Advanced Numerical Methods – 5/23/2013 EXAM RESOLUTION

Part 1, Exercise 1

a) The nodes are evenly spaced (h=2), so we will construct the table of *finite* differences, which will hopefully reveal what the wrong data are. The way to fill in the table is extremely simple: each finite difference is made to be equal to the number on its left minus the one above it. Since we are building it column-by-column rightwards, we will stop writing numbers as soon as we see no need to continue:

i	xi	fi	Δfi	Δ^2fi	Δ^3fi
0	-8	-178			
1	-6	-77	101		
2	-4	-3	74	-27	
3	-2	49	52	-22	5
4	0	84	35	-17	5
5	2	108	24	-11	6
6	4	122	14	-10	1
7	6	137	15	1	11
8	8	154	17	2	1
9	10	179	25	8	6
10	12	217	38	13	5
11	14	273	56	18	5
12	16	352	79	23	5

We stop here because the column of finite differences of order 3 starts and ends being constant, as corresponds to a polynomial of degree 3, so the numbers in its middle (the ones that are not constant, i.e. different from 5) are most likely the result of error(s) in the data. (Of course the 5s could also be wrong, but we have several ones of them so that's very unlikely.) Furthermore, we know the triangular shape in which errors propagate (which is a direct consequence of what each number depends on). If we underline the wrong data in the rightmost column and, leftwards, all the ones they depend on, we end up with all the wrong numbers underlined (as can be seen above). We can remove all of them and substitute the ones on the right (6, 1, 11, 1, 6) by 5s (for the whole rightmost column to be constant). From there, we can fill in the missing data with the same law (every number = the one on its left minus the one above it). The new, correct table (with the original error into brackets after every corrected number–even if this was not asked) is:

i	xi	fi	Δfi	Δ^2fi	Δ^3fi
0	-8	-178			
1	-6	-77	101		
2	-4	-3	74	-27	
3	-2	49	52	-22	5
4	0	84	35	-17	5
5	2	107(-1)	23(-1)	-12(-1)	5(-1)
6	4	123(+1)	16(+2)	-7(+3)	5(+4)
7	6	137	14(-1)	-2(-3)	5(-6)
8	8	154	17	3(+1)	5(+4)
9	***10***	***179***	25	8	5(-1)
10	***12***	***217***	***38***	13	5
11	*14*	273	56	*18*	5
12	16	352	79	23	*5*

(3.5p)

b) One easy way to obtain the leading coefficient of $p_3(x)$ without calculating the polynomial is to use the known value of the finite differences of third order (5) to obtain the (constant) third derivative of $p_3(x)$, and then relate that derivative with the coefficient sought, namely:

$$\underline{f^{n)}(\xi) = \frac{\Delta^n f(x_i)}{h^n}} \quad \Rightarrow \quad f^{3)}(\xi) = \frac{\Delta^3 f(x_i)}{h^3} = \frac{5}{2^3} = \frac{5}{8}$$

<u>for some ξ in between the nodes from which the finite difference is calculated</u>, and this is true for any function $f(x) \in C^n$ on some interval including the nodes. In the particular case that $f(x) = p_3(x)$ (i.e. a polynomial of degree 3), the third derivative is constant, so ξ is irrelevant.

Regarding the relationship between the leading coefficient and the derivative, it is trivial:

$$p_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \quad \Rightarrow \quad p_3'(x) = 3a_3x^2 + \cdots, \quad p_3''(x) = 6a_3x + \cdots, \quad p_3^{3)}(x) = 6a_3$$

(In general, <u>the n-th derivative of a polynomial of degree n is constant and equal to n! times its leading coefficient</u>.)

Hence: $$6a_3 = 5/8 \quad \Rightarrow \quad \underline{a_3 = 5/48 = 0.1041\hat{6}}$$ **(1.5p)**

Another way is to use the relationship between finite differences and divided differences, the leading coefficient being the last one.

c) There exists one polynomial $p_3(x)$ of degree 3 generating all the data in the corrected table, and it is also the only polynomial of degree 3 generating any 4 of the 13 nodes in the table. Therefore, any 4 nodes can be used to obtain $p_3(x)$ uniquely. We are asked to use the last 4 nodes in the table, i.e. the data printed in bold above. Specifically, we will use the numbers in bold italics.

We will use the Newton representation of the interpolation polynomial with evenly spaced nodes (i.e. in terms of finite differences). First we define the change of variable:

$$t(x) = (x-x_9)/h = (x-10)/2 = x/2 - 5$$

The interpolation polynomial is $$p_3(x) = q(t(x))$$

where: $$q(t) = \sum_{i=0}^{3} \Delta^i f(x_9) \binom{t}{i} = 179 + 38t/1! + 18t(t-1)/2! + 5t(t-1)(t-2)/3! =$$

$$= 179 + 38t + 9t(t-1) + (5/6)t(t-1)(t-2) =$$

Its Hörner-like form is: $$q(t) = \{[(5/6)(t-2) + 9](t-1) + 38\}\, t + 179$$

This reorganization of the polynomial represents an optimal algorithm to evaluate it from the points of view of computational cost, memory storage, and roundoff error propagation. For $x = 5$:

$$t = x/2 - 5 = -2.5$$

Substituting: $$p_3(5) = q(-2.5) =$$

$$= \{[(5/6)(-2.5-2) + 9](-2.5-1) + 38\}(-2.5) + 179 =$$

$$= \{[(5/6)(-4.5) + 9](-3.5) + 38\}(-2.5) + 179 =$$

$$= \underline{129.9375}$$ **(2p)**

Part 1, Exercise 2

The explanation can be found in Section 1.3.3.- of the Classroom Notes (p. 13 in the English version).

The writing of the algorithm that implements the process could be done in any computer language, but the most natural thing to do is to write it in MATLAB, which is the language we have done the computer labs with. The following code is good enough to actually work (and has been available throughout the course in the file `anm_tabledivdif.m` in the usual cloud folder accessible from the moodle course):

```
function F = anm_tabledivdiff(X,F)
%ANM_TABLEDIVDIFF  Calculate table of divided differences from nodal data.
%        F = anm_tabledivdiff(X,F)
%
% INPUT:
% X  = nodes as column vector [x0;x1;x2;...;xn].
% F  = nodal values of f(x) on nodes X as column vector.
%
% OUTPUT:
% F  = divided differences as column vector [f00;f11;f22;...;fnn].
%
% The algorithm is optimal from the point of view of memory storage; that is why F
% is both input (with the nodal values) and output (with the divided differences).
%
% EXAMPLE (mark with mouse and hit F9):
%   X=[0;1;3;4], F=X.^3, F=anm_tabledivdiff(X,F)      % Classroom Notes Exercise 6
%
% APROPOS: anm_tableddr, anm_tabledivdiff, anm_prettyprintable.
% Function anm_tableddr is preferable to anm_tabledivdiff. It can do more things.

% Felipe Jiménez, 2013.
% For subject "Advanced Numerical Methods" (code 26677 of GITECI30, UPV/EHU).


nn = length(F);             % number of nodes (nn=n+1, with n as in subject theory)
n  = nn-1;                  % order of interpol polynomial / of last divided diff.
for k = 1:n                 % CONSTRUCT COLUMN BY COLUMN FROM SECOND (k=1) TO k=n.
    for i = nn:-1:k+1       % FROM LAST ELEMENT OF COL. UPWARDS UNTIL PPAL DIAGONAL
        F(i) = (F(i)-F(i-1))/(X(i)-X(i-k));   % FROM FORMULA (12)
    end
end
```

(3.5p)

Part 1, Exercise 3

This is what $S(x)$ must satisfy:

a) Pass by all 4 data points, with continuity at both interior points;
b) Continuity of the first derivative at both interior points;
c) Continuity of the second derivative at both interior points;
d) Zero second derivative at both end points (for the spline to be natural).

Calling $p_0(x)$ the first polynomial piece, $p_1(x)$ the second, and $p_2(x)$ the third; and calling $x_{0,1,2,3} = \{0, 1, 3, 4\}$ and $y_{0,1,2,3} = \{26, 7, 7, 25\}$:

a)
$$p_0(x_0) = p_0(0) = 26 = y_0 \qquad \text{ok;}$$
$$p_0(x_1) = p_0(1) = 7 = y_1 \qquad \text{ok;}$$
$$p_1(x_1) = p_1(1) = 7 = y_1 \qquad \text{ok;}$$
$$p_1(x_2) = p_1(3) = 7 = y_2 \qquad \text{ok;}$$
$$p_2(x_2) = p_2(3) = 7 = y_2 \qquad \text{ok;}$$
$$p_2(x_3) = p_2(4) = 25 = y_3 \qquad \text{ok.}$$

b)
$$p_0'(x) = 6x^2 + 2x - 22\,; \quad p_1'(x) = 14x - 28\,; \quad p_2'(x) = -9x^2 + 68x - 109$$
$$p_0'(x_1) = p_0'(1) = -14 = p_1'(1) \qquad \text{ok;}$$
$$p_1'(x_2) = p_1'(3) = 14 = p_2'(3) \qquad \text{ok.}$$

c)
$$p_0''(x) = 12x + 2\,; \quad p_1''(x) = 14\,; \quad p_2''(x) = -18x + 68$$
$$p_0''(x_1) = p_0''(1) = 14 = p_1''(1) \qquad \text{ok;}$$
$$p_1''(x_2) = p_1''(3) = 14 = p_2''(3) \qquad \text{ok.}$$

d)
$$p_0''(x_0) = p_0''(0) = 2 \neq 0 \qquad \text{not ok!}$$

So $S(x)$ cannot be and therefore is *not* the natural cubic spline corresponding to those data. **(4.5p)**

(Incidentally, the second derivative at the right end is not 0 either; it is $p_2''(x_3) = p_2''(4) = -4 \neq 0$; however, $S(x)$ is the spline with boundary conditions $S'(0) = p_0'(0) = -22$, $S'(4) = p_2'(4) = 19$.)

Part 1, Exercise 4

Done by hand. See scan below.

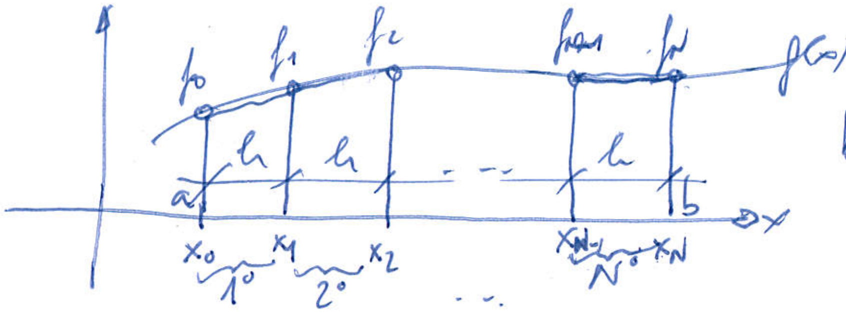
Part 1, Exercise 5

Done by hand. See scan below.

PART 2

Done by hand. See scan below.

1.4 a) N-Trap:



$b-a = Nh$

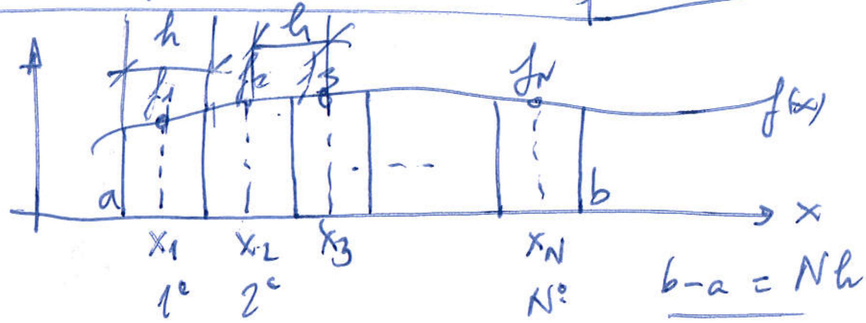
$$Q_{NT} = \frac{h}{2}(f_0+f_1) + \frac{h}{2}(f_1+f_2) + \ldots + \frac{h}{2}(f_{N-1}+f_N) =$$

$$\boxed{Q_{NT} = \frac{h}{2}(f_0+f_N) + h(f_1+f_2+f_3+\ldots+f_{N-1})}$$

$$\text{w̃ } h = \frac{b-a}{N}, \quad f_i = f(x_i), \quad x_i = a + ih$$

1.5

N-Midp.:



$b-a = Nh$

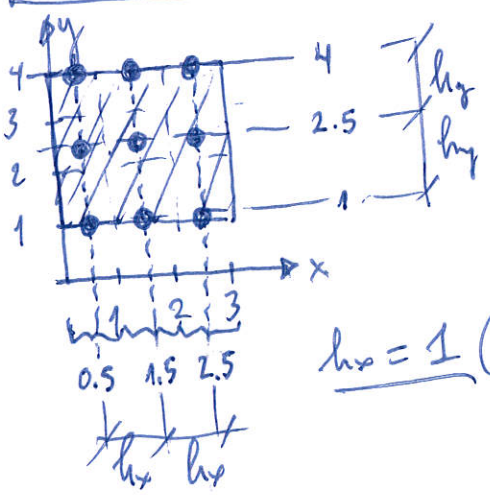
$$Q_{NM} = hf_1 + hf_2 + \ldots + hf_N =$$

$$\boxed{Q_{NM} = h(f_1+f_2+\ldots+f_N)}$$

$$\text{w̃ } h = \frac{b-a}{N}, \quad f_i = f(x_i), \quad x_i = a - \frac{h}{2} + ih$$

1.5

b)



$h_y = 1.5$ (Trap.)

$h_x = 1$ (Midp.)

$$Q_b = f(0.5,4)\,h_x\frac{h_y}{2} + f(1.5,4)\,h_x\frac{h_y}{2} + f(2.5,4)\,h_x\frac{h_y}{2}$$
$$+ f(0.5,2.5)\,h_x h_y + f(1.5,2.5)\,h_x h_y + f(2.5,2.5)\,h_x h_y$$
$$+ f(0.5,1)\,h_x\frac{h_y}{2} + f(1.5,1)\,h_x\frac{h_y}{2} + f(2.5,1)\,h_x\frac{h_y}{2} =$$

$6 \times 0.75 + 3 \times 2 \times 0.75$

$= 12 \times 0.75 = 9 =$

$= 3 \times 3$ OK

$$\boxed{\begin{aligned} Q_b = {} & 0.75\,f(0.5,4) + 0.75\,f(1.5,4) + 0.75\,f(2.5,4) + \\ & + 1.5\,f(0.5,2.5) + 1.5\,f(1.5,2.5) + 1.5\,f(2.5,2.5) + \\ & + 0.75\,f(0.5,1) + 0.75\,f(1.5,1) + 0.75\,f(2.5,1) \end{aligned}}$$

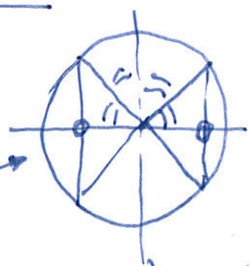
2

(1.5) $I = \int_{-2}^{2} \frac{x^3+2x+1}{\sqrt{4-x^2}}\,dx \underset{x=2t}{=} \int_{-1}^{1} \frac{(2t)^3+2(2t)+1}{\sqrt{4-4t^2}}\,2dt =$

$$= \int_{-1}^{1} \frac{8t^3+4t+1}{\cancel{2}\sqrt{1-t^2}}\,\cancel{2}dt = \int_{-1}^{1} \frac{\overbrace{8x^3+4x+1}^{g(x)}}{\sqrt{1-x^2}}\,dx$$

Gauss-Chebyshev integral will be exact with 2 nodes because for $n=1$, $2n+1 = 3 \geq \text{degree}(g(x)) = 3$.

Nodes: $x_{0,1} = \left\{\cos\frac{3\pi}{4}, \cos\frac{\pi}{4}\right\} = \left\{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\}$ according to



Weights: Their sum is π (prove it if you want via $g(x) \equiv 1$) and they are equal (uniformity of Chebyshev) $\Rightarrow w_{0,1} = \left\{\frac{\pi}{2}, \frac{\pi}{2}\right\}$

$$\Rightarrow I = Q = w_0 g(x_0) + w_1 g(x_1) = \frac{\pi}{2} g\left(-\frac{\sqrt{2}}{2}\right) + \frac{\pi}{2} g\left(\frac{\sqrt{2}}{2}\right) =$$

$$= \frac{\pi}{2}\left[\cancel{\text{odd} + \text{odd}} + 1\right] + \frac{\pi}{2}\left[\cancel{\text{odd} + \text{odd}} + 1\right] = \boxed{\pi}$$

2.1

$$y'' - 0.05\,\underset{\substack{\shortparallel\\ y_2}}{y'} + 0.15\,\underset{\substack{\shortparallel\\ y_1}}{y} = e^t$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -0.15 y_1 + 0.05 y_2 + e^t \end{cases}$$

$$\underline{y}' = \overbrace{\begin{pmatrix} 0 & 1 \\ -0.15 & 0.05 \end{pmatrix}}^{J} \underline{y} + \overbrace{\begin{pmatrix} 0 \\ e^t \end{pmatrix}}^{\underline{g}(t)} = \underline{f}(t, \underline{y})$$

$$t_0 = 0, \quad \underline{y}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h = 0.1$$

$$\left.\begin{aligned} \underline{k}_1 &= \underline{f}(t_0, \underline{y}_0)\,h \\ \underline{k}_2 &= \underline{f}\left(t_0 + \tfrac{h}{2}, \underline{y}_0 + \tfrac{\underline{k}_1}{2}\right) h \\ \underline{k}_3 &= \underline{f}\left(t_0 + \tfrac{h}{2}, \underline{y}_0 + \tfrac{\underline{k}_2}{2}\right) h \\ \underline{k}_4 &= \underline{f}(t_0 + h, \underline{y}_0 + \underline{k}_3)\,h \end{aligned}\right\} \longrightarrow \underline{y}_1 = \underline{y}_0 + \frac{\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4}{6}$$

$$\underline{k}_1 = \left[\begin{pmatrix} 0 & 1 \\ -0.15 & 0.05 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] 0.1 = \begin{pmatrix} 0.1 \\ 0.105 \end{pmatrix}$$

$$\underline{k}_2 = \left[\begin{pmatrix} 0 & 1 \\ -0.15 & 0.05 \end{pmatrix}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.1 \\ 0.105 \end{pmatrix}/2\right) + \begin{pmatrix} 0 \\ e^{0.05} \end{pmatrix}\right] 0.1 = \begin{pmatrix} 0.10525 \\ 0.10963961 \end{pmatrix}$$

$$\underline{k}_3 = \left[J\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.10525 \\ 0.10963961 \end{pmatrix}/2\right) + \begin{pmatrix} 0 \\ e^{0.05} \end{pmatrix}\right] 0.1 = \begin{pmatrix} 0.10548198 \\ 0.10961183 \end{pmatrix}$$

$$\underline{k}_4 = \left[J\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.10548198 \\ 0.10961183 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ e^{0.1} \end{pmatrix}\right] 0.1 = \begin{pmatrix} 0.11096118 \\ 0.11448292 \end{pmatrix}$$

$$\underline{y}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 0.1 \\ 0.105 \end{pmatrix} + 2\begin{pmatrix} 0.10525 \\ 0.10963961 \end{pmatrix} + 2\begin{pmatrix} 0.10548198 \\ 0.10961183 \end{pmatrix} + \begin{pmatrix} 0.11096118 \\ 0.11448292 \end{pmatrix}}{6} = \begin{pmatrix} 0.10540419 \\ 1.10966430 \end{pmatrix}$$

$$\boxed{\begin{aligned} y(0.1) &\simeq 0.105404 \\ y'(0.1) &\simeq 1.109664 \end{aligned}}$$

6p

2) A ver qué ponen (1.5 + 0.25 + 0.25)

3) " " " " (1 + 1 + 1 + 1)

4) a) " " " " (2)

b) $|E_T| = |E_t + E_r| \leq |E_t| + |E_r| \leq \frac{h^4}{90} M + \varepsilon \cdot FA$

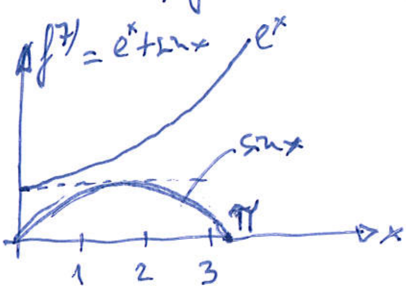
where $\begin{cases} M \geq |f^{6)}(x)| \ \forall \ x \in [0,2] \\ \varepsilon = 5e-6 \\ FA = \sum_{i=0}^{4} |A_i| = \frac{+1+16+3+16+1}{12h^2} = \frac{37}{12h^2} \end{cases}$

Regarding M: $f(x) = e^x + \cos x \rightarrow f^{4)} = f \rightarrow f^{6)} = e^x - \cos x, \ f^{7)} = e^x + \sin x.$

$f^{7)}(x) = e^x + \sin x \Rightarrow f^{7)}(x) > 0 \ \forall x \in [0,2]$

$\sin x \geq 0 \ \forall x \in [0,2]$

$e^x > 0 \ \forall x \in [0,2]$



$\Rightarrow f^{6)}(x)$ increasing monotonic in $[0,2]$.

$f^{6)}(0) = 1 - 1 = 0 \longrightarrow M = f^{6)}(2) = e^2 - \cos 2 = 7.8052029$

$|E_T| \leq \frac{h^4}{90} M + \varepsilon \frac{37}{12h^2} = g(h)$ minimum $\Rightarrow g'(h) = 0$

$\frac{4h^3}{90} M - 2\varepsilon \frac{37}{12h^3} = 0$

$h^6 = \frac{90 \ 2\varepsilon \ 37}{4M \ 12} = \frac{\cancel{90}^{\cancel{45}\,15} \ \cancel{2}\varepsilon \ 37}{\cancel{4} M \ \cancel{12}\,4} = \frac{15 \times 37 \varepsilon}{4M} = \frac{555 \varepsilon}{4M}$

$\Rightarrow h_{opt} = \sqrt[6]{\frac{555 \varepsilon}{4M}} = \sqrt[6]{\frac{555 \times 5e-6}{4 \times (e^2 - \cos 2)}} = \boxed{h_{opt} = 0.21125312}$