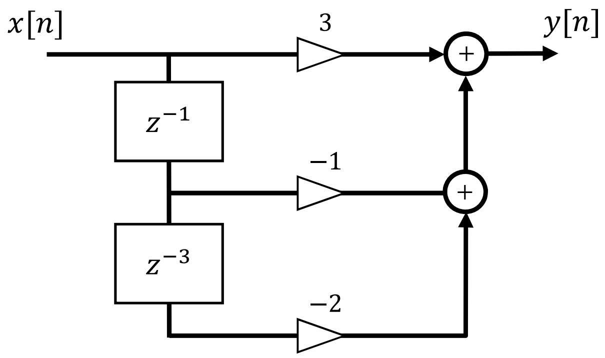
3. Sea el sistema con la siguiente ecuación en diferencias (condiciones iniciales nulas):

$$y[n] = x[n-1] + ay[n-1]$$

   Hallar $h[n]$, la respuesta impulsional del sistema y dar una expresión analítica cerrada. Obtener la condición que debe cumplir a para que el sistema sea estable.

PROBLEMA 2 (10 puntos, 30 minutos)

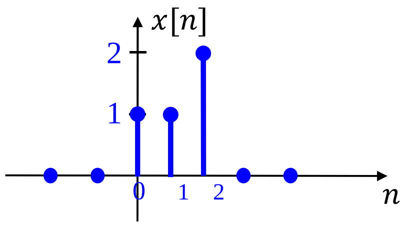
Sea el sistema de la figura.



a. Escribir la ecuación en diferencias que relaciona $y[n]$ con $x[n]$ y especificar el tipo y el orden del sistema. **(3 puntos)**

b. Dibujar $h[n]$, la respuesta impulsional del sistema. **(1 puntos)**

c. Calcular $y[n]$, si la señal de entrada es la de la figura. **(3 puntos)**



d. Dibujar $y[n]$, si la señal de entrada es $u[n]$. **(3 puntos)**

PROBLEMA 3 (10 puntos, 30 minutos)

Se dispone de 3 sistemas LTI con las siguientes respuestas impulsionales:

$$h_1(t) = 3\prod(t/2) - 2\delta(t-4)$$
$$h_2(t) = \delta(t-1)$$
$$h_3(t) = 2\delta(t-5)$$

Los sistemas $h_1(t)$ y $h_2(t)$ se conectan en cascada, y el sistema resultante en paralelo con $h_3(t)$.

a. Analiza la causalidad y estabilidad de cada uno de los sistemas. **(3 puntos)**
b. Dibuja el diagrama de bloques del conjunto**.** **(2 puntos)**
c. Calcula y dibuja la respuesta impulsional del sistema completo y analiza su causalidad y estabilidad. **(2 puntos)**
d. Obtener la salida si la entrada es: $x(t) = \prod(t/2) + \delta(t-10)$ **(3 puntos)**

SEINALEEN PROZESAKETA: AZKEN AZTERKETA
(Lehen partziala)

Azterketak 3 ariketa ditu. Ariketa bakoitzak 10 puntu balio ditu, eta 1. Ariketako galdera guztiek pisu berdina dute. Azterketaren iraupena ordu bat eta 30 minutu da.

1. ARIKETA (10 puntu, 30 minutu)

1. Ondoko sistemak oihartzuna gehitzen dioa sarrera-seinaleari. Sistemaren pultsu-erantzuna ere ezaguna da:

$$y[n] = x[n] + \alpha x[n-k] \qquad \text{eta} \qquad h[n] = \{\underline{1}, 0,0,0,0, \tfrac{1}{5}\}$$

   Kalkulatu sistemaren pultsu-erantzuna α eta k balioen menpe. Lortu α eta k.

2. LTI sistema baten irteera-seinalea $y(t) = 2 \wedge\left(\frac{t-1}{4}\right)$ da, eta sistemaren pultsu-erantzuna berriz: $h(t) = \prod\left(\frac{t-2}{4}\right)$. Irudikatu sistemaren sarrera-seinalea, $x(t)$.

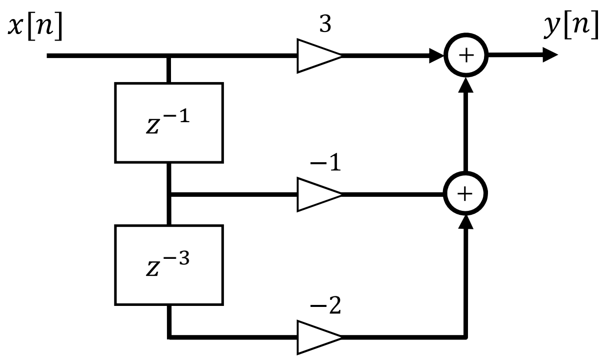
3. Izan bedi hurrengo diferentzia-ekuazioa duen sistema (hasierako baldintza nuluak):

$$y[n] = x[n-1] + ay[n-1]$$

   Lortu $h[n]$, sistemaren-pultsu erantzuna eta lortu pultsu-erantzunaren adierazpide analitikoa. Lortu a-k bete behar duen baldintza sistema egonkorra izan dadin.

2. ARIKETA (10 puntu, 30 minutu)

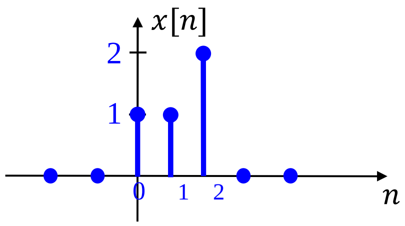
Izan bedi irudiko sistema.



a. Lortu sistemaren diferentzia ekuazioa, $y[n]$ eta $x[n]$ erlazionatzen dituena, eta zehaztu sistema mota eta maila. **(3 puntu)**

b. Irudikatu $h[n]$, sistemaren pultsu-erantzuna. **(1 puntu)**

c. Kalkulatu $y[n]$, sarrera-seinalea irudikoa bada. **(3 puntu)**



d. Irudikatu $y[n]$, sarrera-seinalea $u[n]$ bada. **(3 puntu)**

3. ARIKETA (10 puntu, 30 minutu)

Izan bitez ondoko pultsu-erantzunak dituzten 3 sistema LTI:

$h_1(t) = 3\prod(t/2) - 2\delta(t-4)$

$h_2(t) = \delta(t-1)$

$h_3(t) = 2\delta(t-5)$.

$h_1(t)$ eta $h_2(t)$ sistemak jauzian konektatzen dira, eta horrela lortutako sistema paraleloan konektatu da $h_3(t)$-rekin.

a. Aztertu sistema bakoitza kausala edota egonkorra den. **(3 puntu)**

b. Irudikatu multzoaren bloke diagrama**.** **(2 puntu)**

c. Kalkulatu eta irudikatu sistema osoaren pultsu-erantzuna eta aztertu sistema osoa kausala edota egonkorra den. **(2 puntu)**

d. Lortu irteera-seinalea, sarrera-seinalea hau denean: $x(t) = \prod(t/2) + \delta(t-10)$ **(3 puntu)**

SIGNAL PROCESSING: FINAL EXAM
(First mid-term)

The exam scores a total of 30 points equally divided in 3 problems. All the short questions in the first problem have the same value. The duration of the exam is one hour and 30 minutes.

PROBLEM 1 (10 points, 30 minutes)

1. The following system adds an echo to the input signal. We know the system's impulse response:

   $$y[n] = x[n] + \alpha x[n-k] \qquad \text{and} \qquad h[n] = \{\underline{1}, 0,0,0,0, \tfrac{1}{5}\}$$

   Obtain the analytical expression of the system's impulse response in terms of α and k. Compute α and k.

2. The output signal of a LTI system is $y(t) = 2\wedge\left(\frac{t-1}{4}\right)$, and the system's impulse response is $h(t) = \prod\left(\frac{t-2}{4}\right)$. Sketch the input signal to the system: $x(t)$.

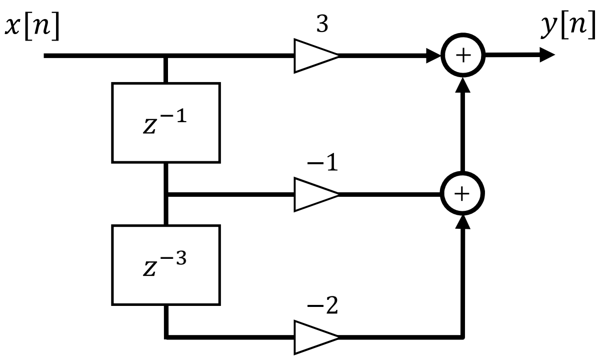
3. Consider a system with the following difference equation (null initial conditions):

   $$y[n] = x[n-1] + ay[n-1]$$

   Obtain h[n], the system's impulse response and give its analytic or closed form expression. Obtain the condition in a so that the system is stable.

PROBLEM 2 (10 points, 30 minutes)

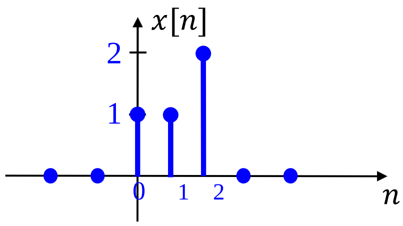
Consider the following system.



a. Write the difference equation that relates $y[n]$ and $x[n]$, and state the type and the order of the system. **(3 points)**

b. Sketch $h[n]$, the system's impulse response. **(1 point)**

c. Compute $y[n]$, if the input signal is the one shown in the picture. **(3 points)**



d. Sketch $y[n]$, if the input signal is $u[n]$. **(3 points)**

PROBLEM 3 (10 points, 30 minutes)

Consider 3 LTI systems with the following impulse responses:

$$h_1(t) = 3\prod(t/2) - 2\delta(t-4)$$
$$h_2(t) = \delta(t-1)$$
$$h_3(t) = 2\delta(t-5)$$

The systems $h_1(t)$ and $h_2(t)$ are connected in cascade, and the resulting system is connected in parallel with $h_3(t)$.

a. Analyze the causality and stability of each of the systems. **(3 points)**
b. Sketch the block diagram of the complete system**.** **(2 points)**
c. Compute and sketch the impulse response of the complete system, and analyze whether it is causal and/or stable. **(2 points)**
d. Obtain the output signal for the following input: $x(t) = \prod(t/2) + \delta(t-10)$. **(3 points)**

PROBLEMA 1 / ARIKETA 1 / PROBLEM 1

1. $y[n] = x[n] + \alpha x[n-K]$; $x[n] = \delta[n] \Rightarrow y[n] = h[n] \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow h[n] = \delta[n] + \alpha\delta[n-K] \\ h[n] = \{ \underline{1}, 0, 0, 0, 0, \frac{1}{5} \} \end{array}\right\} \alpha = \frac{1}{5} ; K = 5$$

2. $$\left.\begin{array}{l} y(t) = 2\Lambda\left(\frac{t-1}{4}\right) \\ h(t) = \Pi\left(\frac{t-2}{4}\right) \end{array}\right\} x(t)?$$

$y(t) = x(t) * h(t)$

$y(t) = x(t) * \Pi\left(\frac{t-2}{4}\right) = 2\Lambda\left(\frac{t-1}{4}\right.$

$$2\Lambda\left(\frac{t-1}{4}\right) = K\,\Pi\left(\frac{t-t_0}{W}\right) * \Pi\left(\frac{t-2}{4}\right) \quad \Big\|\ \Pi\left(\frac{t}{T}\right) * \Pi\left(\frac{t}{T}\right) = T\Lambda\Big($$

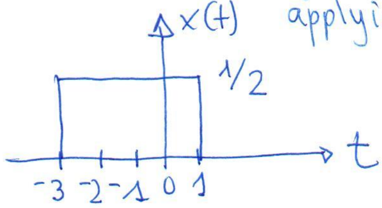
Aldagaiak egokitiz eta
Konboluzioaren propietateak erabiliz:

Identificando: $K = \frac{1}{2}$; $W = 4$; $t_0 = -1$

y aplicando propiedades
de la convolución

By comparing, and
applying convolution
properties:

$$x(t) = \frac{1}{2}\Pi\left(\frac{t+1}{4}\right) \Rightarrow$$



3. $y[n] = x[n-1] + a\,y[n-1]$

$\downarrow x[n] = \delta[n]$; $y[n] = h[n]$

$h[n] = \delta[n-1] + a\,h[n-1]$

$$\left.\begin{array}{l} h[0] = \delta[-1] + a h[-1] = 0 \\ h[1] = \delta[0] + a h[0] = 1 \\ h[2] = \delta[1] + a h[1] = a \\ h[3] = \delta[2] + a h[2] = a^2 \\ h[4] = a^3 \\ \vdots \end{array}\right\} h[n] = a^{n-1} \cdot u[n-1]$$

Estable
Egonkorra
Stable

$\Rightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$; $\sum_{n=1}^{\infty} |a^{n-1}| = 1 + |a| + |a|^2 + |a|^3 + \ldots + |a|^{\infty}$

$\Rightarrow$ Serie geométrica, condición: $r = |a| < 1$] $\hookrightarrow = \frac{1}{1-|a|}$

geometrikoa; batdintza:
geometric series, condition:

PROBLEMA 2 / ARIKETA 2 / PROBLEM 2

a) $y[n] = 3x[n] - x[n-1] - 2x[n-4]$

Sistema FIR, orden 4

" sistema, 4. maila

" system, 4th order

b) $h[n] = 3\delta[n] - \delta[n-1] - 2\delta[n-4]$

$$\| \; x[n] = \delta[n] \Rightarrow y[n] = h[n] \; \|$$

c) $x[n] = \delta[n] + \delta[n-1] + 2\delta[n-2]$

$h[n] = 3\delta[n] - \delta[n-1] - 2\delta[n-4]$

$$y[n] = x[n] * h[n] = 3\delta[n] - \delta[n-1] - 2\delta[n-4] + 3\delta[n-1] -$$

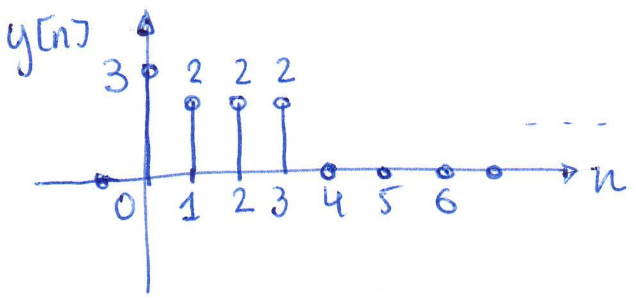
$$- \delta[n-2] - 2\delta[n-5] + 6\delta[n-2] - 2\delta[n-3] - 4\delta[n-6] =$$

$$= 3\delta[n] + 2\delta[n-1] + 5\delta[n-2] - 2\delta[n-3] - 2\delta[n-4] - 2\delta[n-5]$$

$$-4\delta[n-6]$$

d) $x[n] = u[n]$

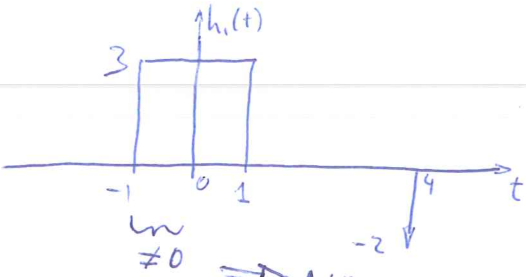
$y[n] = x[n] * h[n] = 3u[n] - u[n-1] - 2u[n-4]$



3. ARIKETA . PROBLEM 3

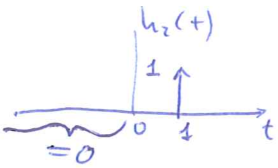
a. Causal? $h(t) \overset{?}{=} 0 \quad \forall t < 0$

$h_1(t) = 3\Pi\left(\frac{t}{2}\right) - 2\delta(t-4)$



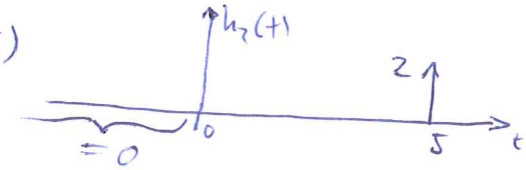
$\Rightarrow$ NO CAUSAL

$h_2(t) = \delta(t-1)$



$\Rightarrow$ CAUSAL

$h_3(t) = 2\delta(t-5)$

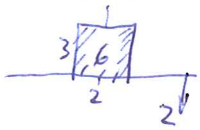


$\Rightarrow$ CAUSAL

Estable?
Egonkorra?

$\int_{-\infty}^{\infty} |h(t)|\, dt \overset{?}{<} \infty$

$h_1(t) \to \int = 8$

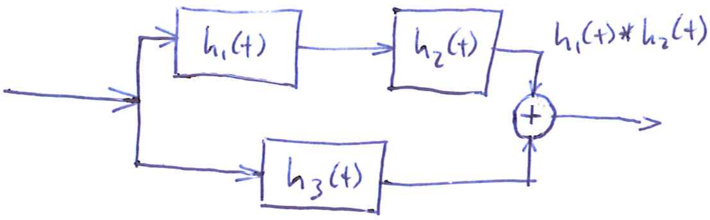


ESTABLE

$h_2(t) \to \int_{-\infty}^{\infty} \delta(t-1)\, dt = 1 \Rightarrow$ ESTABLE

$h_3(t) \to \int 2\delta(\sim) = 2 \Rightarrow$ ESTABLE

b.

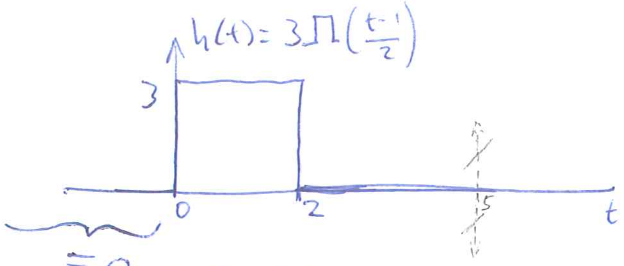


c. $h(t) = h_1(t) * h_2(t) + h_3(t) = \left(3\Pi\left(\frac{t}{2}\right) - 2\delta(t-4)\right) * \delta(t-1) + 2\delta(t-5) =$

$= 3\Pi\left(\frac{t}{2}\right) * \delta(t-1) - 2\delta(t-4) * \delta(t-1) + 2\delta(t-5) =$

$= 3\Pi\left(\frac{t-1}{2}\right) - 2\delta(t-5) + 2\delta(t-5) = 3\Pi\left(\frac{t-1}{2}\right)$

se cancelan mutuamente



$= 0 \Rightarrow$ CAUSAL

$$\int_{-\infty}^{\infty} \left| 3\Pi\left(\frac{t-1}{2}\right) \right| dt = \int_0^2 3\,dt = 6 \Rightarrow \text{ESTABLE}$$

d.

$$y(t) = x(t) * h(t) = \left(\Pi\left(\frac{t}{2}\right) + \delta(t-10)\right) * 3\Pi\left(\frac{t-1}{2}\right) =$$

$$= 3\Pi\left(\frac{t}{2}\right) * \Pi\left(\frac{t-1}{2}\right) + 3\delta(t-10) * \Pi\left(\frac{t-1}{2}\right) =$$

$$= 3\underbrace{\Pi\left(\frac{t}{2}\right) * \Pi\left(\frac{t}{2}\right)}_{2\Lambda\left(\frac{t}{2}\right)} * \delta(t-1) + 3\Pi\left(\frac{t-11}{2}\right) = 6\Lambda\left(\frac{t-1}{2}\right) + 3\Pi\left(\frac{t-11}{2}\right)$$