5. GAIA

ZENBAKIZKO METODOAK BERO-EROAPENEAN

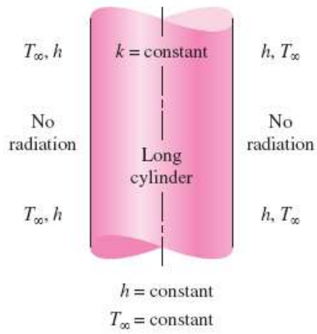
5.0 - HELBURUAK

- Eroapen-problemen ebazpen analitikoen mugak eta ordenagailu bidezko **zenbakizko metodo** intentsiboen **beharra** ulertu.
- Deribatuak diferentzia gisa adierazi eta **diferentzia finituko formulazioak** lortu.
- Dimentsio bakarreko edo biko eroapen geldikorreko **problemak numerikoki ebaztea** diferentzia finituen metodoa baliatuta.
- Dimentsio bakarreko edo biko eroapen **iragankorreko problemak** ebaztea diferentzia finituen metodoa baliatuta.

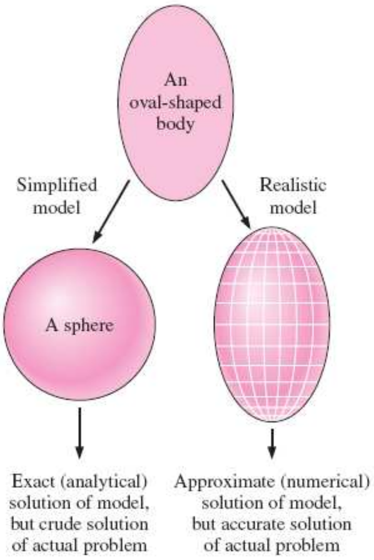
5.1 – ZERGATIK ERABILI ZENBAKIZKO METODOAK?

Ekuazio diferentziala
+
Mugaldeko baldintzak

⇨ **SOLUZIO ANALITIKOA**



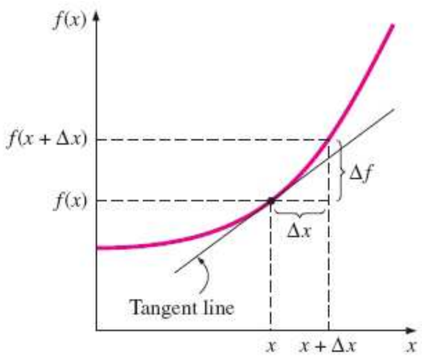
- Mugak
- Eredu hobeak egitea
- Malgutasuna
- Arazoak
- Giza izaera



5.2 – EKUAZIO DIFERENTZIALEN DIFERENTZIA FINITUKO FORMULAZIOA

Ekuazio diferentziala → Ekuazio algebraikoa

Deribatuak → Diferentziak



$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

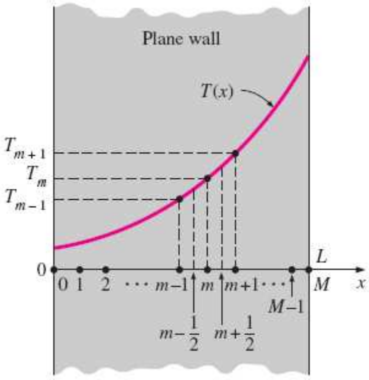
$$\frac{df(x)}{dx} \cong \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Aukera: f(x) funtzioaren Taylorren garapena

$$f(x+\Delta x) = f(x) + \Delta x \cdot \frac{df(x)}{dx} + \frac{1}{2}\Delta x^2 \cdot \frac{d^2 f(x)}{dx^2} + \ldots$$

Errorea Δx-ekiko proportzional

5.2 – EKUAZIO DIFERENTZIALEN DIFERENTZIA FINITUKO FORMULAZIOA



$$\left.\frac{dT}{dx}\right|_{m-\frac{1}{2}} \cong \frac{T_m - T_{m-1}}{\Delta x} \qquad \left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} \cong \frac{T_{m+1} - T_m}{\Delta x}$$

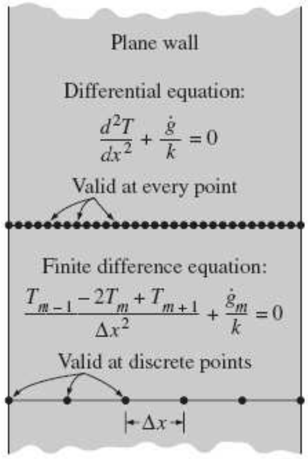
$$\left.\frac{d^2T}{dx^2}\right|_m \cong \frac{T_{m-1} - 2 \cdot T_m + T_{m+1}}{\Delta x^2}$$

$$\frac{d^2T}{dx^2} + \frac{\dot{e}}{k} = 0 \quad \Rightarrow \quad \frac{T_{m-1} - 2 \cdot T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$

m = 1,2,3, …, M-1

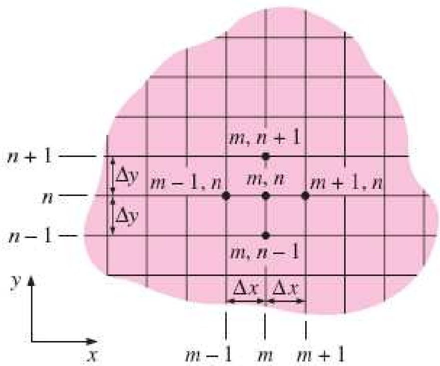
T_0 eta T_M ezagunak: M – 1 ekuazio M – 1 ezezagun

T_0 eta T_M ezezagunak: { M – 1 ekuazio + 2 mugaldeko baldintza } M + 1 ezezagun



5.2 – EKUAZIO DIFERENTZIALEN DIFERENTZIA FINITUKO FORMULAZIOA

Bi dimentsiotan:



$$\frac{T_{m+1,n} - 2 \cdot T_{m,n} + T_{m-1,n}}{\Delta x^2} + \frac{T_{m,n+1} - 2 \cdot T_{m,n} + T_{m,n-1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$

m = 1,2,3, …, M – 1

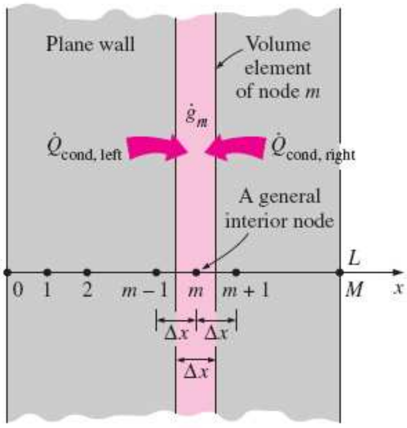
n = 1,2,3, …, N – 1

Energia-balantzearen metodoa

Δx = L/M lodierako M eskualde

Posizioa: $x_m = m \cdot \Delta x$

Tenperatura: $T(x_{m)} = T_m$



$$\dot{Q}_{\text{cond,left}} + \dot{Q}_{\text{cond,right}} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t} = 0$$

$$\dot{E}_{gen,element} = \dot{e}_m \cdot V_{element} = \dot{e}_m \cdot A \cdot \Delta x$$

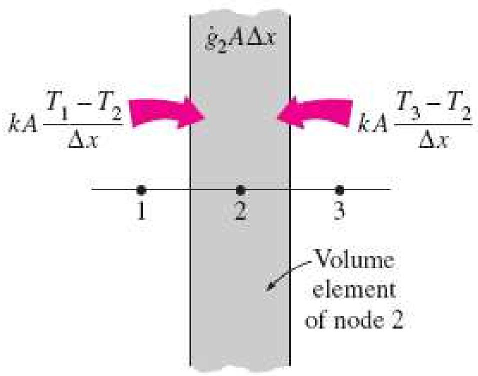
$$\dot{Q}_{\text{cond,left}} = k \cdot A \cdot \frac{T_{m-1} - T_m}{\Delta x} \qquad \dot{Q}_{\text{cond,right}} = k \cdot A \cdot \frac{T_{m+1} - T_m}{\Delta x}$$

$$k \cdot A \cdot \frac{T_{m-1} - T_m}{\Delta x} + k \cdot A \cdot \frac{T_{m+1} - T_m}{\Delta x} + \dot{e}_m \cdot A \cdot \Delta x = 0$$

$$\frac{T_{m-1} - 2 \cdot T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$

m = 1,2,3, …, M-1

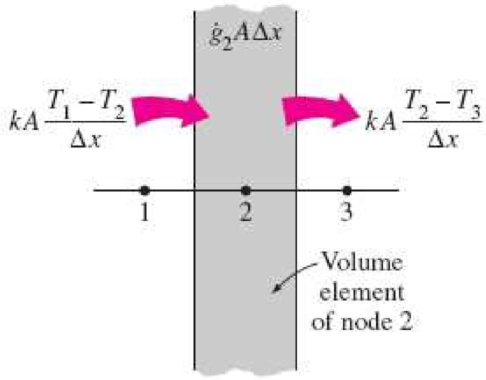
Bolumen elementuaren gainazalean eroapenerako suposatutako norantzak ez du formulazioan eraginik.



$$kA\frac{T_1 - T_2}{\Delta x} + kA\frac{T_3 - T_2}{\Delta x} + \dot{g}_2 A \Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 \;\; \Delta x^2 / k = 0$$



$$kA\frac{T_1 - T_2}{\Delta x} - kA\frac{T_2 - T_3}{\Delta x} + \dot{g}_2 A \Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 \;\; \Delta x^2 / k = 0$$

MUGALDEKO BALDINTZAK

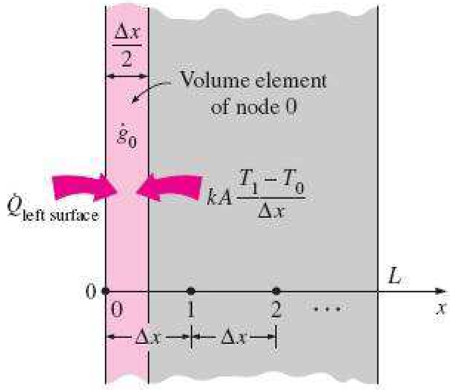
0 nodoa → x = 0 posizioa
M nodoa → x = L posizioa
} Zabalera Δx/2

Tenperatura zehaztua

$$\begin{cases} T(0) = T_0 \\ T(L) = T_M \end{cases}$$

Konbekzio, erradiazio edo konbekzio eta erradiazio konbinatuen, bero-fluxu zehaztua

$$\dot{Q}_{left\ surface} + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$



MUGALDEKO BALDINTZAK

Bero-fluxu zehaztua

$$\dot{q}_0 \cdot A + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

<u>Kasu partikularra</u>: Mugalde isolatua

$$k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

Konbekzio mugaldea

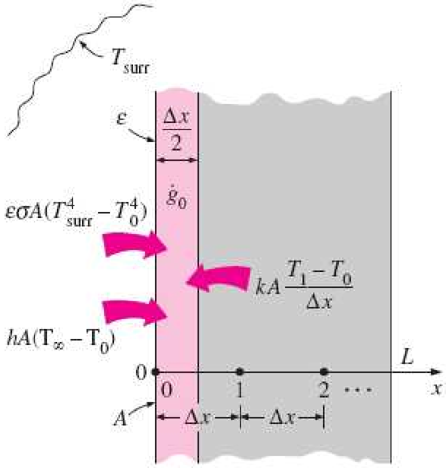
$$h \cdot A \cdot (T_\infty - T_0) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

Erradiazio mugaldea

$$\varepsilon \cdot \sigma \cdot A \cdot (T_{surr}^4 - T_0^4) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

MUGALDEKO BALDINTZAK

Konbekzio eta erradiazio konbinatuaren mugaldea

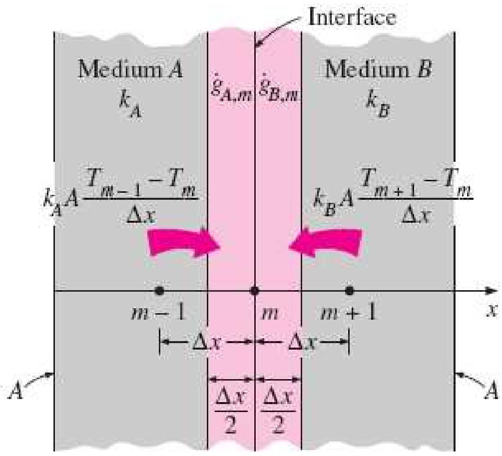


$$h \cdot A \cdot (T_\infty - T_0) + \varepsilon \cdot \sigma \cdot A \cdot (T_{surr}^4 - T_0^4) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

$$h_{combined} \cdot A \cdot (T_\infty - T_0) + k \cdot A \cdot \frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \cdot A \cdot \frac{\Delta x}{2} = 0$$

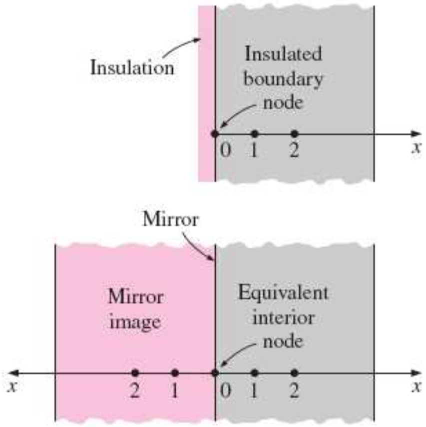
MUGALDEKO BALDINTZAK

Fasearteko mugaldea



$$k_A \cdot A \cdot \frac{T_{m-1} - T_m}{\Delta x} + k_B \cdot A \cdot \frac{T_{m+1} - T_m}{\Delta x} + \dot{e}_{A,m} \cdot A \cdot \frac{\Delta x}{2} + \dot{e}_{B,m} \cdot A \cdot \frac{\Delta x}{2} = 0$$

MUGALDE-NODO ISOLATUAK BARNE-NODOTZAT HARTZEA: ISPILU-IRUDIAREN KONTZEPTUA



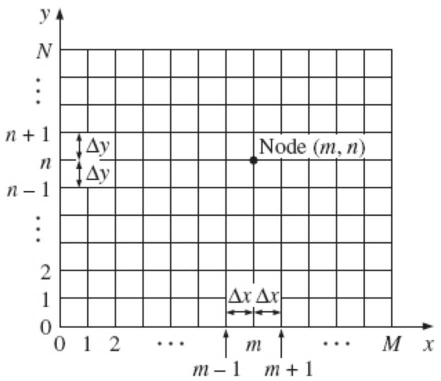
$$\frac{T_{m-1} - 2{\cdot}T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$

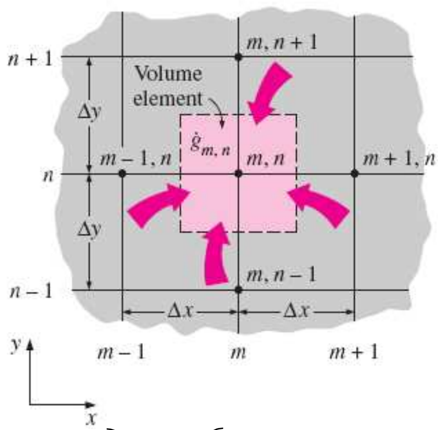
$$\Downarrow$$

$$\frac{T_1 - 2{\cdot}T_0 + T_1}{\Delta x^2} + \frac{\dot{e}_0}{k} = 0$$

Energia-balantzearen metodoaren bidez mugalde isolatuaren kasurako aztertutako adierazpen berdina.

N tenperatura nodoetan N ekuazio
- Ezabapen-metodoa
- Zuzeneko metodoa
- Iterazio-metodoa





(Bero-eroapenaren abiadura ezker-, goi-, eskuin- eta behe-gainazaletan) + (Bero-sorreraren abiadura elementu barruan) = (Elementuaren energia-edukiaren aldaketaren abiadura)

$$\dot{Q}_{\text{cond,left}} + \dot{Q}_{\text{cond,top}} + \dot{Q}_{\text{cond,right}} + \dot{Q}_{\text{cond,bottom}} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t} = 0$$

$$k \cdot \Delta y \cdot \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \cdot \Delta x \cdot \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k \cdot \Delta y \cdot \frac{T_{m+1,n} - T_{m,n}}{\Delta x} +$$

$$+ k \cdot \Delta x \cdot \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{e}_{m,n} \cdot \Delta x \cdot \Delta y = 0$$

$$\frac{T_{m-1,n} - 2 \cdot T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2 \cdot T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$

m = 1,2,3, …, M-1

n = 1,2,3, …, N-1

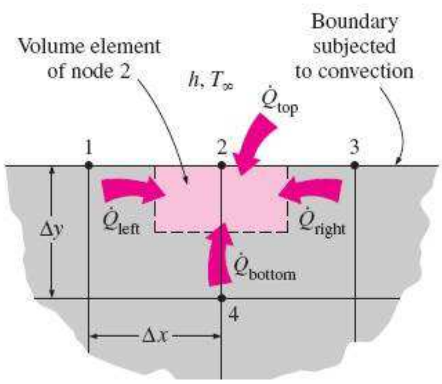
Sarea karratu itxura badauka: $\Delta x = \Delta y = l$

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4 \cdot T_{m,n} + \frac{\dot{e}_{m,n} \cdot l^2}{k} = 0$$

$$T_{left} + T_{top} + T_{right} + T_{bottom} - 4 \cdot T_{node} + \frac{\dot{e}_{node} \cdot l^2}{k} = 0$$

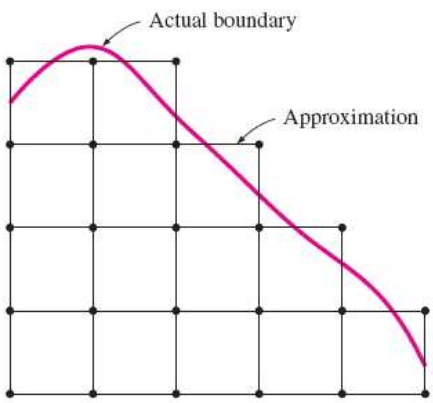
$$T_{node} = \frac{T_{left} + T_{top} + T_{right} + T_{bottom}}{4}$$

Mugalde-nodoak

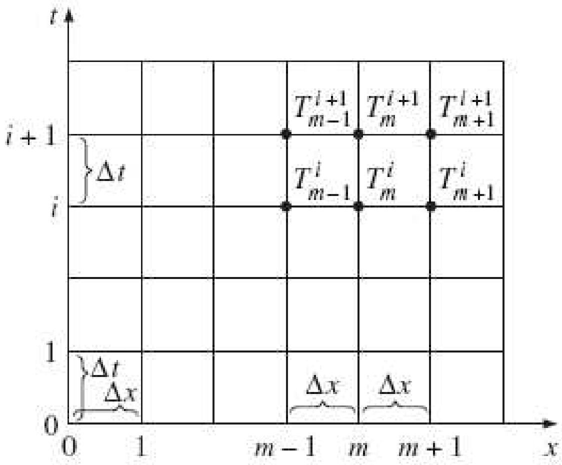


5.4 TEST ARIKETA

Mugalde irregularrak



5.5 – BERO-EROAPEN IRAGANKORRA



i = denbora-urratzen indizea edo zenbatzailea

i = 0; Hasierako baldintza

$t_i = i \cdot \Delta t$

$\mathbf{T_m^i}$ = m nodo eta i denbora-urratseko tenperatura

(Bolumen-elementuaren gainazal guztietatik barrurantz Δt tartean transferitutako beroa) + (Bolumen-elementuaren barnean Δt tartean sortutako beroa) = (Bolumen-elementuan Δt tartean gertatutako energia-edukiaren aldaketa)

$$\sum_{\text{All sides}} \dot{Q} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t} = \rho \cdot V_{elem} \cdot c_p \frac{\Delta T}{\Delta t} = \rho \cdot V_{elem} \cdot c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

5.5 – BERO-EROAPEN IRAGANKORRA

Metodo esplizitua:

$$\sum_{\text{All sides}} \dot{Q}^{i} + \dot{E}^{i}_{gen,element} = \rho \cdot V_{elem} \cdot c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

✔ Ebazteko erreza.

✘ Egonkortasuna mantentzeko Δt mugatuta.

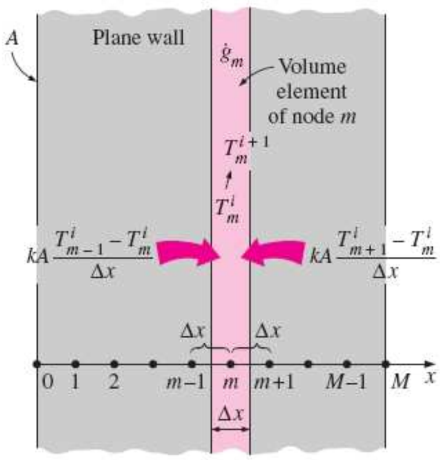
Metodo inplizitua:

$$\sum_{\text{All sides}} \dot{Q}^{i+1} + \dot{E}^{i+1}_{gen,element} = \rho \cdot V_{elem} \cdot c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

✔ Δt ez dago mugatuta.

✘ Nodo-tenperatura guztiak aldi berean ebatzi behar dira.

Horma laua



$$k\cdot A\cdot\frac{T_{m-1}-T_m}{\Delta x}+k\cdot A\cdot\frac{T_{m+1}-T_m}{\Delta x}+\dot{e}_m\cdot A\cdot\Delta x=\rho\cdot A\cdot\Delta x\cdot c_p\,\frac{T_m^{i+1}-T_m^i}{\Delta t}$$

$$\underbrace{T_{m-1}-2\cdot T_m+T_{m+1}+\frac{\dot{e}_m\Delta x^2}{k}}_{\text{Egoera egonkorreko formulazioa}}=\frac{\Delta x^2}{\alpha\cdot\Delta t}(T_m^{i+1}-T_m^i)$$

Egoera egonkorreko formulazioa

$$\tau=\frac{\alpha\cdot\Delta t}{\Delta x^2}$$

Horma laua

Metodo esplizitua:

$$T_{m-1}^i-2\cdot T_m^i+T_{m+1}^i+\frac{\dot{e}_m^i\Delta x^2}{k}=\frac{T_m^{i+1}-T_m^i}{\tau}$$

$$T_m^{i+1}=\tau\cdot(T_{m-1}^i+T_{m+1}^i)+(1-2\tau)\cdot T_m^i+\tau\frac{\dot{e}_m^i\Delta x^2}{k}$$

Egonkortasun-irizpidea

$$\tau=\frac{\alpha\cdot\Delta t}{\Delta x^2}\le\frac{1}{2}\quad\Rightarrow\quad\Delta t\le\frac{1}{2}\frac{\Delta x^2}{\alpha}$$

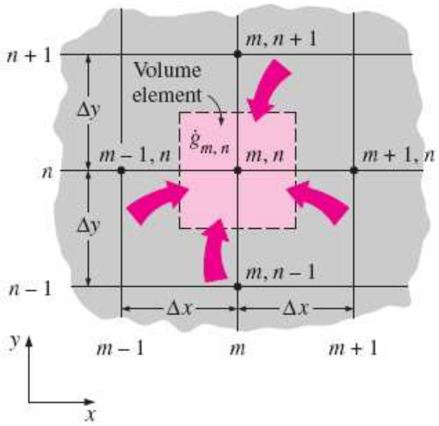
Metodo inplizitua:

$$T_{m-1}^{i+1}-2\cdot T_m^{i+1}+T_{m+1}^{i+1}+\frac{\dot{e}_m^{i+1}\Delta x^2}{k}=\frac{T_m^{i+1}-T_m^i}{\tau}$$

$$\tau\cdot T_{m-1}^{i+1}-(1+2\tau)\cdot T_m^{i+1}+\tau\cdot T_{m+1}^{i+1}+\tau\frac{\dot{e}_m^{i+1}\Delta x^2}{k}+T_m^i=0$$

M-1 barne-nodoentzako formulazio baliogarria

Bi dimentsioko bero-eroapen iragankorra



$$k\cdot\Delta y\cdot\frac{T_{m-1,n}-T_{m,n}}{\Delta x}+k\cdot\Delta x\cdot\frac{T_{m,n+1}-T_{m,n}}{\Delta y}+k\cdot\Delta y\cdot\frac{T_{m+1,n}-T_{m,n}}{\Delta x}$$

$$+k\cdot\Delta x\cdot\frac{T_{m,n-1}-T_{m,n}}{\Delta y}+\dot{e}_{m,n}\cdot\Delta x\cdot\Delta y=\rho\cdot\Delta x\cdot\Delta y\cdot c_p\cdot\frac{T_m^{i+1}-T_m^i}{\Delta t}$$

Sarea karratu itxura badu: Δx = Δy = l

$$T_{m-1,n}+T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}+\frac{\dot{e}_{m,n}\cdot l^2}{k}=\frac{T_m^{i+1}-T_m^i}{\tau}$$

$$\underbrace{T_{left}+T_{top}+T_{right}+T_{bottom}-4T_{node}+\frac{\dot{e}_{node}\cdot l^2}{k}}=\frac{T_{node}^{i+1}-T_{node}^i}{\tau}$$

Egoera egonkorreko formulazioa

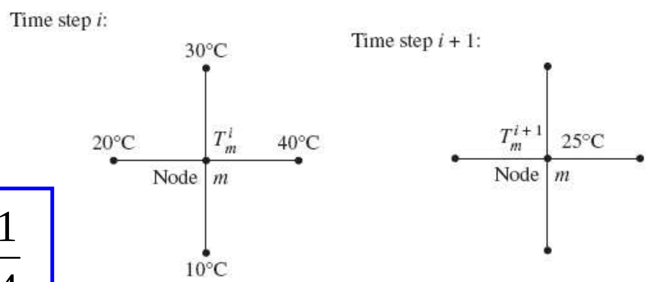
Bi dimentsioko bero-eroapen iragankorra

Metodo esplizitua

$$T_{left}^i+T_{top}^i+T_{right}^i+T_{bottom}^i-4T_{node}^i+\frac{\dot{e}_{node}^i\cdot l^2}{k}=\frac{T_{node}^{i+1}-T_{node}^i}{\tau}$$

$$T_{node}^{i+1}=\tau\cdot(T_{left}^i+T_{top}^i+T_{right}^i+T_{bottom}^i)+(1-4\tau)\cdot T_{node}^i+\tau\cdot\frac{\dot{e}_{node}^i\cdot l^2}{k}$$

$$T_{node}^{i+1}=\frac{T_{left}^i+T_{top}^i+T_{right}^i+T_{bottom}^i}{4}$$



Egonkortasun-irizpidea

$$\tau=\frac{\alpha\cdot\Delta t}{l^2}\leq\frac{1}{4}$$